Group actions on curves and the lifting problem
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Preliminary version

1 Outline

1.1 The lifting problem The problem we are concerned with in our lectures and which we shall refer to as the *lifting problem* was originally formulated by Frans Oort in [12]. To state it, we fix an algebraically closed field κ of positive characteristic p. Let $W(\kappa)$ be the ring of Witt vectors over κ. Throughout our notes, $\mathfrak{o}$ will denote a finite local ring extension of $W(\kappa)$ and $k = \mathrm{Frac}(\mathfrak{o})$ the fraction field of $\mathfrak{o}$. Note that $\mathfrak{o}$ is a complete discrete valuation ring of characteristic zero with residue field κ.

Definition 1.1 Let C be a smooth proper curve over κ. Let $G \subset \mathrm{Aut}_\kappa(C)$ be a finite group of automorphisms of C. We say that the pair (C, G) *lifts to characteristic zero* if there exists a finite local extension $\mathfrak{o}/W(\kappa)$, a smooth projective $\mathfrak{o}$-curve $\mathcal{C}$ and an $\mathfrak{o}$-linear action of G on $\mathcal{C}$ such that

(a) $\mathcal{C}$ is a lift of C, i.e. there exists an isomorphism $\lambda : \mathcal{C} \otimes_\mathfrak{o} \kappa \cong C$, and

(b) the G-action on $\mathcal{C}$ restricts, via the isomorphism λ, to the given G-action on C.

Problem 1.2 (The lifting problem) Which pairs (C, G) as in Definition 1.1 can be lifted to characteristic zero?

Remark 1.3 It is easy to find pairs (C, G) which cannot be lifted to characteristic zero. To see this, assume that $(\mathcal{C}, G)$ is an equivariant lift of (C, G). Then both the special fiber C and the generic fiber $\mathcal{C}_k := \mathcal{C} \otimes_\mathfrak{o} k$ of $\mathcal{C}$ are smooth projective curves of the same genus g. Since the characteristic of k is zero, the classical *Hurwitz bound* applies and shows that for $g \geq 2$ we have

$$|G| \leq 84(g-1). \tag{1}$$

See e.g. [18], Exercise IV.2.5. However, in characteristic p there exist pairs (C, G) violating this bound (see Exercise 1.11 for an example). It follows that such pairs (C, G) cannot be lifted.

Another way to produce counterexamples is to take $C := \mathbb{P}^1_\kappa$ and use that $\mathrm{Aut}_L(\mathbb{P}^1_L) = \mathrm{PGL}_2(L)$ for any field L, see Exercise 1.10

1.2 The local lifting problem and the local-global principle At first sight, Remark 1.3 seems to suggest that the liftability of a pair (C, G) is a global issue, as the Hurwitz bound depends on the genus. However, Theorem 1.5 below states that, on the contrary, liftability depends only on the (finite) set of closed points $y \in C$ with nontrivial stabilizers $G_y \subset G$ and the action of G_y on the formal neighborhood of y. The lifting problem is thus reduced to the following *local lifting problem*.

Definition 1.4 A *local action* is a pair $(\bar{A}, G)$, where $\bar{A} = \kappa[[z]]$ is a ring of formal power series in κ and $G \subset \mathrm{Aut}_\kappa(\bar{A})$ is a finite group of automorphisms of $\bar{A}$. We say that the pair $(\bar{A}, G)$ *lifts to characteristic zero* if there exists a finite extension $\mathfrak{o}/W(\kappa)$ and an action of G on $A := \mathfrak{o}[[z]]$ lifting the given G-action on $\bar{A}$.

The *local lifting problem* is the question 'which local actions $(\bar{A}, G)$ lift to characteristic zero?'.

Theorem 1.5 (The local-global principle) *Let (C, G) be as in Definition 1.1. Then (C, G) lifts if and only if for all closed points $y \in C$ the induced local action $(\hat{\mathcal{O}}_{C,y}, G_y)$ lifts. (Note that $\hat{\mathcal{O}}_{C,y}$ is a ring of formal power series since C is smooth over κ.)*

Proof: One direction is more or less obvious: if $(\mathcal{C}, G)$ is a lift of (C, G), then smoothness of $\mathcal{C}$ shows that $\hat{\mathcal{O}}_{\mathcal{C},y}$ is a ring of formal power series over $\mathfrak{o}$. Therefore, $(\hat{\mathcal{O}}_{\mathcal{C},y}, G_y)$ is a lift of $(\hat{\mathcal{O}}_{C,y}, G_y)$, for all $y \in C$.

For the proof of the converse, see e.g. [14], [9], or [4]. □

Corollary 1.6 *Suppose that for all $y \in C$ the order of the stabilizer G_y is prime to p. Then (C, G) lifts.*

Proof: Assume that $p \nmid |G_y|$. Then Exercise 1.12 (a) shows that G_y is cyclic. Moreover, one can choose $z \in \hat{\mathcal{O}}_{C,y}$ such that $\hat{\mathcal{O}}_{C,y} = \kappa[[z]]$ and $\sigma(z) = \bar{\zeta} \cdot z$ (here σ is a generator of G and $\zeta \in \kappa$ a primitive nth root of unity). Since $(p, n) = 1$, Hensel's Lemma shows that $\bar{\zeta}$ lifts uniquely to an nth root of unity $\zeta \in \mathfrak{o}$. So the rule $\sigma(z) := \zeta \cdot z$ defines a lift of the natural G_y-action on $\hat{\mathcal{O}}_{C,y} = \kappa[[z]]$ to the ring $\mathfrak{o}[[z]]$. Now apply Theorem 1.5. □

Remark 1.7 Corollary 1.6 corresponds to a well known fact from Grothendieck's theory of the *tame fundamental group*, see [16]. Let $D := C/G$ denote the quotient curve and $x_1, \ldots, x_r \in D$ the images of the points on C with nontrivial stabilizers. Then the quotient map $f : C \to D$ is a G-Galois cover, which is tamely ramified in $x_1, \ldots, x_r$ under the hypothesis of Corollary 1.6. Let $\mathcal{D}$ be a lift of C to a smooth proper $\mathfrak{o}$-curve (which exists by [16], Chapter 3). Choose sections $x_{\mathfrak{o},i} : \operatorname{Spec} \mathfrak{o} \to \mathcal{D}$ lifting the points x_i. Now Grothendieck's theory shows that there exists a *unique* lift of the cover π to a G-Galois $f_{\mathfrak{o}} : \mathcal{C} \to \mathcal{D}$ tamely ramified along the sections $x_{\mathfrak{o},i}$. By construction, $(\mathcal{C}, G)$ is a lift of (C, G).

The standard proof of this result (and of Theorem 1.5) uses *formal patching* (see e.g. the lectures by Hartmann and Harbater).

Remark 1.8 Let G be a finite group of automorphisms either of $\kappa[[z]]$ or of $\mathfrak{o}[[z]]$. Then G is a so-called *cyclic-by-p group*, i.e. $G = P \rtimes C$, where P is the Sylow p-subgroup of G and C is a cyclic group of order prime to p (see Exercise 1.12).

This results significantly cuts down the classes of groups we have to consider for the local lifting problem. But it does not give any obstruction against liftability, because it applies to both rings $\kappa[[z]]$ and $\mathfrak{o}[[z]]$.

1.3 Group actions versus Galois covers It turns out to be extremely difficult to approach the lifting problem by working directly with automorphisms of $\kappa[[z]]$ and $\mathfrak{o}[[z]]$ in terms of explicit power series (see Exercise 1.13). To really get our hands on the problem we need a shift of perspective.

Let C be a smooth projective curve over κ and $G \subset \operatorname{Aut}_\kappa(C)$ a finite group of automorphisms. We have already remarked that the quotient map $f : C \to D := C/G$ is a finite Galois cover. Knowing the pair (C, G) is equivalent to knowing the cover $f : C \to D$, and in principal we can replace *group actions* by *Galois covers* everywhere.

An advantage of this point of view is that we have more tools to construct these objects. For instance, to construct a Galois cover $f : C \to D$ of a given curve D it suffices to define a Galois extension L/K of the function field $K := \kappa(D)$; the corresponding curve C is then simply the normalization of C in L. The same approach works in the local setting, see Exercise 1.14.

Lifting group actions is also equivalent to lifting Galois covers. However, the two points of view may lead to very different techniques for solving instances of the lifting problem. For instance, if one proves the Local-Global-Principle (Theorem 1.5) with formal patching (as e.g. in [9]) one uses the perspective of covers. In contrast to this, the proof given in [4] works directly with a pair (C, G).

1.4 Stable models of Galois covers and Hurwitz trees We try to describe, as briefly as possible, our approach to the lifting problem, which is based on the study of semistable reduction and group actions on semistable curves. The origin of this method is the work of Raynaud, Green, Matignon and Henrio ([15], [19]).

We start with a Galois cover $f : Y \to X$ of smooth projective curves over the local field k. (Recall that k is a complete discrete valuation field of characteristic zero, whose residue field has characteristic $p > 0$.) Let G denote the Galois group of f. Let $y_1, \ldots, y_r$ be the ramifications points of f (we assume that all of them are k-rational). Assuming that $2g(Y) - 2 + r > 0$ and that k is sufficiently large, there exists a canonically defined $\mathfrak{o}$-model $\mathcal{Y}$ of Y called the *stably marked model*. It is the minimal $\mathfrak{o}$-model of Y which is semistable and such that the points y_i specialize to pairwise distinct smooth points on the special fiber $\bar{Y} := \mathcal{Y} \otimes_{\mathfrak{o}} \kappa$. The action of G on Y extends to $\mathcal{Y}$. The quotient scheme $\mathcal{X} := \mathcal{Y}/G$ is a semistable model of X. We call the quotient map $\mathcal{Y} \to \mathcal{X}$ the *stable model* of the Galois cover $f : Y \to X$ and its restriction to the special fiber $\bar{f} : \bar{Y} \to \bar{X}$ the *stable reduction* of f.

We say that the cover f has *tame good reduction* if $\bar{Y}$ and $\bar{X}$ are smooth. If this is the case, then $\bar{X}$ and $\bar{Y}$ are irreducible and $\bar{f}$ is an at most tamely ramified G-Galois cover. We can consider f (resp. the pair (Y, G)) as a lift of $\bar{f}$ (resp. of the pair $(\bar{Y}, G)$). Grothendieck's theory of the tame fundamental group (compare Corollary 1.6 and Remark 1.7) says that the lift f is uniquely determined by $\bar{f}$, the curve X and the choice of points $x_j \in X$ lifting the branch points of $\bar{f}$.

We are of course more interested in the case that f has bad reduction. Then the map $\bar{f}$ is still a finite G-invariant map, but it will typically reveal phenomena of wild ramification. Firstly, if $\bar{Y}_1 \subset \bar{Y}$ is an irreducible component, then the restriction of $\bar{f}$ to $\bar{Y}_1$ is in general inseparable. Even if it is separable, it may be wildly ramified.

Problem 1.9 Can one characterize the finite G-invariant maps $\bar{f} : \bar{Y} \to \bar{X}$ between semistable curves over κ which arise as the stable reduction of a G-Galois cover $f : Y \to X$?

In some sense, the lifting problem is just a special case of this problem, for the following reason. Suppose $\bar{f}_0 : \bar{Y}_0 \to \bar{X}_0$ is a G-Galois cover between smooth curves over κ. Let $f : Y \to X$ be a lift of $\bar{f}$, defined over the local field k. Let $\bar{f} : \bar{Y} \to \bar{X}$ be the stable model of f. Then $\bar{f}_0$ can be recovered from $\bar{f}$ by contracting all but one component of $\bar{Y}$ and $\bar{X}$. More precisely, we have a commutative diagram

$$\begin{array}{ccc} \bar{Y} & \longrightarrow & \bar{Y}_0 \\ {\scriptstyle \bar{f}}\downarrow & & \downarrow{\scriptstyle \bar{f}_0} \\ \bar{X} & \longrightarrow & \bar{X}_0, \end{array}$$

where the horizontal arrows are contraction maps. In this situation we will say that the cover f has *good reduction* (as opposed to tame good reduction). In the light of Problem 1.9 we regard $\bar{f}$ as an 'enhancement' of $\bar{f}_0$ which encodes information on the lift f. Thus, in order to show that a lift of $\bar{f}_0$ exists it is natural to first try to enhance $\bar{f}_0$ to a map $\bar{f}$ with certain good properties and then try to lift $\bar{f}$.

In the stated generality, Problem 1.9 is very hard. If p does not divide the order of G, a complete answer is known by the theory of *tame admissible covers*.

The only other case that is reasonably well understood is when the Sylow p-subgroup of G has order p, and we will often make this assumption during our lectures. But the general philosophy can be described, without any assumption, as follows. Using *higher ramification theory* we attach to the map $\bar{f}$ certain extra data (called *Swan conductors*, *deformation data* and the like). We then try to find enough rules that these extra data must satisfy (with respect to the map f, the action of G etc.) Finally, we try to show that any map $\bar{f}$ together with enough extra data satisfying all known rules occur as the stable reduction of a Galois cover in characteristic zero.

1.5 Content and focus of our lectures In our lectures we do not try to give a comprehensive survey of known results on the lifting problem. Instead, we focus on a few particular results (both positive and negative) and the methods they rely on. Our choice of results and the way we present them will be very much biased by our own contributions to the topic.

It is sometimes useful to distinguish between negative and positive results. Here we call a result *negative* if it gives some obstruction against liftability which shows that certain pairs (C, G) (resp. $(\bar{A}, G)$) cannot be lifted. A result is called *positive* if it shows that certain pairs can be lifted. In these notes, however, we treat both aspects simultaneously, and stress the principals underlying positive and negative results. We plan to treat the following topics in some detail.

- **Obstructions:** A systematic way to find necessary conditions for liftability of a pair $(\bar{A}, G)$ is to study group actions on semistable curves. Using ramification theory, we can attach certain invariants to such an action which 'live on the special fiber'. Compatibility rules connecting these invariants then lead to necessary conditions for liftability of local actions $(\bar{A}, G)$. These can be roughly classified into three types, which we call *combinatorial*, *metric* and *differential*. We will focus on the general approach and on some specific but enlightening examples.
- **p-Sylow of order p:** Let $(\bar{A}, G)$ be a local action such that p strictly divides the order of G (then $G \cong \mathbb{Z}/p\mathbb{Z} \rtimes \mathbb{Z}/n\mathbb{Z}$, with $(p, n) = 1$ by Remark 1.8). In this case there is an if-and-only-if condition for liftability which only depends on certain discrete invariants attached to $(\bar{A}, G)$. In other words, $(\bar{A}, G)$ lifts if and only if the *Bertin obstruction* vanishes. This was proved in [5] and [6]. We will explain the main steps of the proof.
- **Cyclic actions and the Oort conjecture:** It is expected that for a cyclic group G all local actions $(\bar{A}, G)$ lift. This expectation is traditionally called the *Oort conjecture*. It has been proved for cyclic groups of order $p^n m$, where $(p, m) = 1$ and $n \leq 3$ (see [35] for $n = 1$, [14] for $n = 2$ and [32] for $n = 3$). We will present some elements of the recent work of Obus and the second author which gives a sufficient condition for liftability of general cyclic actions and in particular settles the case $n \leq 3$ of the Oort conjecture.

1.6 Prerequisites and reading list We will assume that students are more or less familiar with the following material.

- Artin–Schreier and Kummer Theory, Witt vectors, Hensel's Lemma. These can be found in most algebra book on the graduate level. A more advanced approach to Artin–Schreier and Kummer Theory can be found in [30].
- Ramification theory of local fields, in particular higher ramification groups and conductors. The standard reference in [38], Chapters 3–5. Section 2 of our notes will contain a review of all the results that we will need. These include the case of a non-perfect residue field (see [40] and the course notes of Saito and Mieda), but we will not assume that this material is already known.
- Blowing-up, arithmetic surfaces, models of curves. The definition of blowing-up can for example be found in [28], Section 8.1. Section 10.1 of [28] contains more material on arithmetic surfaces than we will require. Concrete examples of blowing-ups of arithmetic surfaces can for example be found in [39], Chapter IV.
- The p-adic disk (this is the rigid analytic spaces associated to the ring of power series $\mathfrak{o}[[z]]$). We will not use rigid analysis in any deep way, but you should know at least some basic facts about power series over p-adic rings, such as the Weierstrass Preparation Theorem and properties of the Newton polygon. See e.g. [26]. Their rigid-geometric interpretation will be explained in our notes.

For a recent overview on the lifting problem we recommend [31].

1.7 Project description The goal of our project is to solve the local lifting problem for the group A_4 when $p = 2$. This result was announced in [5], but the proof has never been written up. Note that the Sylow 2-subgroup of A_4 has order 4, and hence this case is not covered by the results of [5] and [6].

There are several steps involved in this project which may be treated separately. Most of the material explained during the lectures will appear at some stage of the project. (The strange terms occurring in the following description will be explained in detail in our notes.)

(a) Classify all local A_4-action $A_4 \subset \mathrm{Aut}_\kappa(\kappa[[z]])$ over an algebraically closed field κ of characteristic 2 in terms of filtration of higher ramification groups (or equivalently, in terms of the Artin conductor).

(b) Show that the Bertin Obstruction vanishes for every local A_4-action.

(c) Construct Hurwitz trees for every local A_4-action.

(d) Show that every Hurwitz tree constructed in (c) can be lifted to an A_4-action in characteristic zero.

If the project succeeds, there is the possibility of expanding the result into a publishable paper.

1.8 Exercises Here are some warm up exercises which should be helpful to get started.

Exercise 1.10 Prove the following statements (which give an example of a pair (X, G) that does not lift).

(a) For any field K the group of automorphisms of $\mathbb{P}^1_K$ is $\mathrm{PGL}_2(K)$.

(b) Let κ be an infinite field of characteristic p. Then there exists a subgroup $G \subset \mathrm{PGL}_2(\kappa)$ isomorphic to $(\mathbb{Z}/p\mathbb{Z})^n$, for all $n \geq 1$.

(c) If K is a field of characteristic zero, and $G \subset \mathrm{PGL}_2(K)$ is isomorphic to $(\mathbb{Z}/p\mathbb{Z})^n$, then $n \leq 1$ or $p^n = 4$.

Exercise 1.11 Let X be the smooth projective curve over $\kappa := \bar{\mathbb{F}}_p$ with affine plane model

$$y^p - y = x^{p+1}.$$

(a) Compute the genus of X (e.g. by using the Riemann–Hurwitz formula).

(b) Fix a primitive (p^2-1)th root of unity $\zeta \in \kappa$ and let $\sigma, \tau \in \mathrm{Aut}_\kappa(X)$ be the automorphisms given by

$$\sigma^*(x) = x, \quad \sigma^*(y) = y + 1$$

and

$$\tau^*(x) = \zeta \cdot x, \quad \tau^*(y) = \zeta^{p+1} \cdot y.$$

Compute the order of the subgroup $G \subset \mathrm{Aut}_\kappa(X)$ generated by σ and τ. Show that G violates the Hurwitz bound (1) for $p \gg 0$.

Exercise 1.12 (a) Let $G \subset \mathrm{Aut}_\kappa(\kappa[[z]])$ be a finite group of automorphisms of $\kappa[[z]]$. Then G is cyclic-by-p, i.e. of the form $G = P \rtimes H$, where P is the Sylow p-subgroup of G and H is cyclic of order prime to p. (This result is well-know and can for example be found in [38].)

Show that if $\sigma \in G$ is of order prime to p there exists a parameter $z' = z + a_2 z^2 + \ldots$ such that $\sigma(z') = \zeta \cdot z'$, where $\zeta \in \kappa$ is a root of unity.

(b) Now let $G \subset \mathrm{Aut}_{\mathfrak{o}}(\mathfrak{o}[[z]])$ be a finite subgroup. Prove the same statement as in (a).

(c) Verify (a) for the nontrivial local actions induced by the examples in Exercise 1.10 and 1.11.

Exercise 1.13 (a) Show that the automorphism $\sigma : \kappa[[z]] \xrightarrow{\sim} \kappa[[z]]$ given by

$$\sigma(z) := z/(1+z)$$

has order p.

(b) Assume $p = 2$ or $p = 3$. Lift the automorphism σ in (a) to an automorphism of order p of $\mathfrak{o}[[z]]$, for a suitable ring extension $\mathfrak{o}/W(\kappa)$.

Exercise 1.14 Fix $h \in \mathbb{N}$, $(h, p) = 1$. Set $A := \kappa[[t]]$ and $K := \mathrm{Frac}(A)$. Let $L := K(y)$ be the Galois extension given by the Artin-Schreier equation

$$y^p - y = t^{-h}.$$

Let B be the integral closure of A in L.

(a) Find $z \in B$ such that $B = \kappa[[z]]$.

(b) Let $\sigma \in \mathrm{Gal}(L/K) \cong \mathbb{Z}/p\mathbb{Z}$ be the generator with $\sigma(y) = y+1$. Determine $z' := \sigma(z) \in B$ as a power series in z.

(c) Compare with the automorphism σ from Exercise 1.13 (a).

2 Ramification theory

In this section, we recall the necessary basics from ramification theory. We distinguish three situations, which we label the classical case (Case A), the case of residual dimension one (Case B) and the 2-local case (Case B). The 2-local case is a special case of the residual-dimension-one case.

The classical case deals with local fields whose residue field is algebraically closed (and hence perfect). The ramification theory for Galois extensions of such a field is well known, and we treat it here mainly to motivate the definitions in the other two cases, where the residue field is imperfect. The generalization of classical ramification theory to a local field with imperfect residue field is a topic on its own (see the course of Mieda and Saito). Luckily, under the rather special assumptions made in Case B and C, things become much simpler. Our main references here are [24] and [23].

2.1 Notations and assumptions Let K be a field which is complete with respect to a discrete valuation v. We write $\mathfrak{o}_K$ for the valuation ring, $\mathfrak{p}_K$ for the maximal ideal, and $\bar{K}$ for the residue field. Also, π_K will denote a fixed uniformizing element. We always assume that the residue field $\bar{K}$ has positive characteristic $p > 0$.

Our goal is to study the ramification of finite Galois extensions L/K. Since K is complete, the valuation v extends in a unique way to a discrete valuation on L which we also call v. Otherwise, the notation for L will follow the usual pattern, i.e. we use $\mathfrak{o}_L$, $\bar{L}$, π_L etc.

We let $e = (v(L^\times) : v(K^\times))$ denote the ramification index and $f := [\bar{L} : \bar{K}]$ the degree of the residue field extension of L/K. The fundamental equality ([38], Section III.7, corollary to Prop. 19) says that

$$|G(L/K)| = [L : K] = e \cdot f. \tag{2}$$

Case A: the classical case

The classical ramification theory (which can e.g. be found in [38]) relies on the assumption that the residue field $\bar{K}$ is perfect. For reasons that will become clear later, we make the stronger assumption that $\bar{K}$ is algebraically closed. We can distinguish two subcases.

Suppose first that K has characteristic zero. Then K contains a unique minimal complete subfield $K_0 \subset K$ in which p is a uniformizer. The valuation ring of K_0 is canonically isomorphic to the ring of Witt vectors $W(\bar{K})$. Moreover, K/K_0 is a finite, totally ramified extension. See [38], Sections II.5–6. For instance, the local field k introduced at the beginning of Section 1 is of this form. Since our general assumption in these notes is that the field k is 'sufficiently large', we are not so interested in the ramification theory of k. Therefore, this first subcase may safely be excluded in this section.

Now suppose that K has characteristic p. Then $K = \bar{K}((\pi_K))$ can be identified with the field of Laurent series in the chosen uniformizer π_K, and $\mathfrak{o}_K = \bar{K}[[\pi_K]]$ is the subring of power series in π_K. See [38], Section II.4. We will usually use latin letters like $t, z, \ldots$ to denote a uniformizer of such a field.

If L/K is a finite Galois extension, then $f = 1$ and $e = [L : K]$ (i.e. L/K is totally ramified). We will usually normalize the valuation v such that $v(L^\times) = \mathbb{Z}$ (and hence $v(K^\times) = e\mathbb{Z}$).

Example 2.1 The example to keep in mind is the following. Let $f : C \to D$ be a Galois cover between smooth projective curves over an algebraically closed field κ of characteristic p. Let $y \in C$ be a closed point and $x := f(y)$ its image on D. Let $G_y \subset G$ be the stabilizer of y. Since C and D are smooth over κ, the complete local rings $\hat{\mathcal{O}}_{C,y}$ and $\hat{\mathcal{O}}_{D,x}$ are discrete valuations rings which are (noncanonically) isomorphic to $\kappa[[z]]$. Let $L := \mathrm{Frac}(\hat{\mathcal{O}}_{C,y})$ and $K := \mathrm{Frac}(\hat{\mathcal{O}}_{D,x})$ be the fraction fields. Then L/K is a totally ramified G_y-Galois extension.

Case B: residual dimension one

In this case we make the following assumptions on K (following [40]):

Assumption 2.2

(a) K has mixed characteristic $(0, p)$.

(b) $[\bar{K} : \bar{K}^p] = p$.

(c) $H^1(\bar{K}, \mathbb{Z}/p\mathbb{Z}) \neq 0$, i.e. there exists a nontrivial Galois extension of $\bar{K}$ of degree p.

(d) K is *weakly unramified* over its field of constants.

The geometric interpretation of Conditions (b) and (c) is that the residue field $\bar{K}$ has 'dimension one'. Note that (b) implies that $\bar{K}$ is not perfect. To understand Condition (d) we need the fact that K contains a maximal subfield $k \subset K$ with the property that the residue field of k (with respect to the valuation v) is perfect. We call k the *field of constants* of K. Condition (d) says that $e(K/k) = (v(K^\times) : v(k^\times)) = 1$. The maximality of k implies that the extension of residue fields $\bar{K}/\bar{k}$ is *regular*, i.e. it is separable and $\bar{k}$ is algebraically closed in $\bar{K}$.

Example 2.3 Here is the example to keep in mind. Let k be a local field of characteristic zero with algebraically closed residue field κ of characteristic p (Case A, subcase one). Let X be a smooth proper k-curve and $\mathcal{X}$ a normal $\mathfrak{o}$-model of X. Let $\bar{Z}$ be an irreducible component of the special fiber $\bar{X} := \mathcal{X} \otimes_{\mathfrak{o}} \kappa$. Then $\bar{Z}$ gives rise to a discrete valuation $v_{\bar{Z}}$ on the function field $k(X)$ of X extending the given valuation v of k. Let $K_{\bar{Z}}$ be the completion of $k(X)$ with respect to $v_{\bar{Z}}$. Alternatively, we can define $K_{\bar{Z}} := \mathrm{Frac}(\hat{\mathcal{O}}_{\mathcal{X},\bar{Z}})$ as the field of fractions of the complete local ring of $\mathcal{X}$ at the generic point of $\bar{Z}$.

