Course and Project Description: The rank one abelian Stark conjecture

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In this course and project description, we briefly present the following:

- The statement of Stark's rank one abelian conjecture and of the central motivating problem of this lecture series, namely: *can we give (conjectural) exact formulas for Stark's units?*
- The claim that this question has an affirmative answer via the construction of explicit cycles in the "Eisenstein" cohomology of certain arithmetic groups.
- A discussion of a student project to be pursued at the AWS.
- A list of open research problems suggested by these lines of inquiry.

1 Statement of the Conjecture

Let K/F denote an abelian extension of number fields with associated rings of integers $\mathcal{O}_K$ and $\mathcal{O}_F$. Let S denote a finite set of places of F containing the archimedean places and those which ramify in K. Assume that S contains at least one place v that splits completely in K and that $|S| \geq 2$. For each ideal $\mathfrak{n} \subset \mathcal{O}_F$ not divisible by a prime that ramifies in K, we denote by $\sigma_{\mathfrak{n}}$ the associated Frobenius element in $\mathrm{Gal}(K/F)$. For each element $\sigma \in G := \mathrm{Gal}(K/F)$, we define the partial zeta-function

$$\zeta_{K/F,S}(\sigma, s) := \sum_{\mathfrak{n} \subset \mathcal{O}_F, (\mathfrak{n}, S)=1, \sigma_{\mathfrak{n}} = \sigma} \frac{1}{\mathrm{N}\mathfrak{n}^s}, \qquad s \in \mathbf{C}, \mathrm{Re}(s) > 1.$$

Here $\mathrm{N}\mathfrak{n}$ denotes the norm of the ideal $\mathfrak{n}$. Each function $\zeta_{K/F}(\sigma, s)$ has an analytic continuation to $\mathbf{C}$, with only a simple pole at $s = 1$. The fact that S contains a place v that splits completely in K ensures that $\zeta_{K/F,S}(\sigma, 0) = 0$ for all $\sigma \in G$. Denote by e the number of roots of unity in K. Let $U_{v,S} = U_{v,S}(K)$ denote the set of elements $u \in K^\times$ such that:

- if $|S| \geq 3$, then $|u|_{w'} = 1$ for all $w' \nmid v$;

- if $S = \{v, v'\}$, then $|u|_{w'}$ is the same for all w' above v', and $|u|_{w'} = 1$ for all $w' \notin S$.

The following is the rank one abelian Stark conjecture.

Conjecture 1.1 (Stark [15]). *Fix a place w of K lying above v. There exists an element $u \in U_{v,S}$ such that*

$$\zeta'_{K/F,S}(\sigma, 0) = -\frac{1}{e} \log |u^\sigma|_w \text{ for all } \sigma \in G \tag{1}$$

and such that $K(u^{1/e})/F$ is an abelian extension.

The conditions $u \in U_{v,S}$ and equation (1) together specify the absolute value of u at every place of K. Therefore, if the S-unit u exists, it is unique up to multiplication by a root of unity in $K^\times$. It is possible to state an alternate equivalent version of Conjecture 1.1 in which the relevant unit is actually unique (not just up to a root of unity), but we will not explain this here.

2 Motivating Question

When S contains two primes that split completely in K, the rank one abelian Stark conjecture holds trivially (for instance, if S contains two primes that split completely in K and $|S| \geq 3$, then $\zeta'_{K/F,S}(\sigma, 0) = 0$ for all $\sigma \in G$, so $u = 1$ satisfies the conjecture). Therefore, we need only consider the setting where S contains exactly one prime v that splits completely in K. Since complex places split completely in every extension, we are left with the following possibilities:

- Case TR_∞: F is totally real, and the place v is real. The places of K above v are real, and all other archimedean places are complex.
- Case ATR: F is "almost totally real," i.e. it has exactly one complex place v and all other places are real. The field K is totally complex.
- Case TR_p: F is totally real and the place v is finite. The field K is totally complex.

