QUANTUM UNIQUE ERGODICITY AND NUMBER THEORY

KANNAN SOUNDARARAJAN

In this course I will describe recent progress on the "Quantum Unique Ergodicity" conjecture of Rudnick and Sarnak in a special arithmetic situation. To explain what this conjecture is about, let $\mathbb{H}$ denote the upper half plane $\{x + iy : y > 0\}$. The group $SL_2(\mathbb{R})$ acts on $\mathbb{H}$ by Möbius transformations, and let $\Gamma = SL_2(\mathbb{Z})$. In number theory it is of great interest to study functions on $\mathbb{H}$ that are either invariant under the action of Γ, or transform in some nice way under this action. The classical theory of modular forms of weight k (an even positive integer) considers holomorphic functions f satisfying

$$f(\gamma z) = (cz + d)^k f(z)$$

for all $\gamma = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in SL_2(\mathbb{Z})$. If we also require f to be holomorphic and decay rapidly "at the cusp at infinity" then we get the theory of cusp forms, and the most famous example of this is Ramanujan's Δ-function. In the 1940's and 50's Maass and Selberg developed a nice theory of functions satisfying$f(\gamma z) = f(z)$ for all $\gamma \in SL_2(\mathbb{Z})$. These functions are no longer holomorphic but are real analytic eigenfunctions of the Laplace operator $\Delta = -y^2(\frac{d^2}{dx^2} + \frac{d^2}{dy^2})$. If we also require that these eigen-functions should decay rapidly at ∞, then we get the theory of *Maass* cusp-forms; even their existence is not easy to demonstrate, and was first established by Selberg using his trace formula.

Let ϕ denote such a Maass cusp form, and let λ denote its Laplace eigenvalue, and let ϕ be normalized so that $\int_X |\phi(z)|^2 \frac{dx\ dy}{y^2} = 1$ (where $X = SL_2(\mathbb{Z})\backslash\mathbb{H}$). From work of Zelditch it follows that as $\lambda \to \infty$, for a *typical* Maass form ϕ the measure $\mu_\phi := |\phi(z)|^2 \frac{dx\ dy}{y^2}$ approaches the uniform distribution measure $\frac{3}{\pi} \frac{dx\ dy}{y^2}$. That is, typically the L^2-mass is uniformly spread out over a fundamental domain for $\Gamma\backslash\mathbb{H}$. This statement is referred to as "Quantum Ergodicity." Rudnick and Sarnak have conjectured that an even stronger result holds. Namely, that as $\lambda \to \infty$, for *every* Maass form ϕ the measure μ_ϕ approaches the uniform distribution measure. This is a special case of their "Quantum

Unique Ergodicity" conjecture. Lindenstrauss made great progress towards this conjecture, showing that, for Maass cusp forms that are eigenfunctions of the Laplacian and all the Hecke operators,[1] the only possible limiting measures are of the form $\frac{3}{\pi}c\frac{dx\ dy}{y^2}$ with $0 \le c \le 1$. Recently I showed that $c = 1$, completing the proof of the QUE conjecture for Hecke-Maass forms on $SL_2(\mathbb{Z})\backslash\mathbb{H}$. Lindenstrauss's work is based on ergodic theory and measure rigidity, and these will be explained in Einsiedler's lectures.

In this course, I will explain the proof (due to Holowinsky and me) of the analog of the quantum unique ergodicity conjecture for classical holomorphic modular forms. Let f be a holomorphic modular cusp form of weight k (an even integer) for $SL_2(\mathbb{Z})$. Associated to f we have the measure

$$\mu_f := y^k|f(z)|^2\frac{dx\ dy}{y^2},$$

which is invariant under the action of $SL_2(\mathbb{Z})$, and we suppose that f has been normalized so that

$$\int_X y^k|f(z)|^2\frac{dx\ dy}{y^2} = 1.$$

The space $S_k(SL_2(\mathbb{Z}))$ of cusp forms of weight k for $SL_2(\mathbb{Z})$ is a vector space of dimension about $k/12$, and contains elements such as $\Delta(z)^{k/12}$ (if $12|k$, and where Δ is Ramanujan's cusp form) for which the measure will not tend to uniform distribution. Therefore one restricts attention to a particularly nice set of cusp forms, namely those that are eigenfunctions of all the Hecke operators. The Rudnick-Sarnak conjecture in this context states that as $k \to \infty$, for *every* Hecke eigencuspform f the measure μ_f tends to the uniform distribution measure. For simplicity, we have restricted ourselves to the full modular group, but the conjecture could be formulated just as well for holomorphic newforms of level N. Luo and Sarnak have shown that equidistribution holds for most Hecke eigenforms, and Sarnak has shown that it holds in the special case of dihedral forms. It does not seem clear how to extend Lindenstrauss's work to the holomorphic setting.[2]