It is easy to see that $K_{\bar{Z}}$ satisfies Conditions (a), (b) and (c) of Assumption 2.2. Indeed, property (a) is inherited from the subfield k, and (b) and (c) hold because $\bar{K}_{\bar{Z}} = \kappa(\bar{Z})$ is a function field of transcendence degree one over κ. On the other hand, Condition (d) is not automatic. It is satisfied if and only if $\bar{X}$ is reduced at the generic point of $\bar{Z}$ (this condition is often expressed by saying that $\bar{Z}$ is a *multiplicity one component* of $\bar{X}$). In our course we will usually work with semistable models $\mathcal{X}$. Then $\bar{X}$ is reduced by assumption, and Condition (d) holds automatically.

Definition 2.4 Let K be a field satisfying Assumption 2.2 and let L/K be a finite Galois extension. We say that L/K is *weakly unramified* if the ramification index satisfies $e := [v(L^\times) : v(K^\times)] = 1$. We say that L/K is *fierce* if it is weakly unramified and, moreover, the residue field extension $\bar{L}/\bar{K}$ is purely inseparable.

Note that a fierce extension is *totally ramified*, i.e. the action of the Galois group on the residue field $\bar{L}$ is trivial. If L/K is weakly unramified, then there exists a unique Galois subextension M such that L/M is fierce and M/K is unramified. In fact, $M = L^I$ is the fixed field of the inertia subgroup

$$I := \mathrm{Ker}(\mathrm{Gal}(L/K) \to \mathrm{Aut}(\bar{L}/\bar{K})).$$

We then have a short exact sequence of groups

$$1 \to I = \mathrm{Gal}(L/M) \to \mathrm{Gal}(L/K) \to \mathrm{Gal}(M/K) \to 1.$$

By the fundamental equality (2) we have

$$|I| = [L : M] = [\bar{L} : \bar{M}].$$

Since $\bar{L}/\bar{M}$ is purely inseparable, the degree $[\bar{L} : \bar{M}]$ is a power of p and hence I is a p-group.

Remark 2.5 Let L/K be a finite Galois extension. Then L satisfies Conditions (a), (b) and (c) of Assumption 2.2, but not necessarily (d). In fact, (d) holds if and only if L/K is weakly unramified.

By a theorem of Epp ([13]) there exist a finite extension k'/k such that $L' := Lk'$ is weakly unramified over k'. Therefore, after replacing L by L' and K by $K' := Kk'$, we obtain a finite Galois extension L/K which is weakly unramified and where both L and K satisfy all parts of Assumption 2.2. Extensions of the form K'/K and L'/L are called *almost constant*.

Thus, if we accept to study extensions L/K 'up to an extension of the field of constants', we may always assume that L/K is weakly unramified. By the discussion above, the ramified part of L/K is then fierce.

Notation 2.6 Let $r = a/b \in \mathbb{Q}$ be given. After enlarging the field of constants k of K (as in the previous remark) we may choose an element $\pi_r \in k^\times$ such that $\pi_r^b = p^a$. We may also assume that the choices made for (finitely many) different values of r are compatible, i.e. that we have

$$\pi_r^n = \pi_{nr},$$

for all $r \in \mathbb{Q}$ and $n \in \mathbb{Z}$. We shall use the notation

$$p^r := \pi_r.$$

Note that $v(p^r) = r$.

Case C: the 2-local case

Additionally to the assumptions in Case B, we assume that the residue field $\bar{K}$ of K is itself complete with respect to a normalized discrete valuation $\bar{v} : \bar{K}^\times \to \mathbb{Z}$ and has an algebraically closed residue field κ. We call the triple $(K, v, \bar{v})$ a *2-local field*.

A *local parameter* for the 2-local field $(K, v, \bar{v})$ is an element $x \in K^\times$ with $v(x) = 0$ and $\bar{v}(\bar{x}) = 1$ (here $\bar{x}$ denotes the image of x in $\bar{K}$). It follows that $\bar{K} = \kappa((\bar{x}))$ and that κ is the residue field of the field of constants k of K.

On a 2-local field $(K, v, \bar{v})$ we can define a valuation $\eta : K^\times \to \Gamma$ of rank two, called the *composition* of v and $\bar{v}$. We consider the abelian group $\Gamma := \mathbb{Q} \times \mathbb{Z}$ as an ordered group with respect to the *lexicographic ordering*:

$$(\mu_1, m_1) \leq (\mu_2, m_2) \quad :\Leftrightarrow \quad \mu_1 \leq \mu_2 \text{ and } (\, \mu_1 = \mu_2 \;\Rightarrow\; m_1 \leq m_2 \,).$$

For $f \in K^\times$ we define

$$\#(f) := \bar{v}(p^{-v(f)} f)$$

(here we use Notation 2.6) and

$$\eta(f) := (v(f), \#(f)).$$

It is easy to check that $\#$ and η are well defined and that η is a valuation. It is clear that the knowledge of η is equivalent to the knowledge of the pair $(v, \bar{v})$.

Example 2.7 We can construct 2-local fields as follows. Let (K, v) be as in Case B, and let $\bar{v} : \bar{K}^\times \to \mathbb{Z}$ be a discrete normalized valuation which is trivial on the subfield $\bar{k}$. Let $\hat{\bar{K}}$ denote the completion of $\bar{K}$ with respect to $\bar{v}$. By [29], there exists an extension of complete valued fields $\hat{K}/K$, unique up to unique isomorphism, which is unramified and has residue field extension $\hat{\bar{K}}/\bar{K}$ (here we use that $\hat{\bar{K}}/\bar{K}$ is separable). The field $\hat{K}$ is a 2-local field by construction. We call it the *residual completion* of K with respect to the valuation $\bar{v}$.

Example 2.8 Residual completions occur naturally in the context of Example 2.3. Let k be a local field of mixed characteristic $(0, p)$ with algebraically closed residue field κ. Let X be a smooth projective curve over k, $\mathcal{X}$ a normal integral $\mathfrak{o}_k$-model of X and $Z \subset \bar{X}$ an irreducible component of the special fiber. Let $K_Z := \mathrm{Frac}(\hat{\mathcal{O}}_{\mathcal{X},Z})$ be the complete valued field associated to Z in Example 2.3. By construction, the residue field of K_Z is the function field of Z.

Let $x \in \bar{Z}$ be a smooth closed point of Z. (We do not assume that x is a smooth point of $\bar{X}$!) Then x gives rise to a normalized discrete valuation

$$\mathrm{ord}_x : \bar{K}_Z = \kappa(Z) \to \mathbb{Z} \cup \{\infty\}.$$

By definition, the valuation ring of ord_x is the local ring $\mathcal{O}_{Z,x}$.

Let $\hat{K}_{Z,x}$ be the residual completion of K with respect to the valuation ord_x. By construction, the residue field of $\hat{K}_{Z,x}$ is the complete local ring $\hat{\mathcal{O}}_{Z,x}$. Using the uniqueness of residual completion, it is now easy to see that we can identify $\hat{K}_{Z,x}$ with the completion of the fraction field of $\hat{\mathcal{O}}_{\mathcal{X},x}$ with respect to v_Z.

Let $(K, v, \bar{v})$ be a 2-local field and L/K a finite, weakly unramified Galois extension. Then L is again a 2-local field. Indeed, we have seen in the previous subsection (Case B) that v extends uniquely to a discrete valuation on L (still denoted by v) and that (L, v) satisfies Assumption 2.2. Moreover, the extension of residue fields $\bar{L}/\bar{K}$ is finite and $\bar{K}$ is complete with respect to the discrete valuation $\bar{v}$. It follows that $\bar{v}$ extends uniquely to a discrete valuation on $\bar{L}$ (still denoted by $\bar{v}$) and that $\bar{L}$ is complete with respect to $\bar{v}$. All in all we see that the rank two valuation η on K extends uniquely to L. In this situation, we will usually normalize $\bar{v}$ such that $\bar{v}(\bar{L}^\times) = \mathbb{Z}$ (and hence $\bar{v}(\bar{K}^\times) = [L : K] \cdot \mathbb{Z}$). In terms of η this means that $\eta(L^\times) = \frac{1}{e} \cdot \mathbb{Z} \times \mathbb{Z}$, where e is the absolute ramification index of the field of constants $k/\mathbb{Q}_p$.

2.2 The higher ramification groups In this section, we fix a finite Galois extension L/K, where K is as in Case A or B. We then define the filtration of higher ramification groups on $G = \mathrm{Gal}(L/K)$. A refinement of this filtration in Case C will be discussed in §2.6.

We first recall the definition of the higher ramification groups in Case A. There are two filtrations

$$(G_t)_{t \geq 0}, \qquad (G^t)_{t \geq 0}$$

on $G = \mathrm{Gal}(L/K)$ indexed by $\mathbb{R}_{\geq 0}$ defined as follows. For $\sigma \in G$, we define

$$i_G(\sigma) := \min\{v(\sigma(x) - x) \mid x \in \mathfrak{o}_L\}.$$

It is easy to see that

$$i_G(\sigma) = v(\sigma(\pi_L) - \pi_L). \tag{3}$$

The *lower-numbering filtration* is defined as

$$G_t := \{\sigma \in G \mid i_G(\sigma) \geq t + 1\}.$$

The (finitely many) values of t such that $G_{t+\varepsilon} \neq G_t$ for all $\varepsilon > 0$ are called the *lower jumps*. It is clear that all of these values are integers.

Now assume that K is as in Case B. In principle we could define a filtration $(G_t)_t$ as in Case A. However, it turns out that such a filtration will not have the desired properties. One problem is, for instance, that (3) will in general not hold true anymore. It was discovered by Hyodo and Kato (see [20] and [23]) that the classical theory of higher ramification groups can be extended (with minor changes) to Case B if we assume, additionally, that the Galois extension L/K is weakly unramified. The main point is that, under this additional assumption, the extension of valuation rings $\mathfrak{o}_L/\mathfrak{o}_K$ is monogenic. In fact, using Assumption 2.2 one shows that the extension of residue fields $\bar{L}/\bar{K}$ can be generated by one element $\bar{x} \in \bar{L}$. Moreover, any element $x \in \mathfrak{o}_L$ lifting $\bar{x}$ generates $\mathfrak{o}_L$ over $\mathfrak{o}_K$. Alternatively, x is a unit in $\mathfrak{o}_L$ whose reduction $\bar{x} \in \bar{L}$ is not a pth power. We call such an element a *generator* of L. It is now easy to see that we have

$$i_G(\sigma) := \min\{v(\sigma(y) - y) \mid y \in \mathfrak{o}_L\} = v(\sigma(x) - x),$$

for all $\sigma \in G$ and for an arbitrary generator x of L/K. In this situation the *lower filtration* on $G = \mathrm{Gal}(L/K)$ is defined as

$$G_t := \{\sigma \in G \mid i_G(\sigma) \geq t\}.$$

Note that the convention on the numbering is different from the classical case. The lower jumps are defined as in the classical situation. Due to the normalization of the valuation v (i.e. the assumption $v(p) = 1$), the lower jumps are in general rational numbers.

In both situations, we can define the upper filtration as follows. Set

$$\varphi_{L/K}(t) := \int_0^t \frac{1}{[G_0 : G_s]} \,\mathrm{d}s.$$

Then $\varphi_{L/K} : \mathbb{R}_{\geq 0} \to \mathbb{R}_{\geq 0}$ is piecewise linear and strictly increasing, hence bijective. Let $\psi_{L/K} := \varphi_{L/K}^{-1}$ be the inverse map. The *upper filtration* is now defined by

$$G^t := G_{\psi_{L/K}(t)},$$

or, equivalently,

$$G^{\varphi_{L/K}(t)} = G_t.$$

One shows that

$$\psi_{L/K}(t) = \int_0^t [G^0 : G^s] \,\mathrm{d}s.$$

The *upper jumps* are the (finitely many) values of t such that $G^{t+\varepsilon} \neq G^t$ for all $\varepsilon > 0$.

Lemma 2.9 *The filtrations $(G^t)_t$ and $(G_t)_t$ have the following properties:*

(a) If $s \le t$, then $G^s \supset G^t$ and $G_s \supset G_t$.

(b) The subgroups G_t and G^t are normal subgroups of G. Moreover, $G = G^0 = G_0$.

(c) For each $\sigma \in G\backslash\{1\}$, the sets $\{t \ge 0 \mid \sigma \in G^t\}$ and $\{s \ge 0 \mid \sigma \in G_s\}$ have a maximum.

(d) Let $H \lhd G$ be a normal subgroup and $M := L^H$ the corresponding subfield. Then

$$(G/H)^t = G^t/(G^t \cap H), \qquad G^t \cap H = H^{\psi_{M/K}(t)},$$

where the filtration $(H^t)_t$ is induced from the isomorphism $H \cong \mathrm{Gal}(L/M)$.

Proof: These properties are for example proved in [38], Chapter 4 in the classical case and in [24], Lemma 2.9 in the case of residual dimension one (Case B). □

In Case B, the following lemma shows that the definition of the filtrations are compatible with enlarging the field of constants of K. Since we use the convention that the field of constants is always 'sufficiently large', this is a crucial point.

Lemma 2.10 *Suppose we are in situation B. Let K'/K be an almost constant extension. Set $L' := LK'$ and $G' := \mathrm{Gal}(L'/K')$. Then the natural isomorphism $G' \stackrel{\sim}{\to} G$ is compatible with the upper and lower filtrations, i.e.*

$$(G')_t \stackrel{\sim}{\to} G_t, \quad (G')^t \stackrel{\sim}{\to} G^t,$$

for all $t \ge 0$.

Proof: Let $x \in \mathcal{O}_L$ be an element whose residue class generates the extension $\bar{L}/\bar{K}$. Then $\mathcal{O}_L = \mathcal{O}_K[x]$ and $\mathcal{O}_{L'} = \mathcal{O}_{K'}[x]$. We conclude that

$$i_G(\sigma) = v(\sigma(x) - x) = i_{G'}(\sigma),$$

for all $\sigma \in G$. The statement of the lemma follows immediately. □

Exercise 2.11 Let L/K be in the classical case (Case A), and suppose that $G = \mathrm{Gal}(L/K) \simeq \mathbb{Z}/p\mathbb{Z}$. Choose a generator σ of G.

Define $\wp : K \to K$ as $f \mapsto f^p - f$. Recall that Artin–Schreier Theory yields a bijection between isomorphism classes of G-Galois covers of K and $K/\wp(K)$. The Artin–Schreier extension L corresponding to $[f] \in K/\wp(K)$ is given by

$$L = K[y]/(y^p - y - f), \qquad \sigma^*(y) = y + 1,$$

(a) Write $K = \kappa((x))$. Show that every Artin–Schreier extension L/K may be represented by a unique $f \in K$ with

$$f = x^{-h} + \sum_{i>-h,\gcd(i,p)=1} c_i x^i,$$

with h prime to p.

(b) Show that we may choose a different generator x' of K such that $f = (x')^{-h}$.

(c) Show that h is the unique lower jump in the filtration of higher ramification groups. Conclude that this jump is prime to p.

Exercise 2.12 Let L/K be in situation A, and suppose moreover that the extension is totally ramified, i.e. $e = n$. Let p be the residue characteristic of K.
(a) Prove that $i_G(\sigma^p)$ is strictly larger than $i_G(\sigma)$.
(b) Suppose that L/K is Galois, and that $\mathrm{Gal}(L, K)$ is cyclic of p-power order. Conclude from (a) that every subgroup H of $G = \mathrm{Gal}(L, K)$ occurs as a step in the filtration (G_t).
(c) Prove that G_1 is the Sylow p-subgroup of G.

Exercise 2.13 Suppose that K is as in Case A and that L/K is a Galois extension of local fields in characteristic 2 with Galois group $G \simeq A_4$. For simplicity, we choose generators σ_1, σ_2, τ of G such that $\sigma_i^2 = \tau^3 = e$ and $\tau\sigma_1\tau^{-1} = \sigma_2, \quad \tau\sigma_2\tau^{-1} = \sigma_3$, where $\sigma_3 = \sigma_1\sigma_2$.

(a) Show that there is a unique lower jump h, i.e. a positive integer h such that

$$G = G_0 \supsetneq G_1 = \ldots = G_h \supsetneq G_{h+1} = \{1\}.$$

(Use Lemma 2.9.)

(b) Define

$$L_i := L^{\langle\sigma_i\rangle}, \qquad M := L^{\langle\sigma_1,\sigma_2\rangle}.$$

Using Kummer theory, show that we may choose a local parameter x of M such that $\tau x = \zeta_3 x$. Moreover, using Artin–Schreier theory, show that $L_i = M[y_i]$, where

$$\begin{aligned} y_1^2 + y_1 &= x^{-h}(1 + xc_1 + \cdots), \\ \tau(y_i) &= y_{i+1}. \end{aligned} \tag{4}$$

(c) Using the fact that $L = M[y_1, y_2]$, show that $\gcd(h, 3) = 1$. Conclude from Exercise 2.11 that $\gcd(h, 6) = 1$ holds.

Remark: the proof of Exercise 2.22 also shows that if a coefficient c_i in (4) is nonzero, then $i \equiv 0 \pmod 3$. A similar result for more general inertia groups can be found in [34], Lemma 2.6.

2.3 The Artin representation The Hasse–Arf theorem is one of the most important results on higher ramification groups. In the classical situation (Case A) it states that for an abelian Galois group $G = \mathrm{Gal}(L/K)$ the jumps of the filtration (G^t) are integers. Equivalently, the jumps for the filtration (G_t) (which are integers by definition) satisfy certain congruence relations. (Exercise 2.21 below contains an explicit statement in the cyclic case.)

More generally, if the Galois group G is nonabelian one obtains, by applying the Hasse–Arf theorem to all abelian subquotients of G, highly nontrivial conditions on the filtrations (G_t) and (G^t). (An example is discussed in Exercise 2.23.) It turns out that the character theory of finite groups provides a surprisingly elegant and powerful tool to formulate these consequences of the Hasse–Arf theorem as one compact result, as follows: the *Artin character* (which encodes the filtrations (G_t) and (G^t)) is the character of a certain representation of G, called the *Artin representation*.

We start by recalling some terminology from character theory. Let G be a finite group. A *class function* on G is a map $f : G \to \mathbb{C}$ such that

$$f(\sigma\tau\sigma^{-1}) = f(\tau), \qquad \text{for all } \sigma, \tau \in G.$$

Put $R(G, \mathbb{C})$ for the set of class functions of G. We regard $R(G, \mathbb{C})$ as a $\mathbb{C}$-vector space in the obvious way.

Let (V, ρ) be a *representation* of G, i.e. V is a finite dimensional $\mathbb{C}$-vector space and

$$\rho : G \to \mathrm{GL}_{\mathbb{C}}(V)$$

is a group homomorphism. The *character* of (V, ρ) is the function

$$\chi_V : G \to \mathbb{C}, \quad \chi_V(\sigma) := \mathrm{trace}(\rho(\sigma)).$$

Obviously, $\chi_V \in R(G, \mathbb{C})$ is a class function. We denote by $R^+(G) \subset R(G, \mathbb{C})$ the subset of characters. This is an additive submonoid of $R(G, \mathbb{C})$, but not a subgroup. We denote by $R(G) \subset R(G, \mathbb{C})$ the subgroup generated by $R^+(G)$; its elements correspond to the *virtual representations* of G. Alternatively, the group $R^+(G)$ may be identified with the Grothendieck group of the category of $\mathbb{C}[G]$-modules of finite type ([37], §14.1).

Example 2.14 We write $1_G \in R^+(G)$ for the *unit character*, $r_G \in R^+(G)$ for the *regular character*, and $u_G = r_G - 1_G \in R^+(G)$ for the *augmentation character*. Explicitly, we have $1_G(\sigma) = 1$ for all $\sigma \in G$,

$$r_G(\sigma) = 0, \;\; u_G(\sigma) = -1 \quad \text{for } \sigma \neq 1$$

and

$$r_G(1) = |G|, \;\; u_G(1) = |G| - 1.$$

To see that these class functions are indeed characters, one has to find the corresponding representations. This is rather trivial for 1_G. The representation corresponding to r_G is the permutation representation associated to the action of G on itself by (left or right) translation. (Exercise: find a representation corresponding to u_G.)

Definition 2.15 The *dimension* of a character $\chi \in R^+(G)$ is the nonnegative integer $m := \chi(1)$. If (V, ρ) is a representation with character χ then $m = \dim_{\mathbb{C}} V$.

Let
$$\hat{G} := \{\chi \in R^+(G) \mid \chi(1) = 1\}$$
be the subset of characters of dimension one. An element $\chi \in \hat{G}$ is actually a group homomorphism $\chi : G \to \mathbb{C}^\times$. Therefore, $\hat{G}$ is an abelian group with unit element 1_G. Note, however, that the group structure on $\hat{G}$ (which we shall write as multiplication) is different from the addition law in $R^+(G)$. Concretely, $\hat{G}$ is equal to the group of units of the *ring* $R^+(G)$, with multiplication of characters corresponding to the tensor product of representations.

Definition 2.16 We say that a character $\chi \in R^+(G)$ is *trivial* if $\chi = m \cdot 1_G$ for some integer $m \geq 0$. (If (V, ρ) is a representation with character χ then this means that the action of G on V is trivial.)

Given two class functions $f_1, f_2 \in R(G, \mathbb{C})$, the scalar product is defined as

$$\langle f_1, f_2 \rangle_G := \frac{1}{|G|} \cdot \sum_{\sigma \in G} \overline{f_1(\sigma)} f_2(\sigma).$$

If f_1 and f_2 are virtual characters then $\langle f_1, f_2 \rangle_G$ is a integer. If f_1, f_2 are true characters then $\langle f_1, f_2 \rangle \geq 0$. Given a group homomorphism $\phi : H \to G$, we obtain $\mathbb{Z}$-linear maps (*restriction* and *induction*)

$$\phi^* : R(G) \to R(H), \qquad \phi_* : R(H) \to R(G).$$

They are related by the *Frobenius reciprocity formula*:

$$\langle \psi, \phi_*(\chi) \rangle_G = \langle \phi^*(\psi), \chi \rangle_H.$$

If H is a subgroup of G and ϕ the canonical injection, then we will simply write $\psi|_H$ instead of $\phi^*(\psi)$ and χ^* instead of $\phi_* \chi$. See e.g. [38], VI, §1.

Definition 2.17 Suppose that L/K is a finite G-Galois extension either in situation A or B. The *Artin character* of L/K is defined by

$$a_{L/K}(\sigma) := \begin{cases} -i_G(\sigma), & \text{if } \sigma \neq 1, \\ -\sum_{\sigma \neq 1} a_{L/K}(\sigma) = \sum_{\sigma \neq 1} i_G(\sigma), & \text{otherwise.} \end{cases}$$

In Case A, the *Swan character* of L/K is defined by $\mathrm{sw}_{L/K} = a_{L/K} - u_G$, i.e.

$$\mathrm{sw}_{L/K}(\sigma) = \begin{cases} 1 - i_G(\sigma), & \text{if } \sigma \neq 1, \\ \sum_{\sigma \neq 1} (i_G(\sigma) - 1), & \text{otherwise.} \end{cases}$$

(In Case B, there is no distinction between the Artin and the Swan character.)

It follows from Lemma 2.9.(b) that $a_{L/K}$ and $\mathrm{sw}_{L/K}$ are class functions.

Theorem 2.18 (Hasse–Arf–Artin) *Suppose L/K is in situation A. Then the class functions $a_{L/K}$ and $\mathrm{sw}_{L/K}$ are characters.*

Proof: For $a_{L/K}$ this is [38], § VI.2, Theorem 1. The proof proceeds by reduction to the case that G is abelian. In this case the statement of the theorem is equivalent to the statement that the upper jumps are integers, i.e. the original statement of Hasse and Arf. This is proved e.g. in [38], §V.7. The statement that $\mathrm{sw}_{L/K}$ is also a character is left as an exercise (see Exercise 2.20). □

Definition 2.19 Suppose we are in Case A and let $\chi \in R^+(G)$ be a character. Then the *Artin conductor* (resp. the *Swan conductor*) of χ is defined as

$$a_{L/K}(\chi) := \langle a_{L/K}, \chi \rangle_G$$

resp.

$$\mathrm{sw}_{L/K}(\chi) := \langle \mathrm{sw}_{L/K}, \chi \rangle_G.$$

It follows from the Hasse–Arf theorem that $a_{L/K}(\chi)$ and $\mathrm{sw}_{L/K}(\chi)$ are non-negative integers (in fact, Theorem 2.18 is equivalent to this statement). Clearly,

$$a_{L/K}(\chi) = \mathrm{sw}_{L/K}(\chi) + \chi(1).$$

Exercise 2.20 Suppose we are in Case A, and let $\chi \in R^+(G)$ be a character.

(a) Show that $a_{L/K}(\chi) = 0$ if and only if χ is trivial.

(b) Show that $\mathrm{sw}_{L/K}(\chi) = 0$ if and only if χ is *tamely ramified*, i.e. the restriction of χ to the Sylow p-subgroup of G is trivial.

(c) Complete the proof of Theorem 2.18, i.e. prove that $\mathrm{sw}_{L/K}(\chi)$ is a non-negative integer for all $\chi \in R^+(G)$.

Exercise 2.21 Suppose we are in Case A and that G is cyclic of order p^n, where p is the residue characteristic of K.

(a) Show that the statement of Theorem 2.18 is equivalent to the following set of congruences for the lower jumps $h_0, h_1, ..., h_{n-1}$:

$$h_{i+1} \equiv h_i \pmod{p^i}.$$

(b) Show that the statement of Theorem 2.18 in this situation is also equivalent to the statement that the upper jumps $\sigma_0, \ldots, \sigma_{n-1}$ are integers.

(c) Fix an embedding $\chi : G \hookrightarrow \mathbb{C}^\times$ and compute $\mathrm{sw}_{L/K}(\chi)$ in terms of $h_0, \ldots, h_{n-1}$.

Exercise 2.22 Let L/K be as in Exercise 2.13.

(a) Compute that Artin character $a_{L/K}$.