In case TR_∞, equation (1) gives an exact formula for u and its conjugates up to sign:

$$u^\sigma = \pm \exp(-2\zeta'_{K/F,S}(\sigma, 0)) \text{ in the real embedding } w. \tag{2}$$

If one is liberal in interpretation, equation (2) can be viewed as an "explicit class field theory" for the extension K/F, in analogy with the explicit class field theory for imaginary quadratic fields provided by the theory of complex multiplication. In computational terms, it is possible to write down the characteristic polynomial of u over F in the real embedding v by taking as coefficients the appropriate elementary symmetric functions of the values in (2), after choosing the correct signs. Then, assuming that a basis for $\mathcal{O}_F$ is known, it would be possible to "recognize" these real numbers as elements of F using standard lattice algorithms (such as LLL) and thereby write down the characteristic polynomial of u as an element of $F[x]$. In this way, Stark's conjecture in case TR_∞ can be viewed as giving progress towards an explicit class field theory for F and has special significance in the study of Hilbert's 12th problem.

Let us now consider case ATR. Since the place w is complex, inverting equation (1) only yields a formula for the absolute value of u and its conjugates:

$$|u^\sigma|_w = \exp(-e\zeta'_{K/F,S}(\sigma, 0)),$$

not for the image of $u \in \mathbf{C}$ under the embedding w itself. The distinction with case TR_∞ is that the group of elements of $\mathbf{C}^\times$ with absolute value 1 is an entire circle, not merely the finite set $\{\pm 1\}$. Unless we can somehow specify the *argument* of the complex number u, it is not possible to directly write down the characteristic polynomial of u as an element of $F[x]$ as simply as we described in case TR_∞; therefore, in case ATR, Stark's conjecture does not directly make contact with explicit class field theory and Hilbert's 12th problem. This leads us to the central motivating problem of this article.

Question 2.1. Can we give, in all three cases of the rank one abelian Stark conjecture, an exact formula for the image of u at the place w rather than just a formula for its absolute value?

We will see that the answer to this question is essentially "yes," though the formulas that arise are not stated as succinctly as Stark's conjecture.

3 Cycles in Group Cohomology

In each of the three cases, there is a setting where classical explicit class field theory provides a concrete construction of Stark units and thereby gives a very elegant solution to our motivating question. Namely,

- Case TR_∞ when $F = \mathbf{Q}$. Stark units are given by cyclotomic units.
- Case TR_p when $F = \mathbf{Q}$. Stark units are given by Gauss sums.
- Case ATR when F is a quadratic imaginary field. (Yes, by our definition, quadratic imaginary fields are "almost totally real"!) Stark units are given by elliptic units, via the theory of complex multiplication.

In each case, the Stark unit is given by an explicit formula that can be interpreted as the value of a certain analytic function at a particular point. For example, in the first case the units are given by the values of the function $f(x) = 2 - 2\cos(2\pi x)$ at rational arguments x. One key fact that makes the cases $F = \mathbf{Q}$ and $F =$ quadratic imaginary easier than other cases is that the group of units $\mathcal{O}_F^\times$ in these situations is finite. In studying Stark's conjecture in other settings, it becomes clear that the presence of a unit group with positive rank provides an obstacle. Recently, two methods have been developed for circumventing this obstacle:

- *Shintani domains:* Following Shintani, one chooses a fundamental domain for the units acting on F cut out by a union of simplicial cones; after formulating a conjectural construction of Stark units, one must then go back and check that the construction does not depend on the choice of fundamental domain taken. See, e.g., [13, 10, 5] in the cases $\mathrm{TR}_\infty, \mathrm{ATR}, \mathrm{TR}_p$, respectively.

- *Arithmetic cohomology*: Instead of looking at values of functions, look at specializations of group cohomology classes. The classes will be in $H^r(G)$ for an arithmetic group G with r the rank of $\mathcal{O}_F^\times$, such that G is equipped with a homomorphism $\varphi : \mathcal{O}_F^\times \to G$. The specialization will be the value of the class on the image of a basis of units under φ.

The second method will be the point of view adopted at the AWS. To give the flavor of our constructions, let us be more precise in one particular case. We consider case TR_p when F is a real quadratic field and p is a prime that is inert in F. Let $\mathbf{X} = \mathbf{Z}_p^2 - p\mathbf{Z}_p^2$, the space of primitive vectors in $\mathbf{Z}_p^2$. Let $M(\mathbf{X})$ denote the space of $\mathbf{Z}$-valued measures on $\mathbf{X}$ with total measure zero. We will describe the construction of a certain Eisenstein group cohomology class $[\kappa] \in H^1(\mathbf{GL}_2(\mathbf{Z}), M(\mathbf{X}))$. (In fact, to define $[\kappa]$ we will need to pass to a certain congruence subgroup of $\mathbf{GL}_2(\mathbf{Z})$, but let us ignore this point here.) This class will not depend on the real quadratic field F. Choose an embedding $\varphi : \mathcal{O}_F \hookrightarrow M_2(\mathbf{Z})$, restricting to a group homomorphism $\mathcal{O}_F^\times \hookrightarrow \mathbf{GL}_2(\mathbf{Z})$. Let $\tau \in F - \mathbf{Q}$, and consider the group of elements $\epsilon \in \mathcal{O}_F^\times$ such that $\varphi(\epsilon)$ has $\binom{\tau}{1}$ as an eigenvector. This is a finite index subgroup of $\mathcal{O}_F^\times$; let ϵ_τ denote a generator. We conjecture that up to an explicit power of p, the p-adic number

