Arithmetic Quantum Unique Ergodicity

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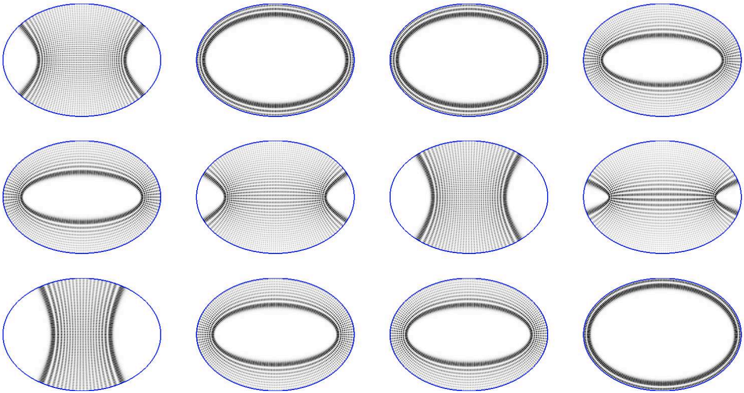

Figure 1E

Figure: Eigenfunctions on an ellipse, picture from
”Recent progress on QUE” by P. Sarnak

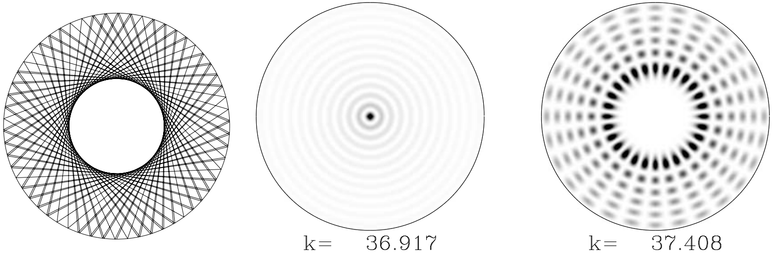

FIGURE 1.1. Left: one orbit of the circular billiard. Center and right: two eigenmodes of that billiard, with their respective frequencies.

Figure: Eigenfunctions on a circle, picture from ”Chaotic vibrations and strong scars” by Anantharaman and Nonnenmacher

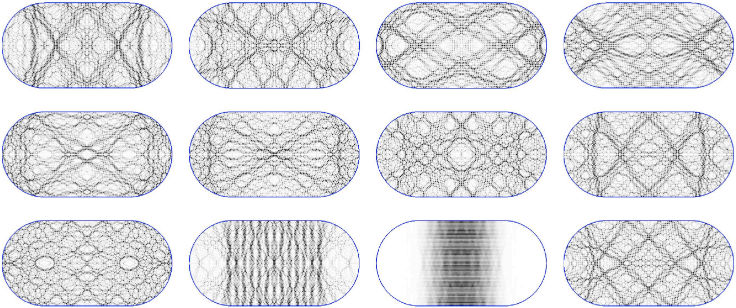

Figure 1S

Figure: Eigenfunctions on the stadium, picture from ”Recent progress on QUE” by P. Sarnak

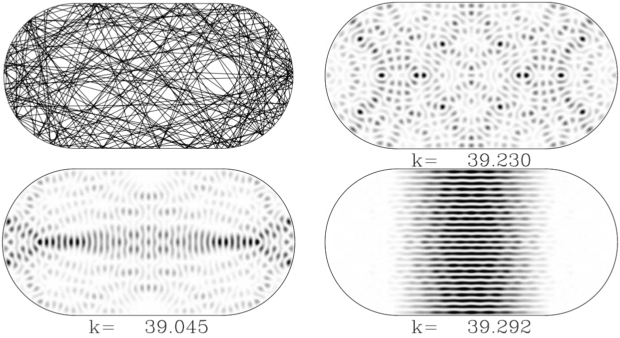

FIGURE 1.2. Top left: one typical “ergodic” orbit of the “stadium”: it equidistribues across the whole billiard. The three other plots feature eigenmodes of frequencies $k_n \approx 39$. Bottom left: a “scar” on the (unstable) horizontal periodic orbit. Bottom right: a “bouncing ball” mode.

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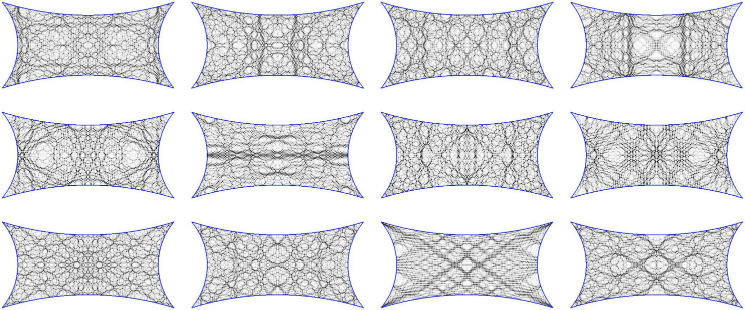

Figure 1B

Figure: Eigenfunctions on a dispersing Sinai billard, picture from "Recent progress on QUE" by P. Sarnak