(b) Determine a representation of A_4 with character χ such that

$$m\,\mathrm{reg}_{A_4} - \chi = a_{L/K},$$

where m is an integer and reg_{A_4} is the regular representation of A_4. (Hint: you might want to use the character table of A_4.)

Exercise 2.23 In this exercise, we determine the higher ramification groups of a class of $G := Q_8$-covers in characteristic 2. We choose two elements σ_1, σ_2 of order 4 which generates G, and put $\sigma_3 = \sigma_1\sigma_2$. Then $-I := \sigma_1^2 = \sigma_2^2 = \sigma_3^2$ is the nontrivial element in the center of G.

(a) Let L/K be a G-Galois extension in situation A. We assume that $K = \kappa((z))$ for some algebraically closed field κ of characteristic 2. Show that that the filtration of higher ramification groups is of the form

$$G = G_0 = \cdots = G_{h_0} \supsetneq \langle -I \rangle = G_2 = \cdots = G_{h_1} \supsetneq \{0\}$$

with $h_0 \equiv h_1 \equiv 1 \pmod 2$.

(b) In the rest of this exercise, we assume that $h_0 = 1$ and write $h_1 = h$. Denote $M := L^{\langle -I \rangle}$. Show that we can choose $x \in M$ such that $M = \kappa((x))$ and a generator y of L/M which satisfies

$$y^2 + y = x^{-h}.$$

Exercise 2.24 In the exercise we prove a partial converse to Exercise 2.23.(c). Namely, we construct local Q_8-extensions in characteristic 2 with conductor $h = 1 + 2^n$ as in Exercise 2.23.(a–b). More general result may be deduced from the results of Lehr–Matignon [27].

Suppose given a degree-2 cover $C \to D := \mathbb{P}^1_\kappa$ of smooth projective curves over κ defined by the affine equation

$$y^2 + y = x^{-h}.$$

Put $w := 1/x$ and $\varphi(w) = w^h$. Note that C has a unique point above $x = 0$.

(a) For $\alpha \in \kappa^*$, we consider $\sigma_\alpha \in \mathrm{Aut}_\kappa(\mathbb{P}^1)$ given by $\sigma_\alpha(w) = w + \alpha$. Assume that σ_α lifts to an automorphism of C. Show that

$$\sigma_\alpha(w, y) = (w + \alpha, y + g(w)),$$

where $g \in \kappa[w]$ satisfies

$$g(x)^2 - g(x) = \varphi(w + \alpha) - \varphi(w).$$

(b) For α as in (a), show that σ_α has order 4.

(c) Now assume that $h = 1 + 2^n$ for some integer $n > 0$. Show that for every $\alpha \in \mathbb{F}_{2^{2n}}^*$ we have $\sigma_\alpha \in \mathrm{Aut}_\kappa(C)$.

(d) Now choose $\alpha \in \mathbb{F}_{2^{2n}}^* \setminus \{1\}$. Show that $\langle \sigma_\alpha, \sigma_1 \rangle \simeq Q_8$.

(e) Choose $h = 7$. (This is the smallest odd integer not of the form $h = 1+2^n$.) Show that for all $\alpha \in \bar{\mathbb{F}}_2^*$ we have $\sigma_\alpha \notin \mathrm{Aut}_\kappa(C)$. Try to generalize this statement as much as possible.

2.4 The depth character and the differential Swan conductor In this section we assume that we are in Case B and that we are given a weakly ramified finite Galois extension L/K. If $\chi \in R^+(G)$ is a character of the Galois group $G = \mathrm{Gal}(L/K)$, we define its *depth conductor*

$$\delta_{L/K}(\chi) := \langle a_{L/K}, \chi \rangle_G \in \mathbb{Q}$$

as the scalar product of χ with the Artin character of L/K. It is not hard to show that $\delta_{L/K}(\chi) \geq 0$ for all $\chi \in R^+(G)$ (and in fact $\delta(\chi) > 0$ if χ is nontrivial). This means that the positivity result which is implicit in the Hasse–Arf theorem is valid in Case B as well. The analogue of the integrality statement of Hasse–Arf theorem is that

$$\delta_{L/K}(\chi) \in v(K^\times) = \frac{1}{e_K}\mathbb{Z}.$$

This follows e.g. from [24] or [23]. However, since we allow arbitrary finite constant extensions of K, the absolute ramification index e_K is not bounded and therefore this result is not useful for us.

Following Kato [23], we will define a refinement of the depth conductor. For this we assume for the rest of this section that L/K is fierce (Definition 2.4). Then

$$|G| = [L : K] = [\bar{L} : \bar{K}] = p^n$$

for some $n \geq 0$. Recall that the Artin character $a_{L/K}$ is defined via the function

$$(\sigma, x) \mapsto v(\sigma(x) - x).$$

For $\sigma \in G\backslash\{1\}$ fixed, the right hand side $v(\sigma(x) - x)$ achieves its minimum $i(\sigma)$ for a generator of L/K. The idea now is to associate to σ not only the integer $i(\sigma)$ but also the function

$$\theta_\sigma : \mathfrak{o}_L \to \bar{L}, \quad x \mapsto \theta_\sigma(x) := \frac{\sigma(x) - x}{p^{i(\sigma)}} \mod \mathfrak{p}_L.$$

Since L/K is fierce by assumption, the action of G on $\bar{L}$ is trivial. It is now easy to check that $\theta_\sigma : \mathfrak{o}_L \to \bar{L}$ is a nontrivial derivation. In particular, $\theta_\sigma(x)$ depends only on the image of x in $\bar{L}$, and $\theta_\sigma(x) \neq 0$ if and only if x is a generator for the extension L/K. Clearly, the pair $(i(\sigma), \theta_\sigma)$ contains more information on the action of σ then the rational number $i(\sigma)$ alone.

In order to turn this idea into a nice formalism of Swan characters, we need some more notation. We define the group S_K as the group of units of the k-algebra

$$\bigoplus_{i,j\in\mathbb{Z}} \mathfrak{p}_K^i/\mathfrak{p}_K^{i+1} \otimes_{\bar{K}} \Omega_{\bar{K}}^{\otimes j}.$$

For an element $x \in K^\times$, let $[x]$ denote the corresponding element of $\mathfrak{p}_K^i/\mathfrak{p}_K^{i+1} \subset S_K$ (with $i := v_K(x)$). Similarly, for an element $\omega \in \Omega_{\bar{K}}^{\otimes j}$, we write $[\omega]$ for the corresponding element of S_K. The group law for S_K is written *additively*. Thus, if we fix a generator x of K, then every element of S_K can be written in the form

$$[f\,(\mathrm{d}\bar{x})^{\otimes^i}] + \frac{m}{e_K}\cdot[p],$$

for unique integers i, m and a unique element $f \in \bar{K}$. In other word, the choice of x yields an isomorphism $S_K \cong \bar{K}^\times \oplus \mathbb{Z}^2$.

We have natural injections

$$\mathfrak{p}_K^i/\mathfrak{p}_K^{i+1} \hookrightarrow \mathfrak{p}_L^i/\mathfrak{p}_L^{i+1}, \qquad \Omega_{\bar{K}} \hookrightarrow \Omega_{\bar{L}}^{\otimes^{p^n}}.$$

The last map sends $f\,\mathrm{d}\bar{x}^{p^n} \in \Omega_{\bar{K}}$ to $f\,(\mathrm{d}\bar{x})^{p^n} \in \Omega_{\bar{L}}^{\otimes p^n}$, where $\bar{x}$ is an arbitrary generator of the extension $\bar{L}/\bar{K}$ (here we use the relation $\bar{K} = \bar{L}^{p^n}$, which follows from Assumption 2.2 (b)). Therefore, we obtain a natural injection

$$S_K \hookrightarrow S_L.$$

One checks easily that the quotient group S_L/S_K is killed by $[L:K]$.

Fix a generator x of L/K. For $\sigma \in \mathrm{Gal}(L/K)$, $\sigma \neq 1$, we define

$$s_{L/K}(\sigma) := [\mathrm{d}\bar{x}] - [x - \sigma(x)] \in S_L.$$

One easily checks that this definition is independent of the choice of x. We also set

$$s_{L/K}(1) := -\sum_{\sigma\neq 1} s_{L/K}(\sigma).$$

We note that

$$v(s_{L/K}(\sigma)) = a_{L/K}(\sigma),$$

which shows that $s_{L/K}$ is indeed a refinement of the Artin character.

Definition 2.25 The element $s_{L/K}(1) \in S_L$ is called the *different* of L/K, and is denoted by $\mathfrak{D}_{L/K}$.

Let H be a normal subgroup of $\mathrm{Gal}(L/K)$, and $M := L^H$. Then for all $\tau \in \mathrm{Gal}(M/K)$, $\tau \neq 1$, we have

$$s_{M/K}(\tau) = \sum_{\sigma\mapsto\tau} s_{L/K}(\sigma), \tag{5}$$

see [23, Proposition 1.9]. In particular, the right hand side of (5) lies in $S_M \subset S_L$. One easily deduces from (5) the transitivity of the different, i.e. the formula

$$\mathfrak{D}_{L/K} = \mathfrak{D}_{L/M} + \mathfrak{D}_{M/K}. \tag{6}$$

We fix a pth root of unity $\zeta \in \mathbb{C}$, and define

$$\epsilon(\zeta) := \sum_{a \in \mathbb{F}_p^\times} [a] \otimes \zeta^a \in S_K \otimes_{\mathbb{Z}} \mathbb{C}.$$

Note that $\epsilon(\zeta^a) = [a] + \epsilon(\zeta)$.

Definition 2.26 Let $\chi : G \to \mathbb{C}$ be a virtual character. The *refined Swan conductor* of χ (with respect to $\zeta \in \mathbb{C}$) is the element

$$\mathrm{sw}_{L/K}(\chi) := \sum_{\sigma \in G} s_{L/K} \otimes \chi(\sigma) + \chi(1) \cdot \epsilon(\zeta) \in S_L \otimes \mathbb{C}.$$

Proposition 2.27 *(a)* $\mathrm{sw}_{L/K}(\chi) \in S_K$.

(b) Let H be a subgroup of G, $M := L^H$, χ a virtual character of H and $\tilde{\chi}$ the induced virtual character on G. Then

$$\mathrm{sw}_{L/K}(\tilde{\chi}) = |G/H| \cdot (\mathrm{sw}_{L/M}(\chi) + \chi(1) \cdot \mathfrak{D}_{M/K}).$$

(c) Let H be a normal subgroup of G, $M := L^H$, χ a virtual character of G/H and χ' the restriction of χ to G. Then

$$\mathrm{sw}_{L/K}(\chi') = \mathrm{sw}_{M/K}(\chi).$$

(d) Let χ be a character of G of rank 1, i.e. a group homomorphism $\chi : G \to \mathbb{C}^*$. Let $H \subset G$ be a normal subgroup. Denote by $M = L^H$. Then

$$\mathrm{sw}_{L/K}(\chi) = \mathrm{sw}_{L/M}(\chi|_H) + \mathcal{D}_{M/K}.$$

Note that (a),(b), (c) and (d) are analogies of well known properties of the classical Swan conductor. Here (b), (c) and (d) are more or less formal consequences of (5) and (6). (Compare to Lemma 2.9.) Property (a) corresponds to the Hasse-Arf Theorem (theorem 2.18) and is quite deep. For a proof, see [23], Proposition 3.3, Theorem 3.4, and Lemma 3.12.

By Proposition 2.27.(a), we can write

$$\mathrm{sw}_{L/K}(\chi) = \delta_{L/K}(\chi) \cdot [p] - [\,\omega_{L/K}(\chi)\,],$$

with $\delta_{L/K}(\chi) \in \mathbb{Q}$ and $\omega_{L/K}(\chi) \in \Omega_{\bar{K}}^{\otimes n}$, $n \in \mathbb{Z}$. Note that $\delta_{L/K}(\chi)$ is the depth conductor of χ defined at the beginning of this section. We call $\omega_{L/K}(\chi)$ the *differential Swan conductor* of χ.

The following lemma gives a useful relation between the differential Swan conductors corresponding to different characters on the same component. The proof can be found in [40], Lemma 3.10.

Lemma 2.28 *Let χ_i, $i = 1, 2, 3$, be characters on K satisfying the relation $\chi_3 = \chi_1 \cdot \chi_2$. Set $\delta_i := \delta_{L/K}(\chi_i)$ and $\omega_i := \omega_{L/K}(\chi_i)$ for $i = 1, 2, 3$. Then the following holds.*

(a) If $\delta_1 \neq \delta_2$ then $\delta_3 = \max\{\delta_1, \delta_2\}$. Furthermore, we have $\omega_3 = \omega_1$ if $\delta_1 > \delta_2$ and $\omega_3 = \omega_2$ otherwise.

(b) If $\delta_1 = \delta_2$ and $\omega_1 + \omega_2 \neq 0$ then $\delta_1 = \delta_2 = \delta_3$ and $\omega_3 = \omega_1 + \omega_2$.

(c) If $\delta_1 = \delta_2$ and $\omega_1 + \omega_2 = 0$ then $\delta_3 < \delta_1$.

Suppose for simplicity that $G = (\mathbb{Z}/p\mathbb{Z})^n$ is an elementary abelian p-group. If $\delta(\chi) = \delta$ for all nontrivial characters $\chi : G \to \mathbb{C}$, then the differential forms $\omega(\chi)$ form an $\mathbb{F}_p$-vector space. An explicit example of such a vector space can be found in Exercise 3.21.

Remark 2.29 It is possible to extend the definition of $\omega_{L/K}(\chi)$ to Galois extensions L/K which are weakly unramified (but not necessarily fierce). Let M/K be the unique unramified subextension of L/K, $I := \mathrm{Gal}(L/M)$ and $\bar{G} := G/I = \mathrm{Gal}(M/K)$. Then for a character $\chi \in R^+(G)$ we simply define

$$\omega_{L/K}(\chi) := \omega_{L/M}(\chi|_I) \in \Omega_{\bar{M}}^{\otimes^m}.$$

It is easy to check that this differential form is invariant under the natural action of $\bar{G} = \mathrm{Gal}(\bar{M}/\bar{K})$ and may therefore be considered as an element of $\Omega_{\bar{K}}^{\otimes^m}$. See [23], Remark 3.15.

Remark 2.30 The following variant of the situation of the previous remark will play an important role. Let L/K be as before, $H \lhd G$ a normal subgroup and $M := L^H$ the corresponding subextension. Assume the following:

(a) We have $H \subset I$. In particular, H is a p-group.

(b) H is elementary abelian, say $H \cong (\mathbb{Z}/p\mathbb{Z})^r$.

(b) The filtration of higher ramification groups on $H = \mathrm{Gal}(L/M)$ has a unique break. Equivalently, the function $i_{L/M}$ is constant on $H\backslash\{1\}$.

Let $\chi \in R^+(G)$ be an irreducible character, and set $m := \chi(1)$. Then the restriction of χ to H decomposes as follows:

$$\chi|_H = k \cdot (\chi_1 + \ldots + \chi_l),$$

with $\chi_1, \ldots, \chi_l \in \hat{H}$ pairwise distinct one-dimensional characters of H, and with $m = kl$ (REFERENCE?). Set $\omega_i := \omega_{L/M}(\chi_i) \in \Omega_{\bar{M}}$. If $H = I$ then by the previous remark we have

$$\omega_{L/K}(\chi) = \big(\omega_1 \otimes \ldots \otimes \omega_l\big)^{\otimes^k} \in \Omega_{\bar{M}}^{\otimes^m}.$$

But the really interesting point about our special assumptions is the following.

Lemma 2.31 *The map*

$$\hat{H} \to \Omega_{\bar{M}}, \quad \psi \mapsto \omega_{L/M}(\psi)$$

is injective, $\mathbb{F}_p$-linear and equivariant with respect to the natural action of $G/H = \mathrm{Gal}(M/K)$ on both sides.

Proof: Injectivity and linearity follow from Assumption (c) and Lemma 2.28. Equivariance is easy to check. □

Let $V_\chi := \langle \chi_1, \ldots, \chi_l \rangle \subset \hat{H}$ be the subgroup generated by the χ_i. Clearly, V_χ is an $\mathbb{F}_p[G/H]$-submodule of $\hat{H}$. By the lemma, we have a natural injection

$$V_\chi \hookrightarrow \Omega_{\bar{M}}$$

of $\mathbb{F}_p[G/H]$-modules whose image is generated by the differential Swan conductors ω_i. As we will see, such *vector spaces of differential forms* are very useful invariants.

2.5 The case of an extension of degree p We assume that L/K is a fiercely ramified extension, as in Situation B, such that L/K is Galois of degree p. Let $G = \mathrm{Gal}(L/K) \simeq \mathbb{Z}/p\mathbb{Z}$ be the Galois group. We assume moreover, that the field of constants k contains a primitive pth root of unity ζ. As usual, we denote the extension of v to L (resp. K) again by v. Since we assume that L/K is fierce, the residue extension $\bar{L}/\bar{K}$ is purely inseparable of degree p. By Kummer theory $L = K(y)$, with $x := y^p \in \mathcal{O}_K^\times$, and we have a generator σ of G such that $\sigma(y) = \zeta y$.

We distinguish two cases. The terminology is explained at the end of this section.

The multiplicative case In the first case, we suppose that $\bar{x} \notin k^p$. Then y is a generator of the extension L/K, and we have

$$s_{L/K}(\sigma^a) = [\,\frac{\mathrm{d}\bar{y}}{\bar{y}}\,] - [\,\lambda\,] - [\,a\,],$$

for all $a \in \mathbb{F}_p^\times$, and with $\lambda := \zeta - 1$. Now if $\chi : G \to \tilde{\mathbb{Z}}$ is a character with $\chi(\sigma) = \zeta^b$, then

$$\begin{aligned} \mathrm{sw}_{L/K}(\chi) &= \left(\sum_{a \in \mathbb{F}_p^\times} \zeta^{ab} - 1\right) \cdot \left([\,\frac{\mathrm{d}\bar{y}}{\bar{y}}\,] - [\,\lambda\,]\right) - \epsilon(\zeta^b) + \epsilon(\zeta) \\ &= -p \cdot ([\,\frac{\mathrm{d}\bar{y}}{\bar{y}}\,] - [\,\lambda\,]) - [\,b\,] \\ &= [\,\lambda^p\,] - [\,b\frac{\mathrm{d}\bar{x}}{\bar{x}}\,]. \end{aligned}$$

Hence, the depth of χ is

$$\delta(\chi) = \frac{p}{p-1}.$$

Here we have used that $e_K = v(p) = 1$ by our chosen normalization. Furthermore, if we choose a suitable root of λ as prime element π_L, then the differential Swan conductor is

$$\omega(\chi) = b \cdot \frac{\mathrm{d}\bar{x}}{\bar{x}}.$$

The additive case For the second case, we suppose that $\bar{x}$ is a pth power in k. Then one can show that $x = z^p(1 + \pi_K^{pn} u)$, with $z, u \in \mathcal{O}_K^\times$, $\bar{u} \notin k^p$ and $0 < n < 1/(p-1)$, see e.g. [21]. Write $y = z(1 + \pi_L^n w)$. Then $\bar{w}^p = \bar{u}$, hence w is a generator of L/K, since $\bar{u}$ is not a pth-power in $\bar{K}$. Therefore, we get

$$s_{L/K}(\sigma^a) = [\,\mathrm{d}\bar{w}\,] - [\,\lambda \pi_L^{-n}\,] - [\,a\,].$$

A similar calculation as above yields

$$\begin{aligned} \mathrm{sw}_{L/K}(\chi) &= -p \cdot ([\,\mathrm{d}\bar{w}\,] - [\,\lambda \pi_L^{-n}\,]) - [\,b\,] \\ &= [\,\lambda^p \pi_L^{-pn}\,] - [\,b\mathrm{d}\bar{u}\,]. \end{aligned}$$

Hence, the depth of χ is

$$\delta(\chi) = \frac{p \cdot e_K}{p-1} - pn,$$

and the differential Swan conductor is

$$\omega(\chi) = b \cdot \mathrm{d}\bar{u}.$$

Note that in the case of fiercely ramified extensions of degree p, the differential form is either of the form $\mathrm{d}f/f$, i.e. ω is *logarithmic*, or is of the form $\mathrm{d}x$, i.e. ω is *exact*. We call the first case the *multiplicative case* and the second the *additive case*. The reason is a relation with actions under finite flat group schemes, which we do not specify here.

Note that $0 < \delta_{L/K}(\chi) \leq p/(p-1)$ and that we are in the multiplicative case if and only if $\delta_{L/K}(\chi) = p/(p-1)$.

The following lemma is proved similarly to the above computations, see for example [7], Lemma 1.4.5.

Lemma 2.32 *Suppose we are in Case B, and let L/K be a fierce Galois extension of degree p. Let χ be a nontrivial character of $G := \mathrm{Gal}(L/K)$. Then*

$$p\mathcal{D}_{L/K} = (p-1)\mathrm{sw}_{L/K}(\chi) + [-1]$$

in S_L.

In case the group G is not cyclic of order p, the structure of the differential Swan conductor is in general more complicated. For example, it is no longer true that the differential part ω is either exact or logarithmic. In Section 5.5, we compute as an example some Swan conductors of certain Q_8-extensions. This may give some feeling for what to expect in general. A detailed description in the case that $G \simeq \mathbb{Z}/p^n\mathbb{Z}$ can be found in [40].

2.6 The case of 2-local fields In this section, we assume that we are in case C, and that L/K is a finite Galois extension which is weakly unramified with respect to the valuation v. Using the rank-two valuation η, we can define another version of the Swan conductor.

We normalize the valuation

$$\eta : L^\times \to \Gamma := \mathbb{Q} \times \mathbb{Z}$$

such that $\eta(x) = \epsilon := (0,1) \in \mathbb{Q} \times \mathbb{Z}$ for a uniformizer x of L/K (i.e. $\bar{x} \in \bar{L}$ is a uniformizer of $\bar{L}$ with respect to the valuation $\bar{v}$). For $\sigma \in G\backslash\{1\}$ we define

$$i^\eta_{L/K}(\sigma) := \min_{y \in \mathcal{O}_L} \eta(\sigma(y) - y) \in \Gamma.$$

One easily shows that

$$i^\eta_{L/K}(\sigma) = \eta(\sigma(x) - x),$$

if x is a uniformizer of η, i.e. $\eta(x) = \epsilon$. We define the *rank-two Swan character* on $G = \mathrm{Gal}(L/K)$ by

$$\mathrm{sw}^\eta_{L/K}(\sigma) := -i^\eta_{L/K}(\sigma) + \epsilon$$

for $\sigma \neq 1$ and by

$$\mathrm{sw}^\eta_{L/K}(1) := -\sum_{\sigma \neq 1} \mathrm{sw}^\eta_{L/K}(\sigma).$$

Definition 2.33 The *rank-two Swan conductor* of a character $\chi \in R^+(G)$ is defined as

$$\mathrm{sw}^\eta_{L/K}(\chi) := \langle \mathrm{sw}^\eta_{L/K}, \chi \rangle_G \in \Gamma \otimes \mathbb{C}.$$

Proposition 2.34 *We have*

$$\mathrm{sw}^\eta_{L/K}(\chi) = (\delta_{L/K}(\chi), -\bar{v}(\omega_{L/K}(\chi)) + \chi(1)) \in \Gamma.$$

Proof: This follows formally from the relation

$$i^\eta_{L/K}(\sigma) = (i_{L/K}(\sigma), -\bar{v}(s_{L/K}(\sigma)) + 1).$$

□

2.7 Globalization (Compare with [38], §VI.3) So far we have assumed that the field K is complete with respect to the valuation v. This was to ensure that v extends uniquely to the field extension L/K. Let us now see what we get if we drop the completeness assumption. What we are going to say applies similarly to all three cases A, B and C.

We assume that we have a field K, a discrete valuation v on K and a finite Galois extension L/K. Let $v_1, \ldots, v_r$ denote the pairwise distinct extensions of v to L. It is well known that the Galois group $G = \mathrm{Gal}(L/K)$ acts transitively on the set $\{v_1, \ldots, v_r\}$. For $i = 1, \ldots, r$ let $G_{v_i} \subset G$ denote the stabilizer of v_i

(also called the *decomposition group* of v_i). If $\tau(v_i) = v_j$ then $\tau G_{v_i} \tau^{-1} = G_{v_j}$. It follows that all decomposition groups G_{v_i} are conjugate to each other.

Let $\hat{K}_v$ denote the completion of K with respect to v. Then v extends uniquely to a continuous and discrete valuation on $\hat{K}_v$ which we still denote by v. Note that the residue field and the value group of v is the same with respect to both K and $\hat{K}_v$. We assume that $(\hat{K}_v, v)$ satisfies the assumptions made in Case A or in Case B. If we are in Case B then we assume, moreover, that K contains a field k which is the field of constants of $\hat{K}_v$.

Let us fix $i \in \{1, \ldots, r\}$. Let $\hat{L}_{v_i}$ denote the completion of L with respect to v_i. Then the action of G_{v_i} on L extends to $\hat{L}_{v_i}$, by continuity, and makes $\hat{L}_{v_i}/\hat{K}_v$ a Galois extension with Galois group G_{v_i}. If we are in Case B then we may (and will) assume, after extending the field of constants k, that $\hat{L}_{v_i}/\hat{K}_v$ is weakly unramified.

Definition 2.35 Let $\chi \in R^+(G)$ be a character of G. If we are in Case A then we define the *Artin conductor* of χ by the formula

$$a_{L/K,v}(\chi) := a_{\hat{L}_{v_i}/\hat{K}_v}(\chi|_{G_{v_i}})$$

(and similarly for the Swan conductor, and for the depth conductor in Case B).

The main point is:

Lemma 2.36 *The definition of $a_{L/K,v}(\chi)$ is independent of the choice of the extension v_i.*

3 Group actions on semistable curves

3.1 The stably marked model In this section, we fix the following notation. Let k be complete discretely valued field of characteristic zero. We assume that the residue field $\bar{k}$ of k is a perfect field of characteristic $p > 0$. We denote the valuation of k by v and the ring of integers by $\mathfrak{o}$.

Let Y/k be a smooth projective curve, and let $G \subset \mathrm{Aut}_k(Y)$ be a finite group. The $f : Y \to X := Y/G$ is a G-Galois cover between smooth projective curves. Denote by $\{y_1, \ldots, y_r\}$ the set of ramification points of f i.e. the points of Y with nontrivial stabilizer.