$$\times\!\!\!\!\!\!\int_{\mathbf{X}} (x - y\tau) d\kappa(\varphi(\epsilon_\tau))(x, y) \in F_p^\times$$

actually lies in the narrow Hilbert class field K/F and is a Stark unit for that extension. Here the integral with an $\times$ through it represents a multiplicative integral. We will define this notion in the lectures. This approach is generalized to totally real fields F of arbitrary degree n in [2] using the Eisenstein cocycle $[\kappa] \in H^{n-1}(\mathbf{GL}_n(\mathbf{Z}), M)$ of Sczech [11], where M is an exotic module reminiscent of $M(\mathbf{X})$.

Another instance of a refinement of Stark's conjecture obtained via arithmetic cohomology is the work of Charollois and Darmon [1]. They consider the ATR situation in the special case where F is a quadratic extension of a totally real field F_0. (In particular, F has even degree.) Charollois and Darmon give a conjectural formula for the argument of the Stark unit in terms of integrals of Eisenstein series on Hilbert modular varieties. It is tempting to attempt to hybridize the approach of Charollois and Dasgupta with that of Charollois and Darmon, attempting to construct Stark units using Eisenstein series on $\mathbf{GL}_n(\mathbf{Z})$. We hope to discuss these issues further in the lectures.

4 Group project and other open questions

In all three cases (TR_p, TR_∞, ATR), alternate conjectural constructions of Stark units have been provided using Shintani's method of choosing fundamental domains consisting of simplicial cones. For example, in [5], Shintani's method was used to give a formula for Stark's units in the case TR_p. Shintani's method will be the topic of the group project at the AWS. The following specific question asks for a link between the two methods currently used to give explicit formulas for Stark units.

Question 4.1. Can one define group cocycles directly using Shintani's method and prove that they are cohomologous to the classes defined at the AWS, thereby proving the equivalence of the two different formulas for Stark units? For example, in the case TR_p, can one use Shintani's method to construct a class in $H^1(\mathbf{GL}_2(\mathbf{Z}), M(\mathbf{X}))$ that on the one hand clearly specializes to the formulas of [5], and on the other hand can be proven to be cohomologous to $[\kappa]$ from the previous section? The work of Hill [9] and Solomon [14] may be relevant for this project.

In addition to the question above, which we hope will lead to a group project at the AWS, we conclude with some open research questions.

Question 4.2. Can the constructions in the three cases discussed here (TR_p, TR_∞, ATR) be unified via an adelic construction? Moreover, can these constructions be unified logically (not just by analogy) with the construction of Stark–Heegner points on elliptic curves?

Question 4.3. Can one prove a rational version (i.e. "tensored wth $\mathbf{Q}$" version) of the exact formula for Stark units in case TR_p using the modular techniques of [6]?

Question 4.4. Can one give explicit formulas for Stark units—not just their regulators—in higher rank cases (we will not consider higher rank cases at the AWS). For example, suppose we are in case TR_p where F contains two primes above p that split completely in K. The relevant group of units breaks up into the group of π-units for each π above p, so the regulator matrix in the (higher rank) abelian Gross–Stark conjecture is well–defined, not just its determinant. Can one give conjectural formulas for the individual entries of the regulator matrix in this case? The methods of [5] give a formula for the diagonal entries of the regulator matrix, but the off-diagonal entries are still a mystery.

5 Background reading

- For an introduction to Stark's conjectures, we recommend [4, 16].
- We encourage participants to familiarize themselves with the basics of Shintani's method. See [12], for instance.
- Familiarity with the methods of [1] is desirable, both for our course as well as for that of Darmon and Rotger.
- For the p-adic perspective, see [7, 8, 3].

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