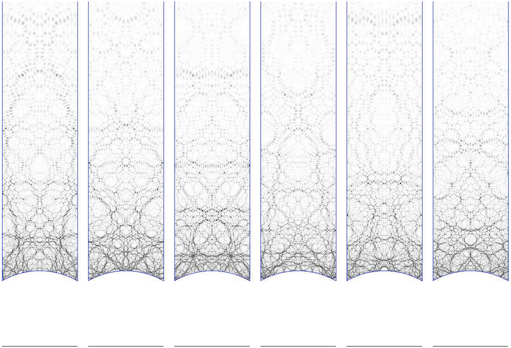

Figure 4a

Figure: Eigenfunctions on the modular surface, picture from ”Recent progress on QUE” by P. Sarnak

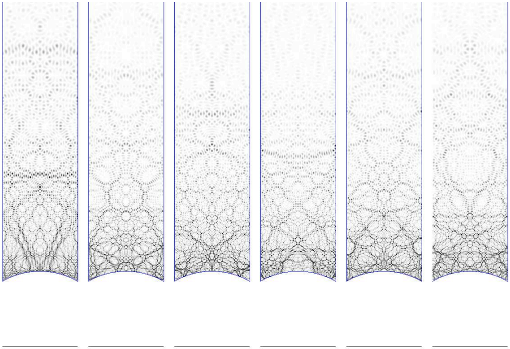

Figure 4b

Figure: Eigenfunctions on the modular surface, picture from ”Recent progress on QUE” by P. Sarnak

Conjecture 1.1 (Quantum Unique Ergodicity; Rudnick–Sarnak). Let Γ be a discrete subgroup of $\mathrm{SL}_2(\mathbb{R})$ such that $M = \Gamma\backslash\mathbb{H}$ is compact. If $\{\phi_i \mid i \in \mathbb{N}\}$ are normalized eigenfunctions for Δ in $C^\infty(M)$ with corresponding eigenvalues $\{\lambda_i \mid i \in \mathbb{N}\}$ such that $\lambda_i \to \infty$ as $i \to \infty$, then

$$|\phi_i|^2 \,\mathrm{dvol}_M \underset{\text{weak}*}{\longrightarrow} \mathrm{dvol}_M \tag{1.1}$$

as $i \to \infty$.

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as $i \to \infty$.

The same should hold for $M = \mathrm{SL}_2(\mathbb{Z})\backslash\mathbb{H}$.

Theorem 1.2. *Let $M = \Gamma\backslash\mathbb{H}$, with Γ a congruence lattice over $\mathbb{Q}$. Then*

$$|\phi_i|^2 \,\mathrm{dvol}_M \underset{weak*}{\longrightarrow} \mathrm{dvol}_M$$

as $i \to \infty$ for any sequence of Hecke–Maass cusp forms for which the Maass eigenvalues $\lambda_i \to -\infty$ as $i \to \infty$.

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(2) In [Lindenstrauss, 2006] it is shown that any limit measure is of the form $c\,\mathrm{dvol}_\mathrm{M}$ for some $c \in [0, 1]$.

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(3) In [Soundararajan, 2010] it is shown that $c = 1$, i.e. that there is no escpace of mass.
(4) Watson has shown before the work of Lindenstrauss that GRH implies the above theorem (with an optimal rate of convergence).

Theorem (Lindenstrauss)

Let Γ be a congruence lattice over $\mathbb{Q}$, let $X = \Gamma \backslash \mathrm{SL}_2(\mathbb{R})$ and let μ be a probability measure satisfying the following properties:

- [I] μ is *invariant* under the geodesic flow,
- $[R]_p$ μ is *Hecke p-recurrent* for a prime p, and
- [E] the *entropy* of every ergodic component of μ is positive for the geodesic flow.

Then $\mu = m_X$ is the Haar measure on X.

Theorem (microlocal lift).

Let $\Gamma \leqslant \mathrm{SL}_2(\mathbb{R})$ be a lattice, and let $M = \Gamma \backslash \mathbb{H}$. Suppose that (ϕ_i) is an L^2-normalized sequence of eigenfunctions of Δ in $C^\infty(M) \cap L^2(M)$, with the corresponding eigenvalues λ_i satisfying $|\lambda_i| \to \infty$ as $i \to \infty$, and assume that the weak*-limit μ of $|\phi_i|^2 \,\mathrm{dvol}_M$ exists. If $\widetilde{\phi}_i$ denotes the sequence of lifted functions defined later, then (possibly after choosing a subsequence to achieve convergence) the weak*-limit $\widetilde{\mu}$ of $|\widetilde{\phi}_i|^2 \,\mathrm{d}m_X$ has the following properties:

[L] Projecting $\widetilde{\mu}$ on $X = \Gamma \backslash G$ to $M = \Gamma \backslash G / K$ gives μ.

[I] $\widetilde{\mu}$ is invariant under the right action of A.

The measure $\widetilde{\mu}$ is called a *microlocal lift* of μ, or a *quantum limit* of (ϕ_i).

Proposition.

For $m, w \in \mathfrak{sl}_2(\mathbb{R})$ we have

$$m \circ w - w \circ m = [m, w]$$

where $[m, w] = mw - wm$ is the Lie bracket, defined by the difference of the matrix products. More concretely, this means that

$$m * (w * f) - w * (m * f) = ([m, w]) * f$$

for any $f \in C^\infty(X)$.