Definition 3.1 (a) An $\mathfrak{o}$-*model* $\mathcal{Y}$ of Y is a normal, flat and proper $\mathfrak{o}$-scheme such that $\mathcal{Y} \otimes_{\mathfrak{o}} k \simeq Y$.

(b) A model $\mathcal{Y}$ of Y is called *semistable* if the special fiber $\bar{Y} := \mathcal{Y} \otimes_{\mathfrak{o}} \bar{k}$ is semistable, which means that it has at most ordinary double points as singularities.

(c) A semistable $\mathfrak{o}$-model $\mathcal{Y}$ of Y is called *G-semistable* if

(i) the action of G on Y extends to $\mathcal{Y}$ and

(ii) the ramification points $y_1, \ldots, y_r$ specialize to pairwise distinct smooth points $\bar{y}_1, \ldots, \bar{y}_r \in \bar{Y}$.

Lemma 3.2 *Let $\mathcal{Y}$ be a G-semistable $\mathfrak{o}$-model of Y. Let $\mathcal{X} := \mathcal{Y}/G$ denote the quotient scheme, $f_{\mathfrak{o}} : \mathcal{Y} \to \mathcal{X}$ the canonical map and $\bar{f} : \bar{Y} \to \bar{X}$ the restriction of $f_{\mathfrak{o}}$ to the special fiber.*

(i) *The $\mathfrak{o}$-scheme $\mathcal{X}$ is a semistable $\mathfrak{o}$-model of $X = Y/G$.*

(ii) *The branch points $x_1, \ldots, x_{r'}$ of f specialize to pairwise distinct smooth points $\bar{x}_1, \ldots, \bar{x}_{r'} \in \bar{X}$.*

(iii) *If $\bar{y} \in \bar{Y}$ is a singular point then $\bar{x} := \bar{f}(\bar{y}) \in \bar{X}$ is singular as well.*

Lemma 3.3 *Assume that $2g(Y) + r - 2 > 0$. Then after replacing k by a finite separable extension, there exists a minimal G-semistable model $\mathcal{Y}$ of Y. It is unique up to unique isomorphism.*

Proof: See [11]. □

Definition 3.4 The minimal G-semistable model $\mathcal{Y}$ from Lemma 3.3 is called the *stably marked model* of Y. The resulting morphism $f_{\mathfrak{o}} : \mathcal{Y} \to \mathcal{X}$ is called the *stable model* of the G-cover f and its restriction to the special fiber $\bar{f} : \bar{Y} \to \bar{X}$ is called the *stable reduction* of f.

From now on we shall make the following assumption.

Assumption 3.5 (i) We have $2g(Y)+r-2 > 0$. Therefore the stable model and the stable reduction of f are well defined.

(ii) Every irreducible component $\bar{Z}$ of $\bar{Y}$ is smooth. (This means that irreducible components of $\bar{Y}$ do not intersect themselves.)

Remark 3.6 For the examples we are interested in, Assumption 3.5 is not very restrictive. Suppose for instance that $G \neq \{1\}$. Then (i) holds except if $g(Y) = 0$ and G is cyclic (in which case $r = 2$). Also, (ii) holds automatically if $g(X) = 0$.

Remark 3.7 Let $H \subset G$ be a subgroup. Then any G-semistable model $\mathcal{Y}$ of Y is also H-semistable. However, if $\mathcal{Y}$ is the (unique) minimal G-semistable model it may not be minimal as a H-semistable model.

In these notes we will often study the geometry of a G-stable model $f_{\mathfrak{o}} : \mathcal{Y} \to \mathcal{X}$ by restricting the G-action to a subgroup $H \subset G$ and looking at the corresponding subcovers $\mathcal{Y} \to \mathcal{Z} := \mathcal{Y}/H$ and $\mathcal{Z} \to \mathcal{X}$. From the remark made above it is clear that the definition of the models $\mathcal{Y}$ and $\mathcal{Z}$ depends on the larger group G. In practice, this should not cause too much confusion.

Example 3.8 Let $p \neq 2$ be a prime. Let Y be the smooth projective curve given by the affine equation

$$y^p = x(x-1) =: g(x). \tag{7}$$

The assumption $p \neq 2$ assures that the genus of Y is greater than or equal to 1. (In fact it is $(p-1)/2$). We assume that k contains a pth root of unity ζ_p. Then $\sigma(x, y) = (x, \zeta_p y)$ defines an automorphism of Y of order p. The curve $X := Y/\langle\sigma\rangle$ is a projective line with function field $k(x)$.

We first consider the 'naive' model $\mathcal{Y}_0$ given by (7) over $\mathfrak{o}$. More precisely, we let $\mathcal{X}_0 := \mathbb{P}^1_{\mathfrak{o}}$ be the 'standard' model of $X = \mathbb{P}^1_k$ with respect to the variable x, and we define $\mathcal{Y}_0$ as the normalization of $\mathcal{X}_0$ in Y. Then the map $f_0 : \mathcal{Y}_0 \to \mathcal{X}_0$ is finite and flat and we have $\mathcal{X}_0 = \mathcal{Y}_0/G$.

However, $\mathcal{Y}_0$ is not semistable. To see this, one shows that the affine piece of $\mathcal{Y}_0$ lying over $\mathbb{A}^1_{\mathfrak{o}} = \operatorname{Spec} \mathfrak{o}[x]$ is

$$\operatorname{Spec} \mathfrak{o}[x, y \mid y^p = g(x)].$$

(One only needs to show that this scheme is normal.) Using the Jacobi criterion one concludes that $\bar{Y}_0 := \mathcal{Y}_0 \otimes_{\mathfrak{o}} \bar{k}$ has a singularity in the (unique) point with coordinate $\bar{x} = 1/2$. (This point corresponds to the zero of $g'(x) \pmod p$.) To obtain a semistable model, we need to take a suitable blow-up $\mathcal{Y}_0$ in the singular point of $\bar{Y}_0$, followed by a normalization. In our situation, this may be performed by a suitable blow-up $\mathcal{X}$ of $\mathcal{X}_0$ together with the normalization of $\mathcal{X}$ in the function field of Y ([19]).

We first define a new parameter $x = x_1 + 1/2$. It follows that the zero of the derivative of $g(x_1) = x_1^2 - 1/4 \pmod p$ is in $x_1 = 0$. We try to find new coordinates

$$y = \lambda y_1 + h(x), \quad x_1 = \mu x_2, \tag{8}$$

with $\lambda, \mu \in \mathfrak{p}_k$ and $h \in k(X) = k(x)$ such that the reduction of the component with this coordinate yields an Artin–Schreier equation in characteristic p, and moreover, the curve defined by this equation has positive genus. One may show that it suffices to consider such substitutions ([19]).

Substituting (8) into (7) yields

$$\lambda^p y_1^p + \cdots + p\lambda h^{p-1} y_1 = \mu^2 x_2^2 - 1/4 + h^p. \tag{9}$$

In order to obtain an Artin–Schreier equation in reduction, we need to choose $h^p = 1/4$ (at least modulo a sufficiently large power of the uniformizing element of k.) Since we assume that the residue field of k is algebraically closed, we may choose $h \in \mathfrak{o}$.

Furthermore, we want

$$v(\lambda^p) = v(p\lambda) = v(\mu^2).$$

Choose λ, μ with $v(p) = (p-1)v(\lambda)$ and $v(\mu) = pv(\lambda)/2$. This is possible after replacing k by a finite extension. Then $v(\binom{p}{i}\lambda^i h^{p-i}) > v(\lambda^p)$ for all $0 < i < p-1$. This shows that there exists nonzero numbers $a, b \in \bar{k}$ such that (9) reduces to

$$\bar{y}_1^p + a\bar{y}_1 = b\bar{x}_2^2. \tag{10}$$

After multiplying y_1 and x_2 by a suitable unit, we may assume that $a = -1$ and $b = 1$.

Define $\mathcal{X}$ to be the blow-up of $\mathcal{X}_0$ in the point $\bar{x}$ with $x_1 = 0$ corresponding to the substitution $x_1 = \mu x_2$. Let $\mathcal{Y}$ be the normalization of $\mathcal{X}$ in the function field L of Y. Its special fiber $\bar{Y}$ consists of two irreducible component. The first component $\bar{Y}_0$ has equation

$$\bar{y}_0^p = \bar{x}_0,$$

which is the normalization of the reduction of the equation given by (7). The second component $\bar{Y}_1$ is birationally given by (10). One calculates that the normalization of $\bar{Y}_1$ has genus $(p-1)/2 = g(Y)$. This implies that $\bar{Y}_1$ is in fact smooth, and $\mathcal{Y}$ is a semistable model of Y. Similarly, the special fiber $\bar{X}$ consists of two projective lines intersecting in the point $x_1 = 0$. We denote by $\bar{X}_i$ the component of $\bar{X}$ underneath $\bar{Y}_i$.

The three ramification points of $f : Y \to X$ clearly specialize to pairwise distinct points on the component $\bar{X}_0$ different from the singularity of $\bar{X}$. This shows that $\mathcal{Y}$ is the stably marked model of Y.

Exercise 3.9 Choose $k = W(\bar{k})[\zeta_p]$. Show that $\lambda = \zeta_p - 1$ satisfies $v(p) = (p-1)v(\lambda)$. Use this to determine an extension of k over which the stably marked model $\mathcal{Y}$ is defined.

The following definition is the key definition needed for formulating the lifting problem (compare to Section 1).

Definition 3.10 Let $f_k : Y \to X$ be a G-Galois cover defined over a local field k as above. Denote by g the genus of Y, and by r the number of ramification points of f_k. Assume that $2g + r - 2 > 0$. Let $f : \mathcal{Y} \to \mathcal{X}$ be the stably marked model of f_k. We say that f_k has *good reduction* (with respect to the valuation v of k) if the reduction $\bar{Y}$ of $\mathcal{Y}$ contains a component $\bar{Y}'$ such that

- the genus of $\bar{Y}'$ equals that of Y,
- we have $D(\bar{Y}') = G$ and $I(\bar{Y}') = \{1\}$.

Note that the fact that $f : Y \to X$ has good reduction, with reduction $\bar{f} : \bar{Y} \to \bar{X}$, also implies that $\bar{Y}$ together with the associated action of $G \subset \mathrm{Aut}_{\bar{k}}(\bar{Y})$ lifts to characteristic zero (Definition 1.1).

Suppose the genus of Y is nonzero and the cover $f_k : Y \to X$ has good reduction. Then the curve $\bar{Y}'$ is the unique component of the stable model $\mathcal{Y}$ of Y of positive genus. If the genus of Y is zero, it may happen that f_k is the lift of a G-Galois cover $\bar{f}' : \bar{Y}' \to \bar{X}'$ in characteristic zero, but that $\bar{Y}'$ is not the component of the stably marked model $\mathcal{Y}$ of Y. However, this can only happen in very special cases, which one may easily exclude. This allows one to give a somewhat simpler (though less intuitive) definition of good reduction, omitting some of the conditions of Definition 3.10.

Example 3.11 The cover of curves described in Example 3.8 has good reduction to characteristic p. Denote by $\bar{Y}_1$ the irreducible component of $\bar{Y}$ of positive genus, and by $\bar{X}_1$ the underlying component of $\bar{X}$. One may define models $\mathcal{Y}'$ (resp. $\mathcal{X}'$) of Y (resp. X) by contracting the components different from $\bar{Y}_1$ (resp. $\bar{X}_1$). Since f_k has good reduction, the model $\mathcal{Y}_1$ is smooth, in particular semistable (but not stably marked). We denote the corresponding map by $f_1 : \mathcal{Y}_1 \to \mathcal{X}_1$.

The reduction $\bar{f}_1 : \bar{Y}_1 \to \bar{X}_1$ is an Artin–Schreier cover branched exactly at one point. The conductor in this case is $h = 2$. The contribution of this branch point to the different is $(h+1)(p-1)$. The generic fiber is $f_k : Y \to X$, which is a Kummer equation branched at $h + 1 = 3$ points. Note that the contribution of these branch points to the different is also $3(p-1) = (h+1)(p-1)$. The three branch points of f_k specialize to the unique branch point in characteristic p. Note that this is a necessary condition for a cover $f_1 : \mathcal{Y}_1 \to \mathcal{X}_1$ to exist. In Section 4.1 we study this more generally.

The advantage of working with the stably marked model $\mathcal{Y}$ rather than with the smooth model $\mathcal{Y}_1$ is that it gives more insight in the exact combinatorics of the reduction of the branch points of f_k. The Example 3.8 is unfortunately to easy to really appreciate this.

3.2 Ramification invariants Let $f : Y \to X$ be a G-Galois cover as in Section **??**. We assume that Assumption 3.5 holds and we denote by $f_{\mathfrak{o}} : \mathcal{Y} \to \mathcal{X}$ the stable model and by $\bar{f} : \bar{Y} \to \bar{X}$ the stable reduction of f. Let $\chi \in R^+(G)$ be a character of G. We shall attach to f and χ certain invariants which 'live' on $\bar{X}$. We use the theory of ramification reviewed in Section **??**.

Let $K := k(X)$ and $L := k(Y)$ be the function fields of X and Y. Then L/K is a Galois extension with Galois group G. Let $Z \subset \bar{X}$ be an irreducible component of $\bar{X}$. It gives rise to a discrete valuation v_Z on K which extends the standard valuation v on k. The completion $\hat{K}_Z$ of K with respect to v_Z is a field as in Case B of the previous section (Example 2.3). We note that the extensions of v_Z to L are in bijection with the irreducible components W of $\bar{Y}$ lying over Z. Combining the results of §2.4 and §2.7 we define the *depth* of the character χ at the component Z as the nonnegative rational number

$$\delta_Z(\chi) := \delta_{K/L,v_Z}(\chi) \in \mathbb{Q}_{\geq 0}.$$

Furthermore, if $\delta_Z(\chi) > 0$ then we can define the *differential Swan conductor* of χ at Z by

$$\omega_Z(\chi) := \omega_{L/K,v_Z}(\chi) \in \Omega^{\otimes m}_{\bar{k}(Z)},$$

with $m := \chi(1)$, see Remark 2.29.

Definition 3.12 A *branch* of $\bar{X}$ is a pair (Z, x), where $Z \subset \bar{X}$ is an irreducible component of $\bar{X}$ and $x \in Z$ is a closed point.

Let (Z, x) be a branch of $\bar{X}$. Then Z gives rise to a discrete valuation v_Z on K with residue field $\bar{k}(Z)$. By Assumption 3.5 (b), $x \in Z$ is a smooth point and therefore gives rise to a discrete valuation $\bar{v}_x$ on $\bar{k}(Z)$. Let $\eta : K^\times \to \mathbb{Q}\times\mathbb{Z}$ be the rank two valuation associated with the pair $(v_Z, \bar{v}_x)$. It is clear that the branch (Z, x) is uniquely determined by the pair of valuations $(v_Z, \bar{v}_x)$ or by the rank two valuation η. In order to simplify notation, we identify from now on the valuation η with the branch (Z, x). Let $\hat{K}_\eta$ be the residual completion of $\hat{K}_Z$ with respect to $\bar{v}_x$ (Example 2.7). As in the rank-one case, one shows that there is a bijection between the set of branches (W, y) of $\bar{Y}$ lying over (Z, x) and the set of extensions ξ of η to L. Therefore, we may use the results of §2.6 and §2.7 to define the Swan conductor of χ at the branch η,

$$\mathrm{sw}^\eta_{L/K}(\chi) \in \Gamma = \mathbb{Q}\times\mathbb{Z}.$$

By definition, the first entry of $\mathrm{sw}^\eta_{L/K}(\chi)$ is equal to the depth $\delta_{L/K,v}(\chi)$. To relate the second entry (which we shall denote by $\#\mathrm{sw}^\eta_{L/K}(\chi)$) to other invariants, we distinguish two cases.

Let us first suppose that $\delta_Z(\chi) > 0$ (i.e. that $\chi|_{I_Z}$ is nontrivial). Then the differential Swan conductor $\omega_Z(\chi) \in \Omega^{\otimes m}_{\bar{k}(Z)}$ is defined, and we have

$$\#\mathrm{sw}^\eta_{L/K}(\chi) = -\bar{v}_x(\omega_Z(\chi)) + \chi(1),$$

see Proposition 2.34. If $\delta_Z(\chi) = 0$ (i.e. if $\chi|_{I_Z}$ is trivial) then ..

Choose an irreducible component $\bar{Y}_i$ of $\bar{Y}$ and let $\bar{X}_i$ be the irreducible component of $\bar{X}$ to which $\bar{Y}_i$ maps via $\bar{f}$. We note that the group G acts on $\bar{Y}$, by the uniqueness of the model $\mathcal{Y}$.

Denote by $G(\bar{Y}_i)$ the *decomposition group* of $\bar{Y}_i$, recall that this is the set of elements in G which send $\bar{Y}_i$ to itself. The *inertia group* $I(\bar{Y}_i)$ of $\bar{Y}_i$ is the normal subgroup of $G(\bar{Y}_i)$ of elements which act trivially on $\bar{Y}_i$. (In other words, $I(\bar{Y}_i)$ is the kernel of $G(\bar{Y}_i) \to \mathrm{Aut}_k(\bar{Y}_i)$.) It is easy to see that $I(\bar{Y}_i)$ is a p-group (compare to Section 1).

The generic point of the irreducible component $\bar{Y}_i$ defines a valuation of L which extends v. We denote this valuation by $v_i := v_{\bar{Y}_i}$. Similarly, we obtain a valuation of K extending v corresponding to $\bar{X}_i$. We denote this valuation also by v_i. Let L_{v_i} (resp. K_{v_i}) be the completion of L (resp. K) with respect to v_i. Then L_{v_i}/K_{v_i} is an extension of degree $|D(\bar{Y}_i)|$. Moreover, the inseparable degree is equal to $|I(\bar{Y}_i)|$. Denote by $L^{\mathrm{insep}}_{v_i}$ the maximal inseparable subextension of L_{v_i}/K_{v_i}. Then $L_{v_i}/L^{\mathrm{insep}}_{v_i}$ is a extension of residual dimension one, which satisfies the assumptions of Section 2.1, Situation B.

Next we choose a point $\bar{y} \in \bar{Y}$, and let $\bar{x} = \bar{f}(\bar{y})$. Choose an irreducible component $\bar{Y}_i$ of $\bar{Y}$ with $\bar{y} \in \bar{Y}_i$. Assumption 3.5 implies that $\bar{y}$ is a smooth point on $\bar{Y}_i$ (though not necessarily of $\bar{Y}_i$.) The point $\bar{y}$ therefore defines a valuation on the residue field $\bar{L}_{v_i}$ of L_{v_i}. We denote this valuation by $\mathrm{ord}_{\bar{y}}$. The two valuations v_i and $\mathrm{ord}_{\bar{y}}$ equip L_{v_i} with the structure of a 2-local field as in Section 2.1, Situation C.

3.3 A more involved example In this section, we discuss a more involved example of the computation of the stably marked model of a cover. In this section, we present the example in a rather ad-hoc way. However, there exists a algorithm for computing it ([2], [1]). While we will not describe this algorithm in these notes, we give some main ideas in the next section. The example presented in this section will serve as a motivation for the rather technical ideas presented in the next section.

Let $G = Q_8$ be the quaternion group with 8 elements. We denote by $\sigma_1, \sigma_2, \sigma_3 \in G$ three elements of G of order 4 satisfying

$$\sigma_1\sigma_2 = \sigma_3, \quad \sigma_1^2 = \sigma_2^2 = \sigma_3^2 = -I,$$

where $-I$ is the nontrivial element of the center of G. Denote by Y the smooth projective curve of genus 2 defined over $\mathbb{Q}(i)$, defined by the affine equation

$$y^2 = x(x^4 - 1) =: \varphi(x). \tag{11}$$

Exercise 3.13 (a) Show that

$$\begin{aligned} \sigma_1(x, y) &= (1/x, iy/x^3), \\ \sigma_2(x, y) &= (-x, iy), \\ \sigma_3(x, y) &= (-1/x, y/x^3) \end{aligned}$$

defines an action of G on Y.

(b) Show that the points of Y with nontrivial stabilizer are exactly the points with x-coordinates $S := \{0, \infty, \pm 1, \pm i\}$. Conclude that we may identify S with the set of ramification points of $Y \to Z := Y/G$. Determine the point stabilizers.

(c) Show that the genus of $X := Y/\langle -I \rangle$ is zero.

(d) More generally, show that $\mathrm{Aut}(S) := \{\sigma \in \mathrm{PSL}_2(\mathbb{C}) \mid \sigma(S) = S\} \simeq S_4$. Conclude that

$$1 \to \{\pm I\} \to \mathrm{Aut}_k(Y) \to S_2 \to 1.$$

Remark: the group $\mathrm{Aut}_k(Y)$ is the unique transitive subgroup of S_6 with 48 elements. The computer algebra package GAP denotes this group by $S_4(6d)$.

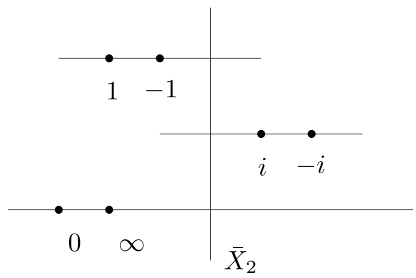
Put $f : Y \to Z = Y/G$. Exercise 3.13.(c) implies that Z is a curve of genus zero. Choose a local field k containing i, whose residue field is an algebraically closed field of characteristic 2. In the course of the example, we may have to replace k by a finite extension. This will not be always mentioned explicitly.

We compute the stable reduction of $Y \to Z$. It turns out, however, to suffice computing the stable reduction of $g : Y \to X$, since $g(X) = 0$. This is easier, since the degree of g is $2 = p$. Let $\mathcal{Y}$ be the stably marked model of Y, and denote by $\mathcal{X} = \mathcal{Y}/G$.

We may consider the 6 points $\{0, \infty, \pm 1, \pm i\}$ as marking on the projective line X. Therefore the stably marked model $\mathcal{Y}_0$ of Y is defined.

Exercise 3.14 (a) For all $x \neq y \in \{0, \infty, \pm 1, \pm i\}$ compute $v(x_i - x_j)$.

(b) Deduce that the special fiber $\bar{X}_0$ of the stably marked model $\mathcal{X}_0 := \mathcal{Y}_0/\langle -I \rangle$ of X looks as follows.



The definition of the stably marked model shows that the irreducible components of $\bar{X}_0$ also show up in $\bar{X}$. Formulated more precisely, there is a natural map $\mathcal{X} \to \mathcal{X}_0$ which is an isomorphism on the generic fiber, contracts certain irreducible components of the special fiber $\bar{X}$, and has degree 1 when restricted to all other components of $\bar{X}$. This implies that for every irreducible component $\bar{X}_1$ of $\bar{X}_0$, there exists a unique component of $\bar{X}$ which maps surjectively to $\bar{X}_1$. We may therefore identify $\bar{X}_1$ with the corresponding component of $\bar{X}$. (Compare to Section 1.)

We first compute the irreducible components of $\bar{Y}$ which map to irreducible components of $\bar{X}$ which are not contracted by the map $\bar{X} \to \bar{X}_0$. We perform this explicitly for one component, and leave the others as exercise.

Let $\bar{X}_2$ the irreducible component of $\bar{X}_0$ as indicated in Picture 3.14). This component is determined by the property that $\{0, \infty\}$ (resp. $\{\pm 1\}$, resp. $\{\pm i\}$) specialize to three distinct points. Let v_2 be the valuation of the function field K of X corresponding to $\bar{X}_2$. We want to compute the irreducible components of $\bar{Y}$ above $\bar{X}_2$. Putting $x = \sqrt{2}x_2 + 1$, it follows from Exercise 3.14.(a) that x_2 defines a coordinate on $\bar{X}_2$. Substituting this into φ, we find

$$\varphi = (\sqrt{2})^5 x_2^5 + 5 \cdot 4x_2^4 + 5(\sqrt{2})^5 x_2^3 + 5 \cdot 4x_2^2 + (\sqrt{2})^5 x_2. \qquad (12)$$

The coefficient of φ with minimal valuation has valuation $v(4) = 2$. Therefore we may define $y = 2y_2$ and divide the equation by 4. The reduction of this equation is

$$\bar{y}_2^2 = \bar{x}_2^4 + \bar{x}_2^2 \equiv \bar{x}_2^2(\bar{x}_2 + 1)^2 =: \varphi_2(\bar{x}_2). \qquad (13)$$

In other words, $\varphi_2(x_2)$ is a 2th power in characteristic 2. (This means that we are in the additive case of Section 2.5.) This means that we have not yet found a description of $\bar{g}|_{\bar{X}_2}$. Therefore we will modify the equation.

We choose a rational function $h(x_2)$ such that $h^2 \equiv \varphi_2(x_2)$ up to a sufficiently high power of the uniformizing element of $\mathfrak{o}$. To make sure we have a sufficiently high precision, we may choose

$$h = cx_2^2 + dx_2, \qquad \text{with } c^2 = 20 = d^2,$$

i.e. such that the coefficients of x_2^4 and x_2^2 equal those of $\varphi/4$ (12). In other words, choose $h = \sqrt{5}x_2(x_2 + 1)$. One computes that $\varphi/4 - h^2$ is divisible by $\sqrt{2}$. Defining

$$\varphi_3 := (\varphi/4 - h^2)/\sqrt{2}, \tag{14}$$

we find in reduction

$$\varphi_3 \equiv \bar{x}_2(1 + \bar{x}_2 + \bar{x}_2^2)^2.$$

Defining $y_2 = \sqrt[4]{2}y_3 + h$ yields in reduction

$$\bar{y}_3^2 = \bar{x}_2(1 + \bar{x}_2 + \bar{x}_2^2)^2. \tag{15}$$

Since the right hand side of (15) is no longer a 2th power, this is the equation we are looking for. It follows that there is a unique irreducible component $\bar{Y}_2$ of $\bar{Y}$ above $\bar{X}_2$. The component $\bar{Y}_2$ is the normalization of the singular curve given by (15). In particular, $\bar{Y}_2$ is a curve of genus zero.

Similarly, one can show that over each of the components $\bar{X}_i$ introduced in Exercise 3.14 there is a unique component of genus zero in $\bar{Y}$.

Recall that the curve (15) has two singularities, namely in the points with $\bar{x}_2^2 + \bar{x}_2 + 1$. The following exercise shows that blowing up $\mathcal{X}$ in these two points, one finds two elliptic curves. This completes the calculation of the stably marked model, since $g(Y) = 2 = 1 + 1$.