The proof of this holomorphic QUE combines two different approaches developed independently by Holowinsky and myself. At their heart,

---

[1] The spectrum of the Laplacian is expected to be simple, so that any eigenfunction of the Laplacian would automatically be an eigenfunction of all Hecke operators. This is far from being proved.

[2] The difficulty from the ergodic point of view concerns the invariance under the geodesic flow of the quantum limits of the micro-local lifts associated to holomorphic forms.

both approaches rely on an understanding of mean-values of multiplicative functions, and I will explain some of the key results in that area. Either of these approaches is capable of showing that there are very few possible exceptions to the conjecture, and under reasonable hypotheses either approach would show that there are no exceptions. However, it seems difficult to show unconditionally that there are no exceptions using just one of these approaches. Fortunately, as we shall explain below, the two approaches are complementary, and the few rare cases that are untreated by one method fall easily to the other method. Both approaches use in an essential way that the Hecke eigenvalues of a holomorphic eigencuspform satisfy the Ramanujan conjecture (Deligne's theorem). The Ramanujan conjecture remains open for Maass forms, and this is the (only) barrier to using our methods in the non-holomorphic setting.[3]

I give below a brief overview of the topics to be covered in the lectures, and a list of some potential projects.

Lecture 1: Motivation and formulation of the conjectures and results. Spectral expansion of functions on $\Gamma\backslash\mathbb{H}$. Expansion in terms of Poincare series.

Lecture 2: Work of Luo and Sarnak connecting QUE with shifted convolution problems. Work of Watson connecting QUE with L-functions and the subconvexity problem and the Generalized Riemann hypothesis.

Lecture 3: Mean values of multiplicative functions and their connection with these problems. Holowinsky's work on QUE.

Lecture 4: My work on weak subconvexity for L-functions. Putting together weak subconvexity and the work of Holowinsky to obtain mass equidistribution for Hecke eigenforms.

Some problems:

1. The one remaining case of arithmetic QUE is the case of holomorphic forms of large weight for compact arithmetic quotients. Here the ergodic approach doesn't seem to work, and since there are no Poincare series there is no obvious analog of Holowinsky's work. Some new ideas seem required. Another interesting example of a compact setting where QUE is not known, is the case of Hecke eigenfunctions for the sphere (see work of Bocherer, Sarnak, and Schulze-Pillott).
2. Develop the analog of QUE for newforms of weight 2 and large level N. Here one projects the measures from $\Gamma_0(N)\backslash\mathbb{H}$ to $\Gamma\backslash\mathbb{H}$ and

[3]Assuming the Ramanujan conjecture for Maass forms, our methods would obtain the stronger micro-local version of QUE. Moreover our methods would then be able to quantify the rate at which equi-distribution is attained.

equidistribution is sought for these projections as $N \to \infty$. In the case of large weight, the equidistribution of measures led by work of Rudnick to the equidistribution of zeros of eigencuspforms. Is there an analog of Rudnick's work for the large level case.

3. To what extent can the assumption of the Ramanujan conjectures be weakened in the work of Holowinsky and Soundararajan. If this could be done fully then one would obtain QUE for Maass forms with a effective rate of convergence.

The three problems above are probably difficult research questions, and I don't expect them to be solved soon (e.g. at the winter school!). However they may offer some motivating and long term questions for the participants. More manageable problems follow:

4. Understand the ideas behind the Duke-Friedlander-Iwaniec amplification method and subconvexity for a special case (e.g. for L-functions of a modular form of large weight k). A general subconvexity result would immediately give QUE, among other applications.

5. Study the L^2-restrictions of eigenfunctions to fixed curves in $\Gamma\backslash\mathbb{H}$.

6. Discussion of my work on "escape of mass." This relies upon an inequality for "Hecke multiplicative" functions, and it would be interesting to develop an analog of that inequality for functions satisfying the multiplicative property that the Fourier coefficients of automorphic forms on $GL(N)$ would satisfy.