Exercise 3.15 (a) By blowing up once more in the points with $\bar{x}_2^2 + \bar{x}_2 + 1$, find two components of genus 1 of the stably marked model $\mathcal{Y}$. (This step is similar to Example 3.8.)

(b) Conclude that we have found all irreducible components of $\mathcal{Y}$.

(c) Describe the action of $G = Q_8$ on $\bar{Y}$. Compute the decomposition and inertia group of each of the irreducible components of $\bar{Y}$.

(d) In particular, show that the restriction of $\bar{f}$ to each of the two irreducible components of $\bar{Y}$ of positive genus is a G-Galois cover totally branched at a unique point. Conclude that $Y \to Z$ has bad reduction to characteristic 2. Compute the filtration of higher ramification groups at this point.

Proposition 3.16 *(a) Let Y be a smooth projective curve of genus 2 whose automorphism group contains Q_8. Then Y is isomorphic to the curve defined by (11) with Q_8-action defined by Exercise 3.13.(a).*

(b) All Q_8-covers $f : Y \to \mathbb{P}^1$ with $g(Y) = 2$ have bad reduction to characteristic 2.

Sketch of the proof: Let Y be as in (a). In particular, Y is hyperelliptic. Denote by $\iota \in \mathrm{Aut}_k(Y)$ the hyperelliptic involution. One can show that ι is contained in the center of $\mathrm{Aut}_k(Y)$. This follows by using that Y admits a unique map $f : Y \to \mathbb{P}^1$ of degree 2 ([18], Proposition 5.3.) Denote by $\mathrm{RA}_k(Y) := \mathrm{Aut}_k(Y)/\langle\iota\rangle$ the *reduced automorphism group* of Y. Since the center of Q_8 is $\{\pm I\}$, it follows that $-I$ acts on Y as the hyperelliptic involution. In particular $(\mathbb{Z}/2\mathbb{Z})^2 \subset \mathrm{RA}_k(Y)$.

Igusa [22], Section 8 classifies all genus-2 curves with $\mathrm{RA}_k(Y) \neq \{1\}$. This classification implies that Y is a specialization of the 1-dimensional family of curves of genus 2 defined by

$$y^2 = x(x^2-1)(x-\lambda)(x-(1-\lambda)^{-1}).$$

Moreover, Igusa shows that in this case $\mathrm{RA}_k(Y) \in \{\mathbb{Z}/2\mathbb{Z} \times \mathbb{Z}/2\mathbb{Z}, D_6, S_4\}$, where D_6 denotes the dihedral group of order 12. Using a similar strategy as in Exercise 3.3.(a), one shows that if $\mathrm{RA}_k(Y) = \mathbb{Z}/2\mathbb{Z}\times\mathbb{Z}/2\mathbb{Z}$ then $\mathrm{Aut}_k(Y) \simeq D_4$. Similarly, it follows that if $\mathrm{RA}_k(Y) = D_6$ then $\mathrm{Aut}_k(Y)$ does not contain a subgroup isomorphic to Q_8. We conclude that $\mathrm{RA}_k(Y) \simeq S_4$. Igusa's classification now implies that Y is as claimed in (a). Part (a) together with the result of Exercise 3.15 immediately imply (b). □

3.4 Compatibility properties of differential Swan conductors In this section, we let k be a complete discretely valuation field of mixed characteristic p, whose residue field $\bar{k}$ is algebraically closed. Suppose given a G-Galois cover $f_k : Y \to X$ over k between smooth curves. After replacing k by a finite extension, we may assume that the stably marked model $f : \mathcal{Y} \to \mathcal{X}$ is defined over $\mathfrak{o}$.

We choose an irreducible component $\bar{Y}_1$ of $\bar{Y}$ with nontrivial inertia group $I_1 := I(\bar{Y}_1)$ (if it exists). Denote by $\bar{X}_1$ the component of $\bar{X}$ underlying $\bar{Y}_1$. Recall that I_1 is a p-group which is a normal subgroup of the decomposition group $D_1 := D(\bar{Y}_1)$. Therefore the restriction of $\bar{f}$ to $\bar{Y}_1$ factors as

$$\bar{f}|_{\bar{Y}_1} : \bar{Y}_1 \to \bar{Z}_1 \to \bar{X}_1, \qquad (16)$$

where $\bar{Z}_1 \to \bar{X}_1$ is a Galois cover with Galois group D_1/I_1.

Let L (resp. K) be the function field of Y (resp. X). Denote by v_1 the valuation of L corresponding to the irreducible component $\bar{Y}_1$. We denote the corresponding valuation of K also by v_1. Passing to the completion yields an extension L_{v_1}/K_{v_1}. This is the extension of local fields corresponding to $\bar{Y}_1 \to \bar{Z}_1$ in the factorization (16). Moreover, it is a fierce extension in the sense of Section 2.1. In particular, the degree of this extension if $|I_1|$.

Using the structure of p-groups, it follows that the purely inseparable map $\bar{Y}_1 \to \bar{Z}_1$ factors into maps of degree p. It is therefore interesting to first consider the case that the order of I_1 is p.

The goal of this section is to make the construction of the differential Swan conductor from Section 2.4 explicit. Recall that the differential Swan conductor consists of two data: a differential form ω_1 and the depth δ_1 which is a rational number. Let $\chi : I_1 \to \mathbb{C}$ be a nontrivial character. As we have seen in Section 2.5, replacing χ by another nontrivial character multiplies ω_1 by a nonzero constant in $\bar{k}$. Therefore we may omit χ from the notation. To illustrate these properties, we compute the differential forms corresponding to some of the components of the Examples 3.8, 3.3.

Example 3.17 (a) We first consider the cover from Example 3.8. Recall that $\bar{Y}$ has a unique component with nontrivial inertia group. This is the component denoted by $\bar{Y}_0$ which corresponds to the (normalization of the) special fiber of the "naive" model $\mathcal{Y}_0$ considered as component of $\bar{Y}$. The restriction of f to this component is given by the normalization of the reduction of the original Kummer equation:

$$\bar{y}^p = \bar{x}(\bar{x} - 1) =: \bar{g}. \tag{17}$$

Since $\bar{f}$ is not a pth power in the function field of $\bar{X}_0$, we are in the multiplicative case of Section 2.5. Therefore the associated differential form is

$$\omega = \frac{\mathrm{d}\bar{g}}{\bar{g}} = \frac{2\bar{x} - 1}{\bar{x}} \mathrm{d}\bar{x}.$$

Note that ω has simple poles in the specialization of the branch points $0, 1, \infty$ of the characteristic-zero cover. Moreover, ω has a zero in the point of $\bar{Y}_0$ which is singular in $\bar{Y}$. Comparing to the calculation from Example 3.8, we see that this is no surprise: the zeros of ω correspond exactly to the singularities of the equation (17). Moreover, we see that the order of the zero of ω is related to the lower jump h of the Artin–Schreier cover (9) by the formula

$$\mathrm{ord}(\omega) + 1 = h.$$

The depth in this case is given by $\delta = p/(p-1)$. We computed this already in Section 2.5.

(b) We now consider the example from Section 3.3, and focus on the component denoted by $\bar{Y}_2$ in that section. Note that $\mathcal{Y}$ is also a stable model of the cover $Y \to X$ with Galois group $\langle -I \rangle \simeq \mathbb{Z}/2\mathbb{Z}$, to which we restrict here for simplicity. Recall that $\bar{Y}_2 \to \bar{X}_2$ is an inseparable cover of degree 2, and that none of the branch points $\{0, \infty, \pm 1, \pm i\}$ specialize to $\bar{X}_2$. We found an equation $\bar{y}^2 = \varphi(\bar{x}_2)$ (13) for this component, where φ_2 is a 2th power in the function field of $\bar{X}_2$, therefore we are in the second case of Section 2.5.

As explained in Section 3.3, writing $y_2 = y_3 + h(x_2)$ for suitable h yields an equation $\bar{y}_3^2 = \varphi_3(\bar{x}_2)$ where $\varphi_3(\bar{x}_2)$ is not a 2th power (15). Therefore the corresponding differential form is

$$\omega_2 = \frac{\mathrm{d}\varphi_3}{h^2} = \frac{(1 + \bar{x}_2 + \bar{x}_2^2)^2}{x_2^2(x_2 + 1)^2} \mathrm{d}\bar{x}_2.$$

We find the same relation between the conductor of the Artin–Schreier covers and the order of the zeros of ω_2 as in the previous example. In particular the fact that ω_2 has two distinct zero of order 2 immediately implies that the cover $Y \to Z$ has bad reduction. For the conclusion of Exercise 3.15.(d) it is therefore not neccessary to explicitly preform the blow-ups from Exercise 3.15.(a). The poles of ω_2 indicate the direction in which to find the specialization of the 6 branch points.

The depth of the component is determined by (14): since we have written $y_2 = h + \sqrt{2}y_3$, we conclude that $pn = v(\sqrt{2}) = 1/2$. Therefore $\delta = p/(p-1) - pn = 3/2$.

Example 3.17 illustrates that the poles and zeros of the differential Swan conductors on a component tells us something about the structure of the other components of $\bar{Y}$.

Suppose that $f_k : Y \to X = \mathbb{P}^1_k$ is a G-Galois cover. Suppose that $\bar{Y}_i$ is an irreducible component of $\bar{Y}$ with nontrivial stabilizer I_i, and let L_v/K_v be the corresponding fiercely ramified extension. As before, we write $\bar{Y}_i \to \bar{Z}_i$ for the maximal inseparable subcover of the restriction of $\bar{f}$ to $\bar{Y}_i$. Recall that this cover has degree $|I_i|$.

Choose a character $\chi : I_i \to \mathbb{C}$. Denote by $(\omega(\chi), \delta(\chi))$ the corresponding differential Swan conductor. Recall that $\omega(\chi)$ is a nontrivial differential form which is defined on the irreducible component of some subcover of $\bar{Y}_i \to \bar{Z}_i$ underlying $\bar{Y}_i$ which corresponds to $\ker(\chi)$. (We sometimes write $\bar{Y}_i/\ker(\chi)$ for this curve, even though I_i acts trivially on $\bar{Y}_i$ in characteristic p, and this therefore should not be considered as the definition of the component.)

Let $\bar{y}$ be a point on $\bar{Y}_i$.

Proposition 3.18 *(a) The zeros and poles of $\omega(\chi)$ are situated in the singularities of $\bar{Y}$ and the specialization of the ramification points of f_k.*

(b) Suppose $\bar{y}$ is the specialization of a ramification point of f_k; in particular $\bar{y}$ is smooth. Then ω has a simple pole in $\bar{x}$ and $\delta(\bar{Y}_i) = p/(p-1)$.

(c) Suppose $\bar{y}$ is a singularity of $\bar{Y}$. Let $\bar{Y}_1, \bar{Y}_2$ be the two components of $\bar{Y}$ intersecting in $\bar{y}$. Let $\chi : I_{\bar{y}} \to \mathbb{C}$ be a character, and write $\omega_i(\chi)$ for the corresponding differential Swan conductor. Then

$$\delta_{\bar{y},1} + 1 = -(\delta_{\bar{y},2} + 1).$$

Property (c) holds, regardless whether the character χ corresponds to an inseparable action on the component $\bar{Y}_i$ or not. Proposition 3.18 is very useful in determining the stable reduction of a cover.

Example 3.19 We consider once more the component $\bar{X}_2$ from Example 3.3. We computed the corresponding differential Swan conductor ω_2 in 3.17.(b). Proposition 3.18.(c), together with the fact that ω_2 has two double zeros, immediately implies that blowing up these points yields a separable component, and moreover, that the conductor of the wild ramification above the intersection point is $2 = \mathrm{ord}(\omega_2) + 1$.

To conclude this, one needs to carefully observe the specialization of the branch point. More precisely, let $\mathcal{X}'$ be the curve obtained by contracting all components of $\bar{X}$ except for $\bar{X}_2$. Then none of the branch points specialize to the zeros of ω_2. This means that the zeros are not caused by the branch points, and hence really correspond to components of $\bar{Y}$ of positive genus.

Exercise 3.20 In this exercise we construct a class of A_4-covers in characteristic zero. Let $G = A_4$ and denote its unique subgroup of order 4 by V. We describe all A_4-covers of order $Y \to \mathbb{P}^1$ of the projective line such that the curve $X := Y/V$ has genus zero. We denote by σ_i the three elements of A_4 of order two, and choose an element τ of order three such that $\tau\sigma_i\tau^{-1} = \sigma_{i+1}$. We denote $X_i = Y/\langle\sigma_i\rangle$.

Let x be a coordinate on X. We may assume that ρ acts on X as $\rho(x) = \zeta_3 x$, where ζ_3 is a primitive 3rd root of unity. Since the branch points of $Y \to X$ form a $\langle\rho\rangle$-set, we may assume they are

$$\{x_{i,j} := \zeta_3^i c_j \mid i = 0, \ldots, 2, j = 1, \ldots, r\}.$$

Moreover, we may assume $c_1 = 1$.

(a) Show that the degree 2 cover $X_i \to X$ is branched exactly at the points $x_{\iota,j}$ with $\iota \neq i$.

(b) Show that the A_4-covers with $g(X) = 0$ bijectively correspond to the unordered tuples $(c_2^3, \ldots, c_r^3) \in (\mathbb{P}^1 \setminus \{0, 1, \infty\})^{r-1}/S_{r-1}$.

We continue with the notations of Exercise 3.20. We choose a complete discretely valued field k of mixed characteristic 2 such that $Y \to X$ may be defined over k. Define $f_i = \prod_{j=1}^r (x - \zeta_3^i c_j)$ and $f = f_0 f_1 f_2$. We have shown that the covers $X_i \to X$ are given by the Kummer equation

$$z_i^2 = \frac{f}{f_i}. \tag{18}$$

Note that the Galois group of $X_i \to X$ is the quotient of V by the subgroup generated by σ_i. There exists a unique character $\chi_i : V \to \mathbb{C}$ with kernel $\langle\sigma_i\rangle$.

Let $\mathcal{Y} \to \mathcal{X}$ be the stable model of $Y \to X$. Let $\mathcal{X}_0 = \mathbb{P}^1_{\mathfrak{o}}$ be the naive model of X which corresponds to the coordinate x, and let $\bar{Y}_0$ be a component of $\bar{Y}$ above $\bar{X}_0$. We compute the differential Swan conductors on $\bar{X}_0$ by using (18), where we denote by (ω_i, δ_i) the Swan conductor corresponding to the character χ_i (or, equivalently, the cover $X_i \to X$.) We find:

$$\omega_i = \left(\sum_{j=1}^r \sum_{\iota \neq i} \frac{1}{\bar{x} - \bar{\zeta}_3^\iota \bar{c}_j} \right) \mathrm{d}\bar{x}.$$

One easily checks that

$$\omega_0 + \omega_1 + \omega_2 = 0,$$

since we are in characteristic two. In other words, the differential forms ω_i form an $\mathbb{F}_2$-vector space, as in Lemma 2.28.(b).

Moreover, we see that the automorphism τ cyclically permutes the differential forms ω_i. In fact, we could have seen this even without explicit calculation, since τ permutes the branch points of the covers $X_i \to X$ which remain distinct when reduced to the component $\bar{X}_0$.

Exercise 3.21 In this exercise, we continue with the above situation, and assume $r = 2$. We may write $c = c_2$. Recall that $c \in \mathbb{P}^1_k \setminus \{0, 1, \infty\}$.

(a) Compute the divisor of the differential forms ω_i. Show that for $c^3 \not\equiv 0, 1, \infty \pmod 2$ the differential forms ω_i have two distinct zeros.

(b) Conclude that a necessary condition for having good reduction is that $c^3 \equiv 0, 1, \infty \pmod 2$.

(c) Suppose that $c \not\equiv 0, 1, \infty \pmod 2$. Show that the stably marked model of Y contains exactly three components of positive genus, which are elliptic curves.

4 Necessary conditions for lifting

In this section, we are given an algebraically closed field κ of characteristic $p > 0$, together with a G-Galois cover $\pi_\kappa : Y_\kappa \to X_\kappa$ defined over κ. We consider necessary conditions for lifting π_κ to characteristic zero (Definition 3.10).

4.1 The Bertin obstruction The first and most important necessary condition for lifting we discuss is the so-called Bertin obstruction. The Bertin Obstruction is a local condition on the Artin characters corresponding to the ramification points of the cover in positive characteristic (see Proposition 4.1 for the precise statement).

We suppose the cover $\pi_\kappa : Y_\kappa \to X_\kappa$ lifts to characteristic zero. We denote the corresponding model by $\pi' : \mathcal{Y}' \to \mathcal{X}'$, and suppose it may be defined over a discrete valuation ring $\mathfrak{o}_k$. Recall that this means that $\kappa = \bar{k}$ is the residue field of $\mathfrak{o}_k$, and that $\pi'_{\mathfrak{o}_k} \otimes_{\mathfrak{o}_k} \bar{k} \simeq \pi_\kappa$. Note that the ramification points of π specialize to ramification points of π_κ.

We denote the generic fiber of π' by $\pi_k : Y \to X$. Choose a ramification point $\bar{y} \in X_\kappa$. Let $\Delta := \Delta_{\bar{y}}$ be the set of ramification points of π which specialize to $\bar{y}$. Obviously the stabilizer $G_{\bar{y}}$ acts on the set Δ. We associate the permutation representation ρ_Δ of $G_{\bar{y}}$. We recall the definition. Let V be a $|G_{\bar{y}}|$-dimensional $\mathbb{C}$-vector space with basis $(e_i)_{i \in \Delta}$. Define

$$\rho_\Delta : G_{\bar{y}} \to \mathrm{GL}(V),$$

by sending $g \in G_{\bar{y}}$ to the linear transformation $e_i \mapsto e_{g(i)}$ induced by the action of G on Δ. We denote by χ_Δ the character of this representation. Denote by r_G the character of the regular representation. (Recall that that the regular representation is the permutation representation of G acting on itself.)

Proposition 4.1 (Bertin) *Suppose that the cover $\pi_\kappa : Y_\kappa \to X_\kappa$ lifts to characteristic zero. Let $\bar{y} \in Y_\kappa$ be a point, and let $a_{\bar{y}}$ be the corresponding Artin character. There exists a $G_{\bar{y}}$-set Δ such that*

$$\chi_\Delta = mr_{G_{\bar{y}}} - a_{\bar{y}}, \tag{19}$$

where m is the cardinality of Δ. Moreover, we require that the point stabilizers of Δ are cyclic.

Proof: This proposition is proved in Bertin ([3]). The set Δ in the statement of Proposition 4.1 is of course exactly the set defined above. The occurrence of the regular representation comes from the way the Artin character is defined for the trivial element. The condition that the point stabilizers are cyclic follows since this always holds for points stabilizers in characteristic zero. $\Box$

Definition 4.2 Let L/K be a Galois extension of local fields in positive characteristic, where K is the localization of the function field of transcendence degree 1 at a point. We say that the *Bertin obstruction vanishes for L/K* if there exists a set Δ such that (19) holds.

A slightly stronger version of the Bertin Obstruction is the so-called Katz–Gabber–Bertin (KGB-)Obstruction. For a discussion, we refer to [10].

Corollary 4.3 *Let $G \simeq \mathbb{Z}/p^n\mathbb{Z}$ be a cyclic group. The Bertin Obstruction vanishes for all local G-Galois extensions as in Definition 4.2 in characteristic p. In particular, the Bertin Obstruction vanishes for a tame actions.*

Proof: Let L/K be a local G-Galois extensions in characteristic p. We denote by $h_1, \ldots, h_n$ be the corresponding lower jumps. The upper jumps j_i are determined by

$$n_{i+1} := (j_{i+1} - j_i) = \frac{h_{i+1} - h_i}{p^i},$$
$$n_1 := j_1 = h_1.$$

It follows from the Hasse–Arf Theorem that the j_i are integers (Exercise 2.21).

We construct the set Δ as disjoint union of sets Δ_i, where the points of Δ_i have the unique subgroup of G of order p^i as stabilizer. Now choose Δ_i consisting of n_i orbits of G. This is possible since the n_i are integers. In particular, Δ_i has cardinality $n_i p^{n-i}$. One easily checks that with this choice for Δ the equality (19) is satisfied. Therefore the Bertin Obstruction vanishes for G. $\Box$

Proposition 4.1 can be used to show that many covers in positive characteristic do not lift to characteristic zero. We discuss a first case. Let p be a prime number and m an integer prime to p. Choose a character $\xi : \mathbb{Z}/m\mathbb{Z} \to \mathbb{F}_p^*$, and denote by m_1 the order of the kernel of ξ. We may write $m = m_1 m_2$ with $m_2 \mid (p-1)$. We denote $G = \mathbb{F}_p \rtimes_\xi \mathbb{Z}/m\mathbb{Z}$, and choose generators σ, τ of G such that

$$\sigma^p = \tau^m = e, \quad \tau\sigma\tau^{-1} = \sigma^a.$$

where $a = \xi(1)$.

Suppose given a G-Galois extension of local rings in characteristic p. We may choose parameters such that the extension is given by

$$\begin{aligned} y^p - y = x^{-h}, \quad x^m = z, \\ \sigma(x,y) = (x, y+1), \quad \tau(x,y) = (\zeta_m x, \zeta_m^{-h} y), \end{aligned} \tag{20}$$

where $\zeta_m \in \kappa$ is a primitive mth root of unity. Computing $\tau\rho\tau^{-1}$ and using the definition of a, one find that $a = \zeta_m^{-h}$. Since $a \in \mathbb{F}_p$, it follows that $m_1 \mid h$ and that $\gcd(h, m_2) = 1$.

In fact, (20) also defines a G-Galois cover of $\mathbb{P}^1$ which is the so-called *Katz–Gabber* cover corresponding to the given local cover ([25]). We denote this cover by $\bar{f}_1 : \bar{Y}_1 \to \bar{X}_1$. Let $\bar{Z}_1 \to \bar{X}_1$ be the subcover of degree m. Note that the genus of $\bar{Z}_1$ is zero.

The filtration of the higher ramification group of the unique point with $x = 0$ is

$$G = G_0 \supsetneq G_1 = \cdots = G_h = \langle \sigma \rangle \supsetneq G_{h+1} = \{0\}.$$

Therefore the unique nontrivial lower (resp. upper) jump is h (resp. $\sigma = h/m$). This implies that the Artin representation is

$$a(g) = \begin{cases} -(h+1) & \text{if } g = \sigma^i \text{ with } i \neq 0, \\ -1 & \text{if } g = \sigma^i \tau^j \text{ with } j \neq 0, \\ pm - 1 + h(p-1) & \text{if } g = e. \end{cases}$$

The following proposition is an explicit version of the Bertin Obstruction (Proposition 4.1) in this case. The cover $\bar{f}_1 : \bar{Y}_1 \to \bar{X}_1$ is totally branched above $x = 0$ and tamely branched at $x = \infty$. By the local-global principal (Theorem 1.5) lifting the local Galois cover at $x = 0$ is equivalent to lifting the Katz–Gabber cover. Moreover, the lift of the tame branch point to characteristic zero also also branched of order m. A special case of the proposition can be found in [33].

Proposition 4.4 *Suppose that the Katz–Gabber $\bar{f}_1 : \bar{Y}_1 \to \bar{X}_1$ lifts to characteristic zero.*

(a) The character ξ is either injective or its image is trivial.

(b) Suppose that χ is injective. The Bertin Obstruction vanishes if and only if $h \equiv -1 \pmod{p}$.

One may reformulate the proposition as follows: we have $m = m_1$ or $m = m_2$. In the first case the group G is cyclic. In the second case G has trivial center. We formulate the proof of the proposition in terms of Galois covers rather than representation theory, but it is possible to rewrite it in those terms as well ([9]).

Proof: Suppose we can lift $\bar{f}_1 : \bar{Y}_1 \to \bar{X}_1$ to a cover $f : Y \to X$. denote by Z the corresponding lift of $\bar{Z}_1$. As remarked before the statement of the proposition, the tame branch point lifts to a branch point of order m of X. Let $e_1, \ldots, e_s$ be the ramification indices of the other tame branch points. The Riemann–Hurwitz formula yields

$$-2 = 2g(Z) - 2 = -2m + m - 1 + \sum_{i=1}^{s} \frac{m}{e_i}(e_i - 1) = (s-1)m - \sum_{i=1}^{s} \frac{m}{e_i}.$$

Since $e_i \geq 2$ for all i, it follows that

$$\frac{s-2}{2}m + 2 \geq 0.$$

This implies that $s = 1$ and $e_1 = m$. We conclude that $Z \to X$ is branched at exactly two points.

We now consider the branch points of $Y \to X$.

Case a: We first assume that the two ramification points of $Z \to X$ are not branch points of $Y \to X$. Since the lift of τ to Z acts on the set of branch points of $Y \to Z$, it follows that the number of branch points of $Y \to Z$ is divisible by m. We write it as rm. Then

$$2g(Y) - 2 = -2p + rm(p-1).$$

Since we assume that Y is a smooth lift of $\bar{Y}_1$, it follows that $2g(Y) - 2 = 2g(\bar{Y}_1) - 2 = -2p + (h+1)(p-1)$. This implies that

$$h \equiv -1 \pmod{m}.$$

We have already shown that the order m_1 of the kernel of χ divides h. It follows that $m_1 = 1$.

Case b: Next we consider that the ramification point of $Z \to X$ which specializes to the point above $x = 0$ is also branched in $Y \to X$. (Note that these are the only two possibilities, as $Y \to X$ is branched of order exactly m at the branch point specializing to $x = \infty$.) Then the number of branch points of $Y \to Z$ is congruent to $1 \pmod{m}$. We write the number as $1 + rm$. Comparing the genus of Y to that of $\bar{Y}_1$ as before, we find that

$$h \equiv 0 \pmod{m}.$$

We have already shown that $\gcd(h, m_2) = 1$, therefore we conclude that $m_2 = 1$ in this case. (Alternatively, we may remark that in this case the lifted cover has a ramification point with ramification index pm. Since we are in characteristic zero, it follows that the inertia group of this ramification point is cyclic.)

We have shown that either $m_1 = 1$ or $m_2 = 1$. Part (a) follows.

Suppose now that $m_1 = 1$, i.e. ξ is injective. We have already show in this case that $h \equiv 1 \pmod{m}$ is a neccessary condition for liftability. Let Δ be the set of ramification points of π_k. Then

$$a_{L/K} = mr_G - \chi_\Delta.$$

Similarly, one checks that such a set Δ does not exist if $h \not\equiv 1 \pmod{m}$. $\square$

Example 4.5 Proposition 4.4 gives a second "local" proof that the cover described in Exercise 1.11 does not lift to characteristic zero. Namely, the cover is totally branched above ∞. The corresponding character $\xi : \mathbb{F}_{p^2}^\times \to \mathbb{F}_p^\times$ describing the extension of the stabilizer of a ramification point is neither injective nor trivial.

The following exercise shows that the Bertin obstruction vanishes for A_4. A more involved proof can be found in [10], Lemma 17.4.

Exercise 4.6 Let L/K be a local Galois extension with Galois group $G = A_4$ in situation A. Denote by h the unique nonzero lower jump in the filtration of higher ramification groups (Exercise 2.13). Recall that we showed in that exercise that $\gcd(h, 6) = 1$. Show that the Bertin Obstruction vanishes for all local A_4-Galois extensions in characteristic 2. (Tip: it suffices to construct a set Δ such that χ_Δ equals the character χ from Exercise 2.22. Alternatively, one may argue as in the proof of Proposition 4.4.)

Exercise 4.7 This exercise is a continuation of Exercises 2.23 and 2.24. Let κ be an algebraically closed field of characteristic 2, and consider $K = \kappa((z))$. Suppose given a $G := Q_8$-Galois extension L/K with lower jumps $h_0 = 1$ and $h_1 = h = 1 + 2n$ (compare to Exercise 2.23).

(a) Show that the Bertin Obstruction vanishes if and only if $h \equiv 1 \pmod 4$ either by explicitly computing the Artin character or by the geometric method of Proposition 4.4.

(b) Conclude from Exercise 2.24 that there exist a Q_8-cover for which the obstruction does not vanish.

(c) Assume $h = 3$. A more elementary proof of the fact that the Q_8-cover from Exercise 2.24 does not lift may be obtained by remarking that $g(C) = 1$ in this case, and using the list of possible automorphism groups of elliptic curves in characteristic zero.

Example 4.8 The Q_8-cover as in Exercise 4.7 with $h = 5$ also does not lift to characteristic zero. This follows from Proposition 3.16. Namely, there is a unique Q_8-cover in characteristic zero which could possibly be a lift, which means that it has exactly the right amount of ramification points of the right order. In Section 3.3 we have shown that this cover has bad reduction to characteristic $p > 0$. It follows from Exercise 4.7 that this does not follow from the Bertin Obstruction.

4.2 The Hurwitz-tree obstruction Let $\bar{\pi}_1 : \bar{Y}_1 \to \bar{X}_1$ be a G-Galois cover of smooth projective curves over an algebraically closed field κ of characteristic $p > 0$. We assume that $g(\bar{Y}_1) > 0$ and that $\bar{\pi}_1$ lifts to a cover $\pi_k : Y \to X$ over a complete discretely valued field k of characteristic zero. Denote by $\pi : \mathcal{Y} \to \mathcal{X}$ the stably marked model. Since we assume that $g(\bar{Y}_1) > 0$, the curve $\bar{Y}_1$ is an irreducible component of special fiber $\bar{Y}$ of $\mathcal{Y}$. Since $g(\bar{Y}_1) = g(Y)$ by assumption, all other components of $\bar{Y}$ have genus zero.

One may define a combinatorial structure describing the irreducible components of $\bar{Y}$, the specialization of the ramification points, the inertia and decomposition groups of the irreducible components. Moreover, we have the differential Swan conductors corresponding to all inseparable subquotients of $\bar{Y} \to \bar{X}$ and the ramification filtration of all ramification points of the separable subquotients. A last ingredient is the thickness of the singularities. These data satisfy several compatibility condition, the most important ones are those from Proposition 3.18. This combinatorial structure is called a *Hurwitz tree*. The existence of such a tree yields a strong neccessary condition for a local action to be liftable to characteristic zero. We call this obstruction the *Hurwitz-tree obstruction*.

This combinatorial structure has only been completely described in the case that the Galois group G is either cyclic of order p ([19]) or the dihedral group of order $2p$ ([5]). In the paper [8] Brewis and Wewers give a partial definition of a Hurwitz tree for arbitrary groups, only using the depth but not the differential form which is part of the differential Swan conductor.

Already in the case $G = \mathbb{Z}/p\mathbb{Z}$ describing the full combinatorial structure of a Hurwitz tree is rather complicated. In these notes we discuss some illustrative examples rather than the full definition. In the next section, we discuss some elements of constructing Hurwitz trees in basic cases.

5 Deformation data and Hurwitz trees

In this section, we discuss the construction of Hurwitz trees. We start by recalling the geometric context. Suppose given a G-Galois cover $Y \to X$ over k. We think of the extension L/K as (a subquotient of) the completion of the function fields of Y and X at the valuation corresponding to an irreducible component of the special fiber $\bar{Y}$ of the stably marked model of Y. The differential Swan conductors associated with such an extension are combinatorial data in characteristic $p > 0$. The goal of this section is to discuss some existence and nonexistence results for these combinatorial structures in positive characteristic. These can be used in explicit cases to prove existence and nonexistence of lifts.

We start by considering the case of a fiercely ramified extension L/K of degree p which has residual dimension one, compare to Section 2.5. We have seen that in the case of degree p there are two different cases: the multiplicative and the additive case. Some of the existence results we prove in the degree-p case can be extended to what we call the *irreducible vector space case*. In this case, we have a vector space of differential forms, together with the action of an automorphism of order prime to p such that the vector space in irreducible with respect to this action.

5.1 Construction of deformation data Let L/K be a fiercely ramified extension which is Galois of degree p and has residual dimension one. For simplicity, we assume that K is the completion of the function field of the projective line X over k at a valuation corresponding to an irreducible component $\bar{X}_0$ of some semistable model $\mathcal{X}$ of X.

After replacing the field of constants k of K by a finite extension if neccessary, we may assume that k contains a primitive pth root of unity ζ_p, which we fix. We use the notation of Section 2.5. In particular, we choose a generator $y \in L$ with $y^p = x \in K$. Moreover, we choose a generator σ of $P := \mathrm{Gal}(L/K) \simeq \mathbb{Z}/p\mathbb{Z}$ such that $\sigma^*(y) = \zeta_p y$, and denote by $\chi : P \to \mathbb{C}^*$ the character which sends σ to ζ_p.

Recall that replacing χ by another nontrivial character of G multiplies the differential Swan conductor $\omega(\chi)$ by a nonzero constant (Section 2.5). To describe the differential Swan conductor in this case, it suffices therefore to describe a single differential form $\omega = \omega(\chi)$.

We need to distinguish two cases. In the multiplicative case, the differential form is logarithmic, i.e. of the form

$$\omega = \frac{\mathrm{d}g}{g}$$

for some $g \in \bar{K}$ which is not a pth-power. Moreover, the depth satisfies $\delta(\chi) = p/(p-1)$. In the additive case, the differential form is exact, i.e. of the form

$$\omega = \mathrm{d}g,$$

for some $g \in \bar{K}$ which is not a pth-power. Moreover, $0 < \delta < p/(p-1)$.

For the application the the lifting problem we discuss in Section 6, we need a slightly more general situation. We suppose that $\tau \in \mathrm{Aut}_k(K)$ is an automorphism of order m with $\gcd(p, m) = 1$ such that $L/K^{\langle\tau\rangle}$ is again Galois. Since p and m are relatively prime, the Galois group, G, is an extension of $\mathbb{Z}/m\mathbb{Z}$ by P. Denote by $\xi : \mathbb{Z}/m\mathbb{Z} \to \mathbb{F}_p^*$ the corresponding character, describing the action of τ on P as explained in Section 4.1 and put $G = P \rtimes_\xi \mathbb{Z}/m\mathbb{Z}$.

We may choose a coordinate $\bar{x}$ of the projective line $\bar{X}_0 \simeq \mathbb{P}^1_{\bar{k}}$ such that $\tau^*\bar{x} = \zeta_m \bar{x}$. This corresponds to choosing an isomorphism $\bar{K} = \bar{k}(\bar{x})$. We may assume that $\bar{x}$ is a *separating parameter* on $\bar{X}_0$. This implies that every meromorphic differential form ω on $\bar{X}_0$ may be written as $g\,\mathrm{d}\bar{x}$ for some $g \in \bar{K}$.

Proposition 4.4 implies that ξ is either injective of trivial. In this section, we assume that ξ is injective. In particular, $m \mid (p-1)$. It easily follows that ω is an eigenvector of the action by τ. More precisely,

$$\tau^*\omega = \zeta_m^c \omega,$$

where $c \in \mathbb{Z}$ is prime to p.

The following definition describes the structure of the differential Swan conductor for $G = P \rtimes_\xi \mathbb{Z}/m/\mathbb{Z}$ concretely.

Definition 5.1 A *deformation datum* for $G = P \rtimes_\xi \mathbb{Z}/m\mathbb{Z}$ on $\bar{X}_0$ is a differential form $\omega = g(\bar{x})\, \mathrm{d}\bar{x} \in \Omega^1_{\bar{X}_0/k}$ with the following properties:

(a) ω is an eigenvector for τ, i.e.

$$\tau^*\omega = \zeta_m^c \omega,$$

for some $c \in \mathbb{Z}$, and

(b) ω is either logarithmic, i.e. $\omega = \mathrm{d}g/g$ for some $g \in \bar{K}^\times$ or exact, i.e. $\omega = \mathrm{d}g$ for some $g \in \bar{K}^\times$ which is not a pth-power.

The deformation datum is called *primitive* if c is relatively prime to m.

A logarithmic differential form has at most simple poles. In the geometric context, these correspond to the specialization to $\bar{X}_0$ of the branch points of the P-Galois cover $Y \to X$ corresponding to L/K. On the other hand, an exact differential form does not have simple poles. Therefore whether we need to put a multiplicative or an additive deformation datum on a given component of $\bar{X}$ depends on whether any of the branch points of the degree-p cover in characteristic zero specialize to the component.

In both cases the differential form may have zeros of arbitrary large order. These indicate the direction in which we may find components of the stable reduction of Y of positive genus, once one knows the position of the specialization of the wild branch points (see Section 3.4 for a discussion of this phenomenon.)

We now discuss a method for showing that a given meromorphic differential form $\omega = g\, \mathrm{d}\bar{x}$ on a curve $\bar{X}_0$ is logarithmic. We first remark that every function $g \in \bar{K}$ may be written uniquely as

$$g = g_0^p + g_1^p\bar{x} + \cdots g_{p-1}^p \bar{x}^{p-1}, \text{ with } g_i \in \bar{K},$$

since $\bar{x}$ is a separating parameter. We define the *Cartier operator* $\mathcal{C} : \Omega^1_{\bar{K}/\bar{k}} \to \Omega^1_{\bar{K}/\bar{k}}$ as follows

$$\mathcal{C}(g\, \mathrm{d}\bar{x}) = g_{p-1}\mathrm{d}\bar{x}.$$

The following properties of the Cartier operator are left as an exercise to the reader ([36]). The properties (a) and (b) state that the operator $\mathcal{C}$ is $(1/p)$-semilinear.

Lemma 5.2 *For $\omega, \omega_i \in \Omega^1_{\bar{K}/\bar{k}}$ and $g \in \bar{K}$ we have that*

(a) $\mathcal{C}(\omega_1 + \omega_2) = \mathcal{C}\omega_1 + \mathcal{C}\omega_2$,

(b) $\mathcal{C}(g^p\omega) = g\mathcal{C}\omega$,

(c) $\mathcal{C}\omega = 0$ *if and only of* $\omega = \mathrm{d}g$ *for some* $g \in \bar{K}$,

(d) $\mathcal{C}\omega = \omega$ *if and only of* $\omega = \mathrm{d}g/g$ *for some* $g \in \bar{K}$.

For the application to the lifting problem, it is particularly important to construct deformation data with a single zero (compare to the examples in Section 3.4).

Definition 5.3 A deformation datum is called *good* if it has a unique zero, i.e. there exists a unique point $P \in \bar{X}_0$ with $\mathrm{ord}_P\,\omega > 0$. If this is the case, the integer

$$h := \mathrm{ord}_P\,\omega + 1$$

is called the *conductor* of ω.

In the rest of this section, we recall some elementary results on the construction of multiplicative deformation data from [6].

Proposition 5.4 *Fix positive integers m and h and assume that there exists a good multiplicative deformation datum ω of conductor h. Then the following holds.*

(a) The conductor h is prime to p.

(b) ω is primitive if and only if h is prime to m. If this is the case, then

$$m|(p-1)$$

(and hence $\zeta_m \in \mathbb{F}_p^\times$) and

$$h \equiv -1 \pmod{m}.$$

(c) If ω is not primitive, then $m|h$ and $\tau^*\omega = \omega$ (i.e. $c \equiv 0 \pmod{m}$ in Definition 5.1 (a)).

This is a special case of [5], Lemma 3.3.(v). A special case of (ii) can be found in [33, §I.1]. For convenience, we recall the proof.

Proof: Let $P \in \bar{X}_0$ be the unique zero of ω. Choose a local coordinate $w \in k(\bar{x})$ at P and a function $g \in k(\bar{x})$ such that $\omega = \mathrm{d}g/g$. By multiplying g with a pth power, if necessary, we can achieve that g has the value 1 in the point P. Writing g as a power series in w and computing $\omega = \mathrm{d}g/g$, one sees that $h = \mathrm{ord}_P\,\omega + 1 \not\equiv 0 \pmod{p}$. This proves (a).

The statements (b) and (c) are trivial for $m = 1$. We may therefore assume $m > 1$. Since the automorphism $\tau : \bar{X}_0 \xrightarrow{\sim} \bar{X}_0$ has order m, it has exactly two fixed points, namely $\bar{x} = 0$ and $\bar{x} = \infty$. The unique zero P of ω is clearly fixed by τ. Replacing the coordinate $\bar{x}$ by $\bar{x}^{-1}$, if necessary, we may assume that P is the point $\bar{x} = \infty$ and that $w = \bar{x}^{-1}$. Now $\tau^* w = \zeta_m^{-1} w$, and Condition (a) of Definition 5.1 implies

$$h = \mathrm{ord}_x(\omega) + 1 \equiv -c \pmod{m}. \tag{21}$$

In particular, ω is primitive if and only if h is prime to m.

The same argument used to prove (21) shows that

$$\mathrm{ord}_{\bar{x}=0}(\omega) + 1 \equiv c \pmod{m}. \tag{22}$$

But $\mathrm{ord}_{\bar{x}=0}(\omega)$ is either equal to -1 or to 0, since ω is logarithmic. In the first case, c and h are divisible by m; this corresponds to (c) of the proposition. In the second case, $c \equiv 1 \pmod{m}$, $h \equiv -1 \pmod{m}$ and ω is primitive. This corresponds to (b) of the proposition.

It remains to prove that $m|(p-1)$ in the second case. Let $\bar{x} = \bar{x}_1$ be a pole of ω and set $a_1 := \mathrm{Res}_{\bar{x}=\bar{x}_1}(\omega)$. Set $\bar{x}_2 := \sigma(\bar{x}_1) = \zeta_m \bar{x}_1$. Condition (a) of Definition 5.1, together with the congruence $c \equiv 1 \pmod{m}$, implies that

$$a_2 := \mathrm{Res}_{\bar{x}=\bar{x}_2}(\omega) = \zeta_m^{-1} a_1.$$

Since ω is logarithmic, the residues a_1, a_2 actually lie in $\mathbb{F}_p^\times \subset k^\times$ (see Lemma 5.6 below). Therefore $\zeta_m \in \mathbb{F}_p^\times$, which is equivalent to $m|(p-1)$. This finishes the proof of the proposition. □

The next result says that the necessary conditions given by Proposition 5.4 are also sufficient, at least if $h < p$. The case $h > p$ is discussed in Section 5.3. To keep the statement simple, we first deal with the case $m|h$ (the non-primitive case). Here one can immediately write down a good multiplicative deformation datum of conductor h (see also [19], §3.5):

$$\omega := \frac{h\,\mathrm{d}\bar{x}}{\bar{x}^{h+1} - \bar{x}} = \frac{\mathrm{d}g}{g}, \qquad \text{with } g := (\bar{x}^h - 1)/\bar{x}^h. \tag{23}$$

It therefore suffices to consider the primitive case (Part (b) of Proposition 5.4).

Proposition 5.5 *Assume $m|(p-1)$ and let h be a positive integer with*

$$h < p \qquad \text{and} \qquad h \equiv -1 \pmod{m}.$$

Then there exists a good multiplicative deformation datum with conductor h.

The proof of the proposition is based on the following well-known lemma.

Lemma 5.6 *Let $\omega = g(\bar{x})\,\mathrm{d}\bar{x} \in \Omega^1_{\bar{K}/\bar{k}}$ be a meromorphic differential form on $\bar{X}_0 = \mathbb{P}^1_{\bar{k}}$. Then ω is logarithmic if and only*

$$\mathrm{ord}_P\,\omega \geq -1 \qquad \text{and} \qquad \mathrm{Res}_P(\omega) \in \mathbb{F}_p,$$

for all $P \in \bar{X}_0$.

Proof: Let $\bar{x}_1, \ldots, \bar{x}_r$ be the set of poles of ω and set $a_i := \mathrm{Res}_{\bar{x}_i}(\omega)$. After a change of coordinates, we may assume that $\bar{x}_i \neq \infty$; then the point $\bar{x}_i \in \bar{X}_0$ is defined by $\bar{x} = \bar{x}_i$, for some $\bar{x}_i \in k$.

Now suppose that ω has at most simple poles and that $a_i \in \mathbb{F}_p^\times$, for all i. Choose a lift $A_i \in \mathbb{Z}$ of a_i. Then

$$\omega = \sum_{i=1}^r \frac{a_i \, \mathrm{d}\bar{x}}{\bar{x} - \bar{x}_i} = \frac{\mathrm{d}g}{g},$$

with

$$g := \prod_{i=1}^r (\bar{x} - \bar{x}_i)^{A_i}.$$

This shows one direction of the claimed equivalence. The other direction is obvious. □

Proof of Proposition 5.5: We fix integers m and h, with $m \geq 1$, $m|(p-1)$, $0 < h < p$ and $h \equiv -1 \pmod{m}$. If $m = 1$ then $m|h$, therefore the existence of the deformation datum follows already from (23). Assume therefore that $m > 1$, and write $h = mr - 1$. The condition $h < p$ ensures that there exist elements $\bar{z}_1, \ldots, \bar{z}_r \in \mathbb{F}_p^\times$ such that

$$\bar{z}_{i,j} := \zeta_m^j \bar{z}_i \in \mathbb{F}_p^\times, \quad i = 1, \ldots, r, \; j = 0, \ldots, m-1,$$

are pairwise distinct. (Here we use that we excluded the case $m = 1$ and $h = p - 1$). Define

$$\omega := \frac{\mathrm{d}\bar{z}}{\prod_i (\bar{z}^m - \bar{z}_i^m)}.$$

We claim that ω is a good multiplicative deformation datum of conductor h.

By construction, we have $\sigma^* \omega = \zeta_m \omega$, where σ is the automorphism of X with $\sigma^* \bar{z} = \zeta_m \bar{z}$. Furthermore, ω has exactly mr simple poles and no zeroes on $\mathbb{A}_k^1 \subset X$. It follows that ω has a zero of order $h = mr - 1$ at ∞. Finally, the residues of ω all lie in $\mathbb{F}_p$. Therefore, ω is logarithmic by Lemma 5.6. This proves the claim and finishes the proof of Proposition 5.5. □

Remark 5.7 Note that the neccessary conditions for the existence for a good deformation datum with conductor h in Proposition 5.5 are exactly the same as the neccessary conditions for liftability of a $\mathbb{F}_p \rtimes_\xi \mathbb{Z}/m\mathbb{Z}$-cover with lower jump h in Proposition 4.4. In Section **??** we use this to show that all $\mathbb{F}_p \rtimes_\xi \mathbb{Z}/m\mathbb{Z}$-covers of the projective line in characteristic p branched at one point with lower jump h lift to characteristic zero.

5.2 Admissible covers We denote by $\bar{w} = 1/\bar{x}$ a local parameter of $\infty \in \bar{X}_0$. Note that τ acts on the complete local ring of the intersection point of $\bar{X}_0$ and $\bar{X}_0$ as

$$k[[\bar{x}_1, \bar{w}]]/(\bar{x}_1 \bar{w}), \qquad \tau(\bar{x}_1, w) \mapsto (\zeta_m \bar{x}_1, \zeta_m^{-1} w).$$

Such an action is called *admissible*. Admissibility of the tame action allows to lift the tame action to

$$\mathfrak{o}_k[[x_1, w]]/(x_1 w - \pi),$$

for some element π is the maximal ideal of $\mathfrak{o}_k$. We refer to [17] or [41] for a detailed discussion, and to [42], §2.1 for a short summary.

5.3 Hurwitz trees for $G = \mathbb{F}_p \rtimes_\xi \mathbb{Z}/m\mathbb{Z}$ In this section, we suppose that $\xi : \mathbb{Z}/m\mathbb{Z} \to \mathbb{F}_p^\times$ is an injective character, and write $G = \mathbb{F}_p \rtimes_\xi \mathbb{Z}/m\mathbb{Z}$. We choose generators σ, τ of G as in Section 4.1. Let h be an integer relatively prime to p with $h \equiv -1 \pmod{m}$. Suppose given a G-Galois cover $\bar{f}' : \bar{Y}' \to \bar{Z}', (\bar{x}, \bar{y}) \mapsto \bar{z}$ defined over an algebraically closed field κ of characteristic $p > 0$ given by the equation

$$\begin{aligned} \bar{y}^p - \bar{y} &= \bar{x}^{-h}, & \bar{x}^h &= \bar{z}, \\ \sigma(\bar{x}, \bar{y}) &= (\bar{x}, \bar{y} + 1), & \tau(\bar{x}, \bar{y}) &= (\zeta_m \bar{x}, \zeta_m^{-h} \bar{y}), \end{aligned} \tag{24}$$

Recall from Section 4.1 that ζ_m is a primitive mth root of unity in $\mathbb{F}_p^*$ and $a := \zeta_m^{-h} = \xi(1)$.

We denote by $\bar{X}' \to \bar{Z}'$ the subcover of $\bar{f}'$ with Galois group $\mathbb{Z}/m\mathbb{Z}$.

The goal of this section is to construct a Hurwitz tree for the cover $\bar{f}'$. Note that the cover $\bar{f}'$ is totally branched at ∞, and that h is the unique nontrivial lower jump, since $\gcd(h, p) = 1$. In Proposition 4.4.(b) we have shown that the Bertin Obstruction vanishes if and only if $h \equiv -1 \pmod{m}$. Constructing a Hurwitz tree implies that the Hurwitz tree obstruction (Section 4.2) also vanishes.

Recall from the proof of Proposition 4.4 that if $\bar{f}'$ lifts to a G-Galois cover $f : Y \to Z$ between smooth projective curves in characteristic zero, then f is branched at 2 points with ramification index m and $(h+1)/m$ points with ramification index p. For a lift f as above, we denote the corresponding $\mathbb{Z}/m\mathbb{Z}$-Galois subcover by $X \to Z$. Note that the cover $Y \to X$ is branched at $h+1$ which form a $\langle\tau\rangle$-set. To avoid a case distinction, we assume in this section that $m > 2$. The case $m = 2$ is similar and can be found in [5].

The first case: $h < p$. In this case one can show that the Hurwitz tree has to consist of two components (INCLUDE RESULT+PROOF) ([14]): the component $\bar{X}'$ defined above, and an additional component $\bar{X}_0$ to which the branch points of the lifted cover $Y \to X$ (if it exists) specialize. We identify the point ∞ on $\bar{X}'$ with the point with $\bar{x} = 0$ on $\bar{X}_0$.

Therefore to obtain a Hurwitz tree, we need to construct a good, multiplicative deformation datum with conductor h. The existence of such a deformation datum is proved in Proposition 5.5. Note that the compatibility conditions of Proposition 3.18 are satisfied.

The second case: $h > p$. Recall that $m \mid (p-1)$. We write $p = 1 + mn$. Since $h \equiv -1 \pmod{m}$, we may write

$$h + 1 = m(\alpha + p\beta).$$

Using $h > p$ and $m > 2$, we may assume that

$$(p + m - 1)/m = 1 + n \leq \alpha \leq p + n.$$

Since we have assumed that $m > 2$, it follows moreover that $p + n < 2p - n$.

Unfortunately, we need a further case distinction.

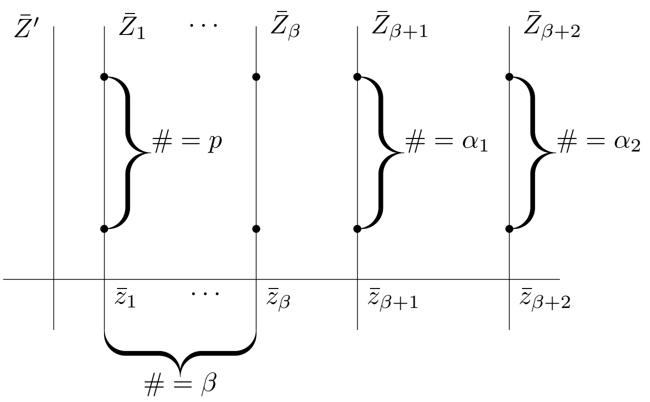
Case 2A: Assume that $p - n + 1 < \alpha < 2p - n$. In this case, we may write $\alpha = \alpha_1 + \alpha_2$ with $1 < \alpha_1 \leq \alpha_2 < p$.

To construct a Hurwitz tree, we first construct a component $\bar{X}_0$ with coordinate $\bar{x}_0$. We identify the point $\bar{x} = \infty$ on $\bar{X}'$ with the point with $\bar{x} = 0$ on $\bar{X}_0$, and let τ act on $\bar{X}_0$ via $\tau\bar{x}_0 = \zeta_m\bar{x}_0$. We may consider $\bar{z}_0 := \bar{x}_0^m$ as coordinate on the curve $\bar{Z}_0 := \bar{X}_0/(\mathbb{Z}/m\mathbb{Z})$.

We first describe the semistable curve $\bar{X}$. We choose $\beta + 2$ distinct points on $\bar{Z}_0 \setminus \{0, \infty\}$. Without loss of generality, we may assume these are

$$\bar{z}_1, \ldots, \bar{z}_\beta, \bar{z}_{\beta+1} := 1, \bar{z}_{\beta+2} =: \lambda.$$

Consider the inverse image $\{\bar{x}_{i,j}\}$ of the set $\{\bar{z}_1, \ldots, \bar{z}_{\beta+2}\}$ on $\bar{X}_0$, where $1 \leq i \leq \beta + 2$ and $0 \leq j \leq m - 1$. (In other words, the points $\{\bar{x}_{i,j}\}$ with i fixed are the inverse image of $\bar{z}_i$.) We connect the component $\bar{X}_0$ with $(\beta + 2)m$ further components, intersecting in each of the chosen points. We label these components $\bar{X}_{i,j}$, with $1 \leq i \leq \beta + 2$ and $0 \leq j \leq m - 1$. The group $\mathbb{Z}/m\mathbb{Z}$ acts on the components $\bar{X}_{i,j}$ by cyclicly permuting those with i fixed. Therefore the semistable curve $\bar{Z} := \bar{X}/(\mathbb{Z}/m\mathbb{Z})$ consists of components $\bar{Z}', \bar{Z}_0, \bar{Z}_1, \ldots, \bar{Z}_{\beta+2}$. The following picture illustrates the components of $\bar{Z}$, together with the position of the specialization of the wild branch points of the cover $f : Y \to Z$.



We construct deformation data on $\bar{X}_0$ and $\bar{X}_{i,j}$. The deformation datum we construct on $\bar{X}_0$ is additive with the points $\{\bar{x}_{i,j}\}$ as poles, where the points above $\bar{z}_i$ have a pole of order p, the points above 1 have a pole of order α_1, and the points above λ a pole of order α_2. Moreover, we require that ω has a zero of order $h+1 = m(\alpha + p\beta)$ in $\bar{x}_1 = \infty$. The existence of this differential form is shown in Lemma 5.8 below. On the components $\bar{X}_{i,0}$, we construct a good, multiplicative deformation datum for the group $\mathbb{Z}/p\mathbb{Z}$ of conductor

$$h_{i,0} = \begin{cases} p-2 & \text{if } 1 \leq i \leq \beta, \\ \alpha_1 - 2 & \text{if } i = \beta+1, \\ \alpha_2 - 2 & \text{if } i = \beta+2. \end{cases}$$

The existence of such a deformation datum is guaranteed by Proposition 5.5. The action of $\mathbb{Z}/m\mathbb{Z}$ on $\bar{X}_0$ cyclicly permutes the components $\bar{X}_{i,j}$ for i fixed. This defines a deformation datum on all components of $\bar{X}$. (Note that the $h_{i,j}$ are prescribed by Proposition 3.18.) Moreover, the conditions of Proposition 3.18 are satisfied.

It remains to construct the deformation datum on $\bar{X}_0$. Define

$$\omega = \frac{\mathrm{d}\bar{x}_0}{(\bar{x}_0^m - 1)^{\alpha_1}(\bar{x}_0^m - \lambda)^{\alpha_2}\prod_{j=1}^{\beta}(\bar{x}_0^m - \bar{z}_j)^p}.$$

Note that this differential form has the required pole and zero orders. Moreover, we have that

$$\tau^*\omega = \zeta_m \omega.$$

Therefore ω defines a good deformation datum if and only if ω is either exact or logarithmic (Definition 5.3). Since ω does not have simple poles, it is certainly not logarithmic. We claim that we may choose λ such that ω is exact. For this we need to introduce some notation.

Define

$$G := (\bar{z}_0 - 1)^{p-\alpha_1}(\bar{z}_0 - \lambda)^{p-\alpha_2},$$

$$Q := (\bar{x}_0^m - 1)(\bar{x}_0^m - \lambda)\prod_{j=1}^{\beta}(\bar{x}_0^m - \bar{z}_j).$$

Note that we may write

$$\omega = \frac{G(\bar{x}_0^m)\,\mathrm{d}\bar{x}_0}{Q^p}.$$

Write $G(\bar{z}_1) = \sum_{i=0}^{2p-\alpha} g_i \bar{z}_0^i$. We may regard the g_i as elements of $\bar{k}[\lambda]$.

Lemma 5.8 *(a) The differential form ω is exact if and only if $g_n(\lambda) = 0$.*

(b) We may choose $\lambda \in \mathbb{P}^1_{\bar{k}} \setminus \{0, 1, \infty\}$ such that $g_n(\lambda) = 0$.

Proof: To prove (a), we use Lemma 5.2.(c). To compute $\mathcal{C}\omega$, we determine the terms of $G(\bar{x}_0^m)$ of the form $(*)\bar{x}_0^{sp-1}$. The assumption on α implies that $\deg_{\bar{z}_1}(G) = 2p - \alpha < p + n - 1$. Therefore the only such possibly nonvanishing term is $g_n\bar{x}_0^{mn} = g_n\bar{x}_0^{p-1}$. It follows from the properties of the Cartier operator (Lemma 5.2) that

$$\mathcal{C}\omega = \frac{g_n^{1/p}\,\mathrm{d}\bar{x}_0}{Q}.$$

Therefore ω is exact if and only if $g_n(\lambda) = 0$.

We consider g_n as polynomial in λ. One easily computes that

$$\begin{aligned} g_n &= (-1)^n \sum_{i+j=n} \binom{p-\alpha_1}{i}\binom{p-\alpha_2}{j}\lambda^{p-\alpha_2-j} \\ &= (-1)^n \sum_{i+j=2p-\alpha-n} \binom{p-\alpha_1}{i}\binom{p-\alpha_2}{j}\lambda^{j}. \end{aligned}$$

To prove (b), we need to show that g_n has a zero $\lambda \neq 0, 1$. The above computation then shows that with this choice of λ, the differential ω is exact. This suffices, since we may choose the other poles $\bar{x}_0^m = \bar{z}_i$ of ω arbitrarily.

We apply Lemma 5.9 below with $a_1 = p-1-n$, $a_2 = p-\alpha_1$, $a_3 = p-\alpha_2$, and $a_4 = -p+n-1+\alpha$. The assumptions we made on the α_i and α imply that $1 \leq a_i < p-1$. Therefore the lemma implies that the order of g_n in $\bar{z}_0 = 1$ is $\max(0, p+1-\alpha)$. A case distinction, together with the assumption 0n α shows that the polynomial g_n has a zero λ different from $0, 1$. □

The following lemma is a well-known lemma on solutions of hypergeometric differential equations in positive characteristic. Note that the lemma proves more precise statements on the zeros of the polynomial g_n then we actually need in the proof of Lemma 5.8.

Lemma 5.9 *Let $1 \leq a_1, a_2, a_3, a_4 < p-1$ be integers with $a_1 + a_2 + a_3 + a_4 = 2(p-1)$. Put $N = (p-1) - a_4$.*

(a) Then the polynomial

$$\Phi := (-1)^N \sum_{i+j=N} \binom{a_2}{i}\binom{a_3}{j}\lambda^j \in \mathbb{F}_p[\lambda]$$

is a solution of the hypergeometric differential equation

$$\Phi'' + [(A+B+1)\lambda - C]\Phi' + AB\Phi = 0, \tag{25}$$

where $A = -N, B = -a_3, C = a_2 + p - N + 1 = a_2 + a_4 - 2(p-1) = -(a_1 + a_3)$.

(b) We have

$$\begin{aligned} \mathrm{ord}_0(\Phi) &= \max(0, a_1 + a_3 - (p-1)), \\ \mathrm{ord}_1(\Phi) &= \max(0, a_2 + a_3 - (p-1)), \\ \deg(\Phi) &= \min(p-1-a_4, a_3). \end{aligned}$$

(c) All other zeros of Φ have multiplicity 1.

Proof: Write $\Phi = \sum_i c_i\lambda^i$. One computes that the c_i satisfy the recursion

$$\begin{aligned}\frac{c_{j+1}}{c_j} &= \frac{(N-j)(a_3-j)}{(a_2-N+j+1)(j+1)} \\ &\equiv \frac{(-N+j)(-a_3+j)}{(a_2-p+1-N+j)(j+1)} \pmod{p}.\end{aligned}$$

To show that Φ satisfies the hypergeometric differential equation from statement (a), we need to choose A, B, C such that

$$\frac{a_{j+1}}{a_j} = \frac{(A+j)(B+j)}{(C+j)(1+j)}. \tag{26}$$

Choosing A, B, C as in the statement of the lemma, fulfills this requirement. This proves (a).

Now consider an arbitrary solution $\Psi := \sum_j c_j\lambda^j \in \bar{\mathbb{F}}_p[\lambda]$ of (26). Then Ψ satisfies the recursion (26). Assume that $c_j \neq 0$ and $c_{j-1} = 0$. The recursion (26) shows that

$$j \equiv -C+1 \pmod{p}, \qquad \text{or} \qquad j \equiv 0 \pmod{p}.$$

Similarly, it follows that if $c_j = 0$ and $c_{j+1} \neq 0$ then

$$j \equiv -A \pmod{p}, \qquad \text{or } j \equiv -B \pmod{p}.$$

For any integer α, we write $[\alpha]$ for the unique integer congruent to $\alpha \pmod{p}$ such that $0 \leq [\alpha] < p$. For our choice of A, B, C, we have that

$$\begin{aligned}&[-A] = p-1-a_4, \quad [-B] = a_3, \\ &[1-C] = \begin{cases} a_1+a_3+1 & \text{if } a_1+a_3+1 \leq p-1, \\ a_1+a_3-(p-1) & \text{otherwise.}\end{cases}.\end{aligned}$$

This implies that the differential equation (25) has a unique monic solution of degree strictly less than p. Obviously, up normalizing the leading term, this solution is the polynomial Φ. Moreover, we have that

$$\begin{aligned}&\deg(\Phi) = \min([-A], [-B]), \\ &\mathrm{ord}_0(\Phi) = \begin{cases} [1-C] & \text{if } 0 \leq [1-C] \leq \deg(\Phi), \\ 0 & \text{otherwise.}\end{cases}\end{aligned}$$

By distinguishing all possibilities and using the assumptions on the a_i one checks that

$$\mathrm{ord}_0(\Phi) = \begin{cases} 0 & if a_1+a_3 < p-1, \\ a_1+a_3-(p-1) & \text{otherwise.}\end{cases}$$

This proves the first and last statements of (b). To prove the middle statement of (b), one considers the expansion of Φ around $\lambda = 1$, and argues similarly. Part (c) follows directly from (a). □

Case 2B: Assume that $1 + n < \alpha \leq p - n + 1 < p$. The construction in this case is very similar, except that we use the differential form

$$\omega = \frac{\mathrm{d}\bar{x}_0}{(\bar{x}_0^m - 1)^\alpha \prod_{j=1}^{\beta}(\bar{x}_0^m - \bar{z}_j)}$$

as deformation data on $\bar{X}_0$. A computation similar to what we did in Case 2A using the Cartier operator and the assumption on α shows that ω is exact.

This finishes the construction of the Hurwitz tree.

5.4 Construction of Hurwitz trees for A_4 This section contains a sketch of the steps one has to take to construct Hurwitz trees for local A_4-actions. This is one of the key steps of the project. We use the notation of Exercises 3.20 and 3.21.

Suppose given an A_4-Galois cover $\bar{f}' : \bar{Y}' \to \bar{Z}'$ in characteristic 2 which is totally branched at ∞, tamely branched above 0, and has no other branch points. Let h be the conductor of the local cover at ∞. Recall from Exercise 2.13 that $\gcd(h, 6) = 1$. We write $\bar{X}' \to \bar{Z}'$ for the subcover with Galois group $\mathbb{Z}/3\mathbb{Z}$.

We distinguish between two cases:

Case a $h \equiv -1 \pmod 6$. In this case, we write $h + 1 = 6\alpha$.

Case b $h \equiv 1 \pmod 6$. In this case, we write $h + 1 = 2 + 6\alpha$.

To describe the Hurwitz tree, we need to construct a tree $\bar{X}$ of projective lines, together with an admissible action of an automorphism τ of order 3 and a marking by $h+1$ points. We may assume that $\bar{X}'$ is one of the components of $\bar{X}$. We denote by $\bar{Y}$ (resp. $\bar{Z}$) the corresponding trees such that $\bar{Y} \to \bar{X}$ (resp. $\bar{X} \to \bar{Z}$) is $V := \mathbb{Z}/2\mathbb{Z} \times \mathbb{Z}/2\mathbb{Z}$- (resp. $\mathbb{Z}/3\mathbb{Z}$-)equivariant.

Moreover, we need to construct differential Swan conductors on all components different from $\bar{X}'$, satisfying the compatibility conditions. We will show that we may construct $\bar{X}$ such that there is a unique component $\bar{X}_0$ which intersects with $\bar{X}'$. Moreover, these two components intersect in the wild branch point ∞ of $\bar{Y}' \to \bar{X}'$. Proposition 3.18 implies that the decomposition group of a component $\bar{Y}_0$ of $\bar{Y}$ above $\bar{X}_0$ is A_4, and that the corresponding inertia group is V.

Exercise 5.10 (a) Show that the decomposition group of a component $\bar{Y}_0$ of $\bar{Y}$ above $\bar{X}_0$ is A_4, and that the corresponding inertia group is V. (Use Proposition 3.18.)

(b) Choose a coordinate $\bar{x}_0$ on $\bar{X}_0$ such that $\tau^*\bar{x}_0 = \zeta_3\bar{x}_0$ for some primitive 3rd root of unity in $\bar{k}$. Let $\chi_i : V \to \mathbb{C}^*$ be the nontrivial characters (as in Exercise 3.20). Construct additive differential forms $\omega_i := \omega(\chi_i)$ on $\bar{X}_0$ such that

   (*) $\omega_0 + \omega_1 + \omega_2 = 0$,

   (*) τ acts on the $\mathbb{F}_2$-vector space $\langle \omega_0, \omega_1 \rangle$,

   (*) and the ω_i have a single zero of order $h - 1$ in ∞.

   (Tip: compare to Exercise 3.21 for an example of such a vector space.)

(c) Construct the Hurwitz tree by using Proposition 5.5. Tip: note that $\langle \tau \rangle$ acts on the components intersecting $\bar{X}_0$. One may construct the tree such that the components are $\bar{X}', \bar{X}_0$ together with the components to which the wild branch points of the lifted cover $Y \to X$ specialize. To guess the decomposition and inertia groups of these last components use Proposition 3.18 together with the description of the ramification of A_4-covers in characteristic zero (Exercise 3.20).

5.5 The Hurwitz obstruction for Q_8-covers In this section, we continue with Example 3.3. To goal is to illustrate that existence of the differential Swan conductors in characteristic p forms a highly nontrivial condition for lifting (in Section 4.2 we called this the Hurwitz-tree Obstruction). The main result of this section is an alternative version of a special case of the result of [8], Section 4.2. This difference with the current approach is that Brewis–Wewers [8] only consider the differents. In this section, we focus on the differential forms. This approach has the advantage that (at least in the special case we treat here) one does not have to consider the full Hurwitz tree, but may work on a single component of the stable reduction.

Let $G = Q_8$. Denote by $V \simeq \mathbb{Z}/2\mathbb{Z} \times \mathbb{Z}/2\mathbb{Z}$ the quotient of G by its center. We consider a G-Galois cover $f : Y \to Z \simeq \mathbb{P}^1$ such that the quotient of Y with respect to the center of G has genus zero. The Riemann–Hurwitz formula implies that $Y \to X := Y/\langle -I \rangle$ is a degree-2 cover branched at $4r+6$ points for some $r \geq 0$. In particular, it follows that $f : Y \to Z$ has exactly 3 branch points with ramification index 4. Moreover, each of the cyclic subgroups of order 4 of G occurs as the inertia group of exactly 2 ramification points. It is easy to see that such covers exist for every value of r. (For example by explicitly writing it down, analogously to what we did in Section 3.3 in the case $r = 0$. Alternatively, one may use the fundamental group of the punctured projective line.)

This description of the ramification implies that the V-Galois cover $X \to Z$ is unique up to isomorphism if we normalize the branch points of this cover to be $0, 1, \infty$. In particular, it is independent of the additional r branch points. Therefore this cover is as described in Section 3.3. (This is the case $r = 0$.)

The strategy of the proof is the following. We denote by $\bar{Y} \to \bar{Z}$ the stable reduction of $Y \to Z$. The stable reduction $\bar{X}' \to \bar{Z}'$ of the cover $X \to Z$, we already computed in Section 3.3, by the previous remark. We denote again by $\bar{X}_2$ the components of $\bar{X}'$ from that section, and write $\bar{X}_2 \to \bar{Z}_2$ for the corresponding purely inseparable cover. We may regards $\bar{Z}$ as a component of the model $\bar{Z}$, and choose a component $\bar{Y}_2$ of $\bar{Y}$ above $\bar{X}_2$. Let v be the valuation of the function field of Y corresponding to $\bar{Y}_2$. We write L (resp. M, resp. K) for the completion with respect to the induced valuations on the functions fields of Y (resp. X, resp. Z).

Let $\sigma_1, \sigma_2, \sigma_3 := \sigma_1\sigma_2$ be the three elements of G of order 4, and put $H_i = \langle \sigma_i \rangle$. We write $\bar{\sigma}_i$ for the action on $X \simeq \mathbb{P}^1$ induced by σ_i, and let $X_i = X/\langle \bar{\sigma}_i \rangle$. We denote by M_i the completion of the function field of X_i with respect to the valuation induced by v. By choosing a suitable coordinate x on X, we may assume as in Section 3.3 that

$$\begin{aligned} \bar{\sigma}_1(x) &= 1/x, \\ \bar{\sigma}_2(x) &= -x, \\ \bar{\sigma}_3(x) &= -1/x. \end{aligned}$$

Exercise 5.11 (a) Show that $z = (x^2-1)^2/4x^2$ is a coordinate on Z.

(b) For z as in (a), show that x_i defined by

$$\begin{aligned} x_1^2 &= 1 + z, \\ x_2^2 &= z/(z-1), \\ x_3^2 &= z. \end{aligned}$$

is a coordinate of X_i.

(c) Show that X_i/Z has multiplicative reduction to characteristic 2, and compute the corresponding Swan conductors.

The next goal is to compute the Swan conductors $\mathrm{sw}_{K/M_i}(\chi)$, where χ is a nontrivial character of H_i. We first assume that $\chi = \chi_1$ is a character of order 2. Proposition 2.27.(c) implies that $\mathrm{sw}_{K/M_i}(\chi_1) = \mathrm{sw}_{M/M_i}(\chi_1)$. This Swan conductor can be computed by calculating a Kummer equation for $X \to X_i$. We present the calculation for $i = 1$, and leave the other two cases as exercises.

Let $i = 1$. One checks that

$$\frac{x^2+1}{2x} = x_1.$$

It follows that a Kummer equation for X/X_1 is given by

$$\tilde{x}^2 = \frac{x_1 - 1}{x_1 + 1} =: f_1, \qquad \text{where} \qquad \tilde{x} = \frac{-x+1}{x+1}.$$

Note that $f_1 \equiv 1 \pmod 2$, therefore we are in the exact case (Section 2.5).

Write $f_1 = 1 + 2/(x_1 + 1)$. Therefore we choose $x = 1 + \sqrt{2}w$ as new coordinate on X. (Note that this is the same choice as we made in Section 3.3

to define the component $\bar{X}_2$; the coordinate was called x_2 in that section.) This yields as equation in reduction

$$\bar{w}^2 = \frac{1}{\bar{x}_1 + 1}.$$

Therefore

$$\begin{aligned} \omega_1 &:= \mathrm{dsw}_{M/M_1}(\chi_1) = \frac{\mathrm{d}\bar{x}_1}{(\bar{x}_1^2 + 1)}, \\ \delta_1 &:= \delta_{M/M_1}(\chi_1) = \frac{p}{p-1} - n = 1 = \frac{1}{p-1}. \end{aligned} \tag{27}$$

A similar computation shows that this formula also holds with M_1 replaced by M_i and x_1 by x_i.

Next we want to compute the Swan conductor $\mathrm{sw}_{L/M_1}(\chi_2)$ for a character of H_1 of order 4. This may be done by using Proposition 2.27.(d), since χ_2 is a character of rank 1. We write $\psi := \chi|_{\langle -I \rangle}$ for the nontrivial character of $\langle -I \rangle \simeq \mathbb{Z}/2\mathbb{Z}$. Proposition 2.27.(d) states that

$$\mathrm{sw}_{L/M_1}(\chi_2) = \mathrm{sw}_{L/M}(\psi) + \mathcal{D}_{M/M_1}. \tag{28}$$

Lemma 2.32 implies that

$$2\mathcal{D}_{M/M_1} = \mathrm{sw}_{M/M_1}(\chi_1) + [-1], \tag{29}$$

since M/M_1 has degree 2. Since k has residue characteristic 2, we may ignore the factor $[-1]$. Substituting (29) in (28) yields

$$2\mathrm{sw}_{L/M_1}(\chi_2) = 2\mathrm{sw}_{L/M}(\psi) + \mathrm{sw}_{M/M_1}(\chi_1).$$

The following example computes $\mathrm{sw}_{L/M_1}(\chi_2)$ in the case that $r = 0$ (Section 3.3).

Example 5.12 Assume that $r = 0$, i.e. $g(Y) = 2$. In this case, we computed $\mathrm{sw}_{L/M}(\psi)$ in Example 3.17.(b). Translated into the notation of the current section, we have

$$\mathrm{sw}_{L/M}(\psi) = [2\sqrt{2}] - \left[\frac{(1 + w^2 + w^4)\,\mathrm{d}w}{w^4(w^4+1)}\right].$$

Write $\omega(\psi)$ for the differential part of $\mathrm{sw}_{L/M}(\psi)$. Then

$$\begin{aligned} 2[\omega(\psi)] &= \left[\frac{(1 + \bar{w}^2 + \bar{w}^4)^2\,(\mathrm{d}\bar{w})^{\otimes 2}}{\bar{w}^8(\bar{w}^8+1)}\right] \\ &= \left[\frac{(\bar{x}_1^2 + \bar{x}_1 + 1)\,\mathrm{d}\bar{x}_1}{\bar{x}_1^2(\bar{x}_1^2+1)}\right]. \end{aligned}$$

Applying (28) yields

$$2[\omega_2] = 2[\omega(\psi)] + [\omega_1] = \left[\frac{(\bar{x}_1^4 + \bar{x}_1^2 + 1)\,(\mathrm{d}\bar{x}_1)^{\otimes 2}}{\bar{x}_1^2(\bar{x}_1+1)^4}\right].$$

Note that addition of elements in S_{M_1} corresponds to multiplication of differential forms in Ω_{M_1}. We conclude that

$$\omega_2 = \frac{(\bar{x}_1^2 + \bar{x}_1 + 1)\,\mathrm{d}\bar{x}_1}{\bar{x}_1(\bar{x}_1+1)^2}.$$

Considering the differents as well, one finds that $\delta_2 = p\delta_1 = 2$.

In [40], Theorem 4.3 describes the relation between a character χ_i of order p^i and $\chi_{i-1} := \chi_i^p$ which has order p^{i-1}. In our situation (i.e. $i = 2$ and $\delta_1 = 1/(p-1) = 1$) that result states that $\delta_2 = p\delta_1 = 2$ and that ω_2 is characterized by

$$\mathcal{C}\omega_2 = \omega_2 + \omega_1. \tag{30}$$

Using the properties of the Cartier operator and the fact that ω_1 is exact (Lemma 5.2) we see that $\omega_1 + \omega_2 = \mathrm{d}g/g$ is a logarithmic differential form. In [32] this statement is reformulated as stating that g satisfies a nonhomogeneous differential equation in characteristic p. This description allows to prove existence results for certain Swan conductors similar to what we did for $G \simeq \mathbb{Z}/p\mathbb{Z}$ in Section 5.1. We do not describe this approach in this notes.

In our situation, we may check (30) explicitly using the properties of the Cartier operator $\mathcal{C}$ (Lemma 5.2):

$$\begin{aligned}\mathcal{C}\omega_2 &= \frac{(\bar{x}_1+1)}{\bar{x}_1+1}\frac{\mathrm{d}\bar{x}_1}{\bar{x}_1}\\ &= \frac{\mathrm{d}\bar{x}_1}{\bar{x}_1} = \omega_2 + \omega_1.\end{aligned}$$

Note that this is a computation in Ω_{M_1} and not in S_{M_1}, therefore addition is here really addition of differential forms and not multiplication.

Since the G-Galois cover is unique in the case that $r = 0$, Example 5.12 computes the only possibly differential Swan conductor corresponding to the character χ_2 which may occur on the component $\bar{Z}_2$. In the rest of this section, we consider the Swan conductor above $\bar{Z}_2$ corresponding to the character χ_2 for an arbitrary G-Galois cover of the type we consider here. We show that the corresponding differential forms has at least two different zeros. As in Example 3.17.(b), this then implies that the cover $Y \to Z$ does not have good reduction.

We now consider the general case that $r \geq 0$, and assume that the cover $Y \to Z$ has good reduction to characteristic 2. We use the notation $\omega_1, \omega_2, \omega(\chi)$ as in Example 5.12. The good-reduction assumption implies that $\omega(\psi)$ has a single zero of order $4r + 6 - 2 = 4(r+1)$, say at $w = \alpha$. For simplicity, we assume that $\alpha \neq \infty$. (This is no restriction, as the situation is symmetric in $0, 1, \infty$.)

Since $\omega(\psi)$ is an exact differential form, the order of its poles is even. We may write

$$\omega(\psi) = \frac{(\bar{w}+\alpha)^{4(r+1)}\,\mathrm{d}\bar{w}}{\bar{w}^2(\bar{w}^2+1)Q^2},$$

for some polynomial $Q \in \bar{k}(\bar{w})$. Let $d = \deg_{\bar{x}_1}(Q)$. As in Example 5.12, we compute that

$$2[\omega(\psi)] = \left[\frac{(\alpha\bar{x}_1+\alpha+1)^{4(r+1)}\,\mathrm{d}\bar{x}_1}{\bar{x}_1^2(\bar{x}_1+1)^{4r+2-2d}\bar{x}_1^2\tilde{Q}^2}\right],$$

where $Q(\bar{w}^2) = \tilde{Q}(\bar{x}_1)/(\bar{x}_1+1)^d$. As in Example 5.12, we conclude that

$$\omega_2 = \frac{(\alpha\bar{x}_1+\alpha+1)^{2(r+1)}\,\mathrm{d}\bar{x}_1}{(\bar{x}_1+1)^{2r+2-d}\bar{x}_1\tilde{Q}}.$$

We now apply (30), which states that $\eta := \omega_1+\omega_2$ is a logarithmic differential form. In particular, η has only simple poles. We compute that

$$\eta = \omega_1+\omega_2 = \frac{(\alpha\bar{x}_1+\alpha+1)^{2(r+1)} - (\bar{x}_1+1)^{2r-d}\bar{x}_1\tilde{Q}\,\mathrm{d}\bar{x}_1}{(\bar{x}_1+1)^{2r+2-d}\bar{x}_1\tilde{Q}}.$$

Since $d \leq 2r$ and $\alpha \neq \infty$, we conclude that η has a pole of order strictly larger than 1 in $\bar{x}_1 = 1$. This yields a contradiction. We conclude that any cover $Y \to Z$ has bad reduction to characteristic p. Moreover, it shows that the Hurwitz-space Obstruction is strictly stronger than the Bertin Obstruction.

6 Lifting results: inertia groups of order p

In this section we state and proof a particular lifting result. Combined with the results of the previous sections it gives a complete solution to the local lifting problem for groups G whose Sylow p-subgroup has order p. We shall do this in the general context described in §1.4.

6.1 Statement of the result We start by formalizing the properties of the stable reduction of a G-cover from Definition 3.4.

Definition 6.1 Let G be a finite group. A *stable G-map* is a finite morphism $\bar{f} : \bar{Y} \to \bar{X}$ between semistable curves over κ, together with a κ-linear action of G on $\bar{Y}$ commuting with $\bar{f}$. Moreover, we assume that the following holds. Let $\bar{W} \subset \bar{Y}$ be an irreducible component, $\bar{Z} := \bar{f}(\bar{W}) \subset \bar{X}$ its image and $G_{\bar{W}} \subset G$ its stabilizer.

(i) The component $\bar{W}$ is smooth.

(ii) The 'inertia group'

$$I_{\bar{W}} := \mathrm{Ker}(G_{\bar{W}} \to \mathrm{Aut}_\kappa(\bar{W}))$$

is a p-group.

(iii) The natural map

$$\bar{W}/G_{\bar{W}} \to \bar{Z}$$

is a homeomorphism, totally inseparabel of degree $|I_{\bar{W}}|$.

(iv) For every singular point $y \in \bar{Y}^{\rm sing}$, the image point $x := \bar{f}(y)$ is a singular point of $\bar{X}$.

Remark 6.2 Let $\bar{f} : \bar{Y} \to \bar{X}$ be a stable G-map.

(i) It follows easily from Part (iii) and (iv) of the definition that the natural map

$$\bar{Y}/G \to \bar{X}$$

is a homeomorphism and that every smooth point of $\bar{Y}$ is mapped to a smooth point of $\bar{X}$.

(ii) The curve $\bar{X}$ and the map $\bar{f} : \bar{Y} \to \bar{X}$ are uniquely determined by the curve $\bar{Y}$ and the action of G on $\bar{Y}$.

The main point of Definition 6.1 is the following. Let $f : Y \to X$ be a G-Galois cover between smooth projective curves over our local field k. Then the stable reduction $\bar{f} : \bar{Y} \to \bar{X}$ of f (see Definition 3.4) is a stable G-map. This suggest the following question.

Problem 6.3 Let $\bar{f} : \bar{Y} \to \bar{X}$ be a stable G-map. Does there exists a G-Galois cover $f : Y \to X$ between smooth projective curves over k whose stable reduction is isomorphic to $\bar{f}$? (As usual, we allow k to be replaced by a sufficiently large finite extension.) If this is the case then we call f a *lift* of $\bar{f}$ and we say that $\bar{f}$ *lifts*.

The above problem is more general than the lifting problem (Problem 1.2). Therefore it is clear that we need more information on $\bar{f}$ before we can expect a lift to exist. The results of §3 and §4 provide us with certain necessary conditions. Namely, if $f : Y \to X$ is a lift of $\bar{f}$ and $\mathcal{Y} \to \mathcal{X}$ the stable model of f, then the ramification invariants attached to the irreducible components of $\bar{X}$ 'live' on the curve $\bar{X}$, and the compatibility conditions they satisfy (with respect to the singular points of $\bar{X}$) can all be formulated purely in terms of the stable G-map $\bar{f}$. A more refined version of Problem 6.3 would ask whether a stable G-map satisfying all these necessary conditions does lift. In full generality, this problem is hard to formulate precisely and, more important, looks very hard to solve. The following theorem only deals with a rather special case.

Theorem 6.4 *Let $\bar{f} : \bar{Y} \to \bar{X}$ be a stable G-map. Assume that*

(i) *For every irreducible component $\bar{Y}_i \subset \bar{Y}$ the inertia group $I_{\bar{Y}_i} \subset G$ has order* 1 *or* p.

(ii) *If $|I_{\bar{Y}_i}| = p$ then the (inseparable) $G_{\bar{Y}_i}$-cover $\bar{Y}_i \to \bar{X}_i := f(\bar{Y}_i)$ carries a deformation datum $(\bar{Z}_i, \omega_i)$.*

(iii) *The datum* $(\bar{f}, (\bar{Z}_i, \omega_i))$ *is admissible.*[1]

Then $\bar{f}$ *lifts to characteristic zero.*

In the special case where $\bar{f}: \bar{Y} \to \bar{X}$ is a *Hurwitz tree* and all inertia groups I_W are either trivial or of order p, this theorem is proved in [5]. The proof of the general case is along the same lines. We will give almost all the details of the proof in this section.

6.2 Formal lifts Let $\bar{f}: \bar{Y} \to \bar{X}$ be a finite map between semistable curves over κ. Let G be a finite group of κ-automorphisms of $\bar{Y}$ which commute with $\bar{f}$. In this section an *algebraic lift* of $\bar{f}$ is a finite morphism $f: \mathcal{Y} \to \mathcal{X}$ between semistable $\mathfrak{o}$-curves, together with an action of G on $\mathcal{Y}$ which commutes with f and a G-equivariant identification of $\bar{f}$ with the special fiber of f.

Let $f: \mathcal{Y} \to \mathcal{X}$ be an algebraic lift of $\bar{f}$. For every $n \geq 0$ we obtain a finite G-invariant morphism

$$f_n : \mathcal{Y}_n := \mathcal{Y} \otimes_{\mathfrak{o}} (\mathfrak{o}/\pi^{n+1}\mathfrak{o}) \to \mathcal{X}_n := \mathcal{X} \otimes_{\mathfrak{o}} (\mathfrak{o}/\pi^{n+1}\mathfrak{o})$$

of $\mathfrak{o}/\pi^{n+1}\mathfrak{o}$-schemes. Note that, as maps between topological spaces, the maps f_n are all *identical* to the map $\bar{f}$. Nevertheless, the lifting f can be reconstructed from the sequence f_n. To do this, we need the language of *formal schemes* (see e.g. [18], Chapter II.9) and invoke Grothendieck's Existence Theorem. Recall that the inverse limits

$$\hat{\mathcal{X}} := \varprojlim_n \mathcal{X}_n, \quad \hat{\mathcal{Y}} := \varprojlim_n \mathcal{Y}_n$$

exist in the category of formal $\mathfrak{o}$-schemes. The (finite) morphism $\hat{f}: \hat{\mathcal{Y}} \to \hat{\mathcal{X}}$ between formal schemes induced by f is called the *formal completion* of f.

Definition 6.5 A *formal lift* of $\bar{f}$ is a finite morphism $\hat{f}: \hat{\mathcal{Y}} \to \hat{\mathcal{X}}$ of flat formal $\mathfrak{o}$-schemes, together with an action of G on $\hat{\mathcal{Y}}$ commuting with $\hat{f}$ and a G-equivariant identification of $\bar{f}$ with $f \otimes_{\mathfrak{o}} \kappa$.

So every algebraic lift of $\bar{f}$ induces a formal lift by the process of formal completion. Conversely,

Theorem 6.6 *Every formal lift of* $\bar{f}$ *is the formal completion of an algebraic lift* $f: \mathcal{Y} \to \mathcal{X}$. *Moreover,* f *is uniquely determined by* $\hat{f}$, *up to unique isomorphism.*

Proof: This follows from [16], ??. □

By the above theorem it suffices to construct a formal lift of $\bar{f}$, which is much easier. The technique we shall use for doing this is called *formal patching*. We should point out that the version of formal patching that we are going to use is quite different from the version explained e.g. in []. Briefly, our version is simpler, but proves a much weaker result. See Remark **??** for a more precise statement.

[1] This notion has not yet been defined! Sorry..

6.3 Formal patching Since we will work mainly with formal $\mathfrak{o}$-schemes, we will write $f : \mathcal{Y} \to \mathcal{X}$ for the formal lift of $\bar{f}$ that we wish to construct. As a first approximation, we will explain how to construct formal lifts $\mathcal{Y}$ of $\bar{Y}$. More precisely, we will show that such lifts correspond bijectively to so-called *patching data* on $\bar{Y}$. We will see later how to make sure that the formal lift $\mathcal{Y}$ is G-equivariant in such a way that $\mathcal{X} := \mathcal{Y}/G$ is a formal lift of $\bar{X}$.

Notation 6.7 Let $\bar{Y}_1, \ldots, \bar{Y}_m$ denote the irreducible components of $\bar{Y}$. By our assumptions made in §3.1, each $\bar{Y}_i$ is a smooth projective curve. Moreover, for every singular point $y \in \bar{Y}^{\rm sing}$ there are exactly two components $\bar{Y}_i, \bar{Y}_j$ which intersect transversally in y. Let $\bar{Y}_i^\circ := \bar{Y}_i \cap \bar{Y}^{\rm sm}$ denote the intersection of $\bar{Y}_i$ with the smooth locus of $\bar{Y}$. Note that $\bar{Y}_i^\circ = \mathrm{Spec}\,(\bar{B}_i)$ is an open affine subset of $\bar{Y}$. It follows that $\bar{L}_i := \mathrm{Frac}(\bar{B}_i)$ is the function field of $\bar{Y}_i$.

Let $y \in \bar{Y}^{\rm sing}$ be a singular point. We write $\bar{D}_y := \hat{\mathcal{O}}_{\bar{Y},y}$ for the complete local ring of $\bar{Y}$ in y. (Recall that $\bar{D}_y \cong \kappa[[u, v \mid uv = 0]]$, where $(u), (v) \lhd \hat{D}$ are the two minimal prime ideals.) If $y \in \bar{Y}_i$ (and hence $(y, \bar{Y}_i)$ is a *branch* of $\bar{Y}$, see §3.2) then there exists a minimal prime ideal $\bar{\mathfrak{p}}_{y,i} \lhd \bar{D}_x$ such that $\bar{D}_y/\bar{\mathfrak{p}}_{y,i} = \hat{\mathcal{O}}_{\bar{Y}_i,y}$. Therefore, we obtain a natural embedding

$$\bar{\beta}_{y,i} : \bar{L}_i := \mathrm{Frac}(\bar{B}_i) \hookrightarrow \bar{L}_{y,i} := \mathrm{Frac}(\bar{D}_y/\bar{\mathfrak{p}}_{y,i}),$$

which identifies $\bar{L}_{y,i}$ with the completion of $\bar{L}_i$ with respect to the valuation corresponding to $y \in \bar{Y}_i$.

Lemma 6.8 *Let $\bar{V} \subset \bar{Y}$ be a open affine subset. Set $\bar{V}_i := \bar{V} \cap \bar{Y}_i^\circ$. We define a morphism of κ-algebras*

$$\bar{\theta}_{\bar{V}} : \prod_i \Gamma(\bar{V}_i, \mathcal{O}_{\bar{Y}}) \times \prod_{y \in \bar{V}} \bar{D}_y \to \prod_{y \in \bar{V}_i} \bar{L}_{y,i}$$

by the formula

$$\bar{\theta}_{\bar{V}}(f_i, g_y) := \big(\bar{\beta}_{y,i}(f_i) - (g_y + \mathfrak{p}_{y,i})\big).$$

Then $\Gamma(\bar{V}, \mathcal{O}_{\bar{Y}}) = \mathrm{Kern}(\bar{\theta}_{\bar{V}})$.

Proof: Left as an exercise. □

Definition 6.9 Let $\bar{A}$ be a κ-algebra. A *lift* of $\bar{A}$ is a flat and complete $\mathfrak{o}$-algebra together with an identification $\bar{A} = A/\pi A$. Here completeness of A means that

$$A = \varprojlim_n A/\pi^n A.$$

Let $f : \mathcal{Y} \to \mathcal{X}$ be a formal lift of $\bar{f}$. Then

$$B_i := \Gamma(\bar{Y}_i^\circ, \mathcal{O}_{\mathcal{Y}})$$

is a lift of $\bar{B}_i$, and the complete local ring

$$D_y := \hat{\mathcal{O}}_{\mathcal{Y},y}$$

is a lift of $\bar{D}_y$. Note that $(\pi) \lhd B_i$ is a prime ideal. Let

$$L_i := \mathrm{Frac}\big((B_i)^{\widehat{}}_{(\pi)}\big)$$

denote the fraction field of the completed localization of B_i at (π). This is a complete discretely valued field with residue field $\bar{L}_i = \mathrm{Frac}(\bar{B}_i)$ (see Notation 6.7), weakly unramified over its field of constants k. It is hence a field of Type B (see §2.1).

For every pair (y, i) such that $y \in \bar{Y}_i$ (which we call a *branch*), let $\mathfrak{p}_{y,i} \lhd D_y$ be the prime ideal corresponding to $\bar{\mathfrak{p}}_{y,i} \lhd \bar{D}_y$ (see Notation 6.7). Let

$$L_{y,i} := \mathrm{Frac}\big((D_y)^{\widehat{}}_{\mathfrak{p}_{y,i}}\big)$$

be the fraction field of the completed localization of D_i at $\mathfrak{p}_{y,i}$. Then $L_{y,i}$ is a complete discretely valued field with residue field $\bar{L}_{y,i}$, weakly unramified over its field of constants k. It is hence a two-local field (Type C).

By Theorem 6.6, there exists an algebraic lift $f^{\mathrm{alg}} : \mathcal{Y}^{\mathrm{alg}} \to \mathcal{X}^{\mathrm{alg}}$ of $\bar{f}$ such that f is the formal completion of f^{alg}. Let $L := k(\mathcal{Y}^{\mathrm{alg}})$ be the function field of $\mathcal{Y}^{\mathrm{alg}}$. (Here we assume, for simplicity, that $\mathcal{Y}^{\mathrm{alg}}$ is irreducible.) For all i the component $\bar{Y}_i$ gives rise to a discrete valuation v_i on L which extends the valuation v on k and whose residue field may be identified with $\bar{L}_i = \kappa(\bar{Y}_i)$. Let $U \subset \mathcal{Y}^{\mathrm{alg}}$ be an open affine subset such that $U \cap \bar{Y} = \bar{Y}_i^{\circ}$. Then $B := \Gamma(U, \mathcal{O}_{\mathcal{Y}^{\mathrm{alg}}})$ is a subring of L such that $L = \mathrm{Frac}(B)$, and B_i has a natural identification with the completion of B with respect to the prime ideal $(\pi) \lhd B$. Moreover, the natural embedding $B \hookrightarrow B_i$ extends to an embedding $L \hookrightarrow L_i$ which identifies L_i with the completion of L at the valuation v_i. Similarly, let $V \subset \mathcal{Y}^{\mathrm{alg}}$ be an open affine subset containing y. Then D_y is equal to the completion of $D := \Gamma(V, \mathcal{O}_{\mathcal{Y}^{\mathrm{alg}}}) \subset L$ at the maximal ideal corresponding to y. Moreover, the natural embedding $D \hookrightarrow D_y$ extends to an embedding $L \hookrightarrow L_{y,i}$ which identifies $L_{y,i}$ with the 2-completion of L with respect to the rank-2-valuation $\eta_{y,i}$ corresponding to the branch $(\bar{Y}_i, y)$ of $\bar{Y}$. It then follows from the uniqueness of the residual completion (see Example 2.7) that there exists a unique embedding

$$\beta_{y,i} : L_i \hookrightarrow L_{y,i}$$

which induces the embedding $\bar{\beta}_{y,i} : \bar{L}_i \hookrightarrow \bar{L}_{y,i}$ from Notation 6.7 on the residue fields, restricts to the identity on the common subfield L and identifies $L_{y,i}$ with the 2-completion of L_i with respect to $\eta_{y,i}$.

The preceeding discussion suggests the following definition.

Definition 6.10 A *patching datum* for $\bar{Y}$ consists of the following objects.

(a) A lift B_i of $\bar{B}_i$, for all i. We set $L_i := \mathrm{Frac}((B_i)^{\widehat{}}_{(\pi)})$.

(b) A lift D_y of $\bar{D}_y$, for all singular points $y \in \bar{Y}$. If $y \in \bar{Y}_i$ then we set $L_{y,i} := \mathrm{Frac}((D_y)\widehat{_{\mathfrak{p}_{y,i}}})$.

(c) For all branches (y,i), a k-linear embedding $\beta_{y,i} : L_i \hookrightarrow L_{y,i}$ which lifts $\bar{\beta}_i$ and identifies $L_{y,i}$ with the 2-completion of L_i with respect to the valuation on $\bar{L}_i$ corresponding to y.

It is clear from the discussion preceeding the definition that every lift $\mathcal{Y}$ of $\bar{Y}$ induced a patching datum.

Proposition 6.11 *Every patching datum for $\bar{f}$ is induced by a formal lift f, which is unique up to unique isomorphism.*

Proof: (sketch) Let $(B_i, D_y, \beta_{y,i})$ be a patching datum for $\bar{Y}$. Let $\bar{V} \subset \bar{Y}$ be an open affine subset. For all i we set $\bar{V}_i := \bar{V} \cap \bar{Y}_i^\circ$, $\bar{B}_{i,\bar{V}} := \Gamma(\bar{V}_i, \mathcal{O}_{\bar{Y}})$ and $B_{i,\bar{V}} := \Gamma(\bar{V}_i, \mathrm{Spf}(B_i))$. Then $B_{i,\bar{V}}$ is a lift of $\bar{B}_{i,\bar{V}}$. Define the $\mathfrak{o}$-linear morphism

$$\theta_{\bar{V}} : \prod_i B_{i,\bar{V}} \times \prod_{y \in \bar{V}} D_y \to \prod_{y \in \bar{V}_i} D_{y,i}$$

by the formula $\theta_{\bar{V}}(f_i, g_y) := (\beta_{y,i}(f_i) - g_y)_{(y,i)}$. It follows easily from Lemma 6.8 that $B_{\bar{V}} := \mathrm{Ker}(\theta_{\bar{V}})$ is a lift of $\bar{B}_{\bar{V}} := \Gamma(\bar{V}, \mathcal{O}_{\bar{Y}})$. Moreover, there exists a (unique) sheaf of $\mathfrak{o}$-algebras $\mathcal{B}$ on $\bar{Y}$ such that $\Gamma(\bar{V}, \mathcal{B}) = B_{\bar{V}}$, for all open affine subsets $\bar{V} \subset \bar{Y}$. This means that there exists a (uniquely determined) formal lift $\mathcal{Y}$ of $\bar{Y}$ such that $\mathcal{O}_{\mathcal{Y}} = \mathcal{B}$. □

By a G-equivariant patching datum for $\bar{Y}$ we mean a patching datum $(B_i, D_y, \beta_{y,i})$, together with a 'G-action' on it which is compatible with the natural G-action on $\bar{Y}$. It should be clear what we mean by that. For instance, for every $\sigma \in G$ and i we have an $\mathfrak{o}$-linear isomorphism $B_{\sigma(i)} \stackrel{\sim}{\to} B_i$ which lifts the isomorphism $\bar{B}_{\sigma(i)} \stackrel{\sim}{\to} \bar{B}_i$ induced by $\sigma : \bar{Y} \stackrel{\sim}{\to} \bar{Y}$. It is also clear from the above proposition that every G-equivariant patching datum is induced from a (uniquely determined) G-equivariant lift $\mathcal{Y}$ of $\bar{Y}$.

Proposition 6.12 *Let $\mathcal{Y}$ be a G-equivariant lift of $\bar{Y}$ and $(B_i, D_y, \beta y, i)$ the induced patching datum. Let $\mathcal{X} := \mathcal{Y}/G$ denote the quotient schemes and $f : \mathcal{Y} \to \mathcal{X}$ the canonical map. Suppose that the following holds.*

(a) The lift D_y of $\bar{D}_y$ is of the form

$$D_y \cong \mathfrak{o}[[u, v \mid uv = a]],$$

with $a \in \mathfrak{o} \backslash \{0\}$, for all $y \in \bar{Y}^{\mathrm{sing}}$.

(b) For all i the action of G_i on B_i is faithfull.

Then the generic fibers of $\mathcal{X}$ and $\mathcal{Y}$ are smooth, the induced map $f_k : \mathcal{Y}_k \to \mathcal{X}_k$ is a G-Galois cover, and $f : \mathcal{X} \to \mathcal{Y}$ is a lift of $\bar{f}$.

Proof: It follows from (a) that $\mathcal{Y}$ is a semistable curve with smooth generic fiber. Then the same holds for the quotient $\mathcal{X} = \mathcal{Y}/G$. Now (b) implies that the map of generic fibers $f_k : \mathcal{Y}_k \to \mathcal{X}_k$ is a G-Galois cover. It follows that the special fiber of f is the stable reduction of f_k, and in particular a finite G-map in the sense of Definition 6.1. By Remark 6.2 (ii), we can identify it with $\bar{f} : \bar{Y} \to \bar{X}$. □

6.4 Lifting the deformation data We have seen that in order to prove Theorem 6.4 it suffices to construct a G-equivariant patching datum for $\bar{Y}$ which satisfies Condition (a) and (b) of Proposition 6.12. We start by constructing the lifts B_i.

Let $\bar{X}_i := \bar{f}(\bar{Y}_i) \subset \bar{X}$ be the component of $\bar{X}$ lying under $\bar{Y}_i$. Then the image of the affine open part $\bar{Y}_i^\circ \subset \bar{Y}_i$ is precisely the affine part $\bar{X}_i^\circ \subset \bar{X}_i$ consisting of those points which are smooth in $\bar{X}$. Let $G_i \subset G$ the stabilizer of $\bar{Y}_i$ and $I_i \lhd G_i$ the inertia group.

We have to distinguish two cases. We first consider the case where the inertia group I_i is trivial. By Definition 6.1 this means that the map $\bar{Y}_i \to \bar{X}_i$ is a G_i-Galois cover between smooth projective curves over $\bar{k}$. Furthermore, the restriction of this cover to the smooth locus of $\bar{Y}$, i.e. the map $\bar{Y}_i^\circ \to \bar{X}_i^\circ$, is at most tamely ramified.[2] Then it follows from [] that $\bar{Y}_i^\circ \to \bar{X}_i^\circ$ lifts uniquely to a tamely ramified G_i-Galois cover

$$\mathcal{Y}_i^\circ \to \mathcal{X}_i^\circ$$

of affine and flat formal $\mathfrak{o}$-schemes. In particular, $\mathcal{Y}_i^\circ = \mathrm{Spf}(B_i)$ for a G_i-equivariant lift B_i of $\bar{B}_i$.

In the second case we have $|I_i| = p$. We set $H_i := G_i/I_i$. By Definition 6.1 the map $\bar{Y}_i \to \bar{X}_i$ factors through an H_i-Galois cover $\bar{Z}_i \to \bar{X}_i$ such that the map $\bar{Y}_i \to \bar{Z}_i$ is an inseparable homeomorphism of degree p (in fact, $\bar{Y}_i \to \bar{Z}_i = \bar{Y}_i^{(p)}$ is the relative Frobenius morphism). Furthermore, the admissibility condition from Theorem 6.4 says that the restriction of the Galois cover $\bar{Z}_i \to \bar{X}_i$ to the affine open subset $\bar{X}_i^\circ$ is at most tamely ramified. As in the first case it follows that $\bar{Z}_i^\circ \to \bar{X}_i^\circ$ lifts uniquely to a tamely ramified H_i-Galois cover

$$\mathcal{Z}_i^\circ \to \mathcal{X}_i^\circ.$$

In order to extend it to a G_i-Galois cover $\mathcal{Y}_i^\circ \to \mathcal{X}_i^\circ$ we use the differential form ω_i on $\bar{Z}_i$. Write $\mathcal{Z}_i^\circ = \mathrm{Spf}(A_i)$.

Proposition 6.13 *There exists a G_i-equivariant lift B_i of $\bar{B}_i$ with the following properties.*

(i) *The ring of invariants $B_i^{I_i}$ is a H_i-equivariant lift of $\bar{A}_i$ and may therefore be identified with A_i.*

[2] This is part of the admissibility condition of Theorem 6.4 that we have not yet spelled out.

(ii) *The I_i-Galois extension* $\mathrm{Frac}(B_i)/\mathrm{Frac}(A_i)$ *is fiercely ramified with respect to the valuation corresponding to the prime ideal* $(\pi) \lhd A_i$ *with differential Swan conductor* ω_i*.*

Proof: (rough sketch) We assume for simplicity that the deformation datum $(\bar{Z}_i, \omega_i)$ is multiplicative. Then $\omega_i = d\bar{u}/\bar{u}$ for an element $\bar{u} \in \bar{A}_i$. Recall that we have $\tau^*\omega_i = \chi(\tau)\omega_i$ for all $\tau \in H_i$ and some character $\chi : H_i \to \mathbb{F}_p^\times$. Let a_τ be the unique integer such that $0 < a_\tau < p$ and $a_\tau \equiv \chi(\tau) \pmod p$. Then

$$\tau^*\bar{u} = \bar{u}^{a_\tau} \cdot \bar{v}_\tau^{-p},$$

for some $\bar{v} \in \bar{A}_i$.

Claim: There exists an element $u \in A_i$ lifting $\bar{u}$ such that for all $\tau \in H_i$ we have

$$\tau^*u = u^{a_\tau} \cdot v_\tau^{-p},$$

with $v_\tau \in A_i$ lifting $\bar{v}_\tau$.

The proof of this claim is left as an exercise (Exercise **??**). Let B_i be the normalization of the A_i-algebra $A_i[w \mid w^p = u]$. We choose a generator $\sigma \in I_i$ and a pth root of unity $\zeta \in \mathfrak{o}$. We let I_i act on B_i via

$$\sigma^*w = \zeta \cdot w, \quad \sigma^*|_{A_i} = \mathrm{Id}_{A_i}.$$

It is now again an exercise we leave to the reader to check that B_i satisfies all the conditions from the proposition. □

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