INTRODUCTION TO WILD RAMIFICATION OF SCHEMES AND SHEAVES

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In classical theory of algebraic numbers, the conductor-discriminant formula asserts that the discriminant of an extension of number fields is the product of local invariants of ramification called the conductor. The conductor of a Galois representation plays also a crucial role in the quantitative formulation of the Langlands correspondences.

In this course, we discuss more geometric aspects of ramification, due to the following reasons. Firstly, we have a clearer picture and more complete theory in the geometric case. Secondly, in the arithmetic case, even in the cases where results similar to the geometric case are obtained, it usually requires more sophisticated technics.

A prototype of the geometric ramification theory is the Grothendieck-Ogg-Shafarevich formula [10] that computes the Euler number of an ℓ-adic sheaf on a curve over an algebraically closed field of positive characteristic different from ℓ. The formula is a sheaf theoretic refinement of the Riemann-Hurwitz formula for a ramified covering of algebraic curves, which is a geometric counterpart of the conductor-discriminant formula, with the analogy between the discriminant of a number field and the genus of a curve. In the Grothendieck-Ogg-Shafarevich formula, the conductor appears as the local contribution of ramification.

We will discuss generalizations of the GOS formula through the following three approaches:

1. Ramified coverings and log products.
2. Characteristic classes and characteristic cycles.
3. Blow-up at the ramification locus in the diagonal.

They are related to each other but can be discussed independently at least in the beginning of the theory. All of the three approaches rely essentially on constructions using the product that make the theory in arithmetic case technically more complicated where we need to work with some substitutes.

Here follows a more concrete description of the course. Using ramified coverings and log products in the first approach, we introduce the Swan class of an ℓ-adic sheaf ramified along the boundary. The Swan class is a generalization of the conductor and defined as an 0-cycle class supported on the boundary. It enables us to generalize the GOS formula computing the Euler number to higher dimension.

For an ℓ-adic sheaf on a variety, its characteristic class is defined as a cohomology class using an abstract formalism and the Lefschetz trace formula applied to a compactification asserts that its trace computes the Euler number. Hence, a second approach to a generalization of the GOS formula is the computation of the characteristic class. The theory of $\mathcal{D}$-modules suggests that the characteristic cycle [11] defined in the cotangent bundle gives the class. We observe that this is the case for rank 1 sheaf at least in a certain favorable situation.

The conductor of a Galois representation of a local field is defined by the filtration of ramification groups on the Galois group. The definition of the filtration is generalized to a local field with imperfect residue field, first using rigid geometry. The interpretation of rigid geometry in terms of blow-up inspired the third approach. We define the filtration of ramification groups without using rigid geometry and observe how the groupoid structure on a blow-up of the product allows us to study the graded subquotients of the filtration.

The course is intended to make an introduction to the subjects discussed in a survey article [15]. More details are found in the following articles; [13] for the first approach, [9], [12], [7], [14] for the second approach and [5], [6], [14], [8] for the third approach.

The audiences are assumed to have some acquaintances to the following subjects:

Number theory ([1, Parties 1, 2]): Galois theory including infinite Galois extensions, discrete valuation rings, ...

Algebraic geometry ([4, Exposés I, V], [3, Sections 1, 2]): schemes, etale morphisms, etale sites, fundamental groups, cohomology, ...

Representations of finite groups ([2]).

We will briefly recall some basic terminologies on etale topology at the beginning of the course.

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Project outline: Cohomology of nearby cycles and ramification

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The Grothendieck-Ogg-Shafarevich (GOS) formula (*cf.* [SGA5, Exposé X]) is one of the most classical results in geometric ramification theory. It describes the Euler characteristic of an smooth ℓ-adic sheaf $\mathcal{F}$ on a smooth curve over an algebraically closed field in terms of a local ramification invariant of $\mathcal{F}$ called the Swan conductor. In [KS08], Kato and Saito obtained a higher-dimensional generalization of it. In this project, we will consider an arithmetic variant of the GOS formula.

Let V be a complete discrete valuation ring with algebraically closed residue field. Put $S = \operatorname{Spec} V$ and consider a scheme X separated of finite type over S whose generic fiber X_η is smooth. For a smooth ℓ-adic étale sheaf $\mathcal{F}$ on X_η, we can attach the nearby cycle complex $R\psi\mathcal{F}$, which is an object of the derived category $D^b_c(X_s, \mathbb{Q}_\ell)$ of ℓ-adic sheaves over the special fiber of X (*cf.* [SGA7, Exposé XIII]). In our project, we will study the compactly supported cohomology $H^i_c(X_s, R\psi\mathcal{F})$, especially its Euler characteristic $\chi_c(X_s, R\psi\mathcal{F}) = \sum_i (-1)^i \dim_{\mathbb{Q}_\ell} H^i_c(X_s, R\psi\mathcal{F})$. If X is proper over S, the proper base change theorem tells us that $H^i_c(X_s, R\psi\mathcal{F})$ is isomorphic to $H^i(X_{\overline{\eta}}, \mathcal{F}_{\overline{\eta}})$, the ℓ-adic cohomology of the generic fiber. In this case, no contribution of ramification of $\mathcal{F}$ appears. However, if X is not proper over S, the nearby cycle cohomology $H^i_c(X_s, R\psi\mathcal{F})$ should reflect arithmetic ramification of $\mathcal{F}$. Note that $H^i_c(X_s, R\psi\mathcal{F})$ is not necessarily isomorphic to $H^i_c(X_{\overline{\eta}}, \mathcal{F}_{\overline{\eta}})$. The Euler characteristic $\chi_c(X_{\overline{\eta}}, \mathcal{F}_{\overline{\eta}})$ of the latter can be described by the classical GOS formula, and only reflects ramification of $\mathcal{F}$ on the generic fiber.

In the language of rigid geometry, the nearby cycle cohomology $H^i_c(X_s, R\psi\mathcal{F})$ can be interpreted as follows. Let $X^\wedge$ be the formal completion of X along X_s and $\mathfrak{X}$ the rigid generic fiber of $X^\wedge$ in the sense of Raynaud. Then, $H^i_c(X_s, R\psi\mathcal{F})$ is naturally isomorphic to the étale cohomology $H^i_c(\mathfrak{X}_{\overline{\eta}}, \mathcal{F}^{\mathrm{rig}}_{\overline{\eta}})$ of the rigid space $\mathfrak{X}$ with a naturally induced coefficient. In this context, when $\dim X_\eta = 1$, Huber [Hub01] proved a formula of GOS type[1]:

$$\chi_c(X_s, R\psi\mathcal{F}) = \chi_c(\mathfrak{X}_{\overline{\eta}}, \mathcal{F}^{\mathrm{rig}}_{\overline{\eta}}) = \operatorname{rank}\mathcal{F} \cdot \chi_c(\mathfrak{X}_{\overline{\eta}}, \mathbb{Q}_\ell) - \sum_{x \in \mathfrak{X}^c_{\overline{\eta}} \setminus \mathfrak{X}_{\overline{\eta}}} \operatorname{Sw}_x \mathcal{F}.$$

Here $\mathfrak{X}^c_{\overline{\eta}}$ is the so-called universal compactification of $\mathfrak{X}_{\overline{\eta}}$. To a point x on the boundary $\mathfrak{X}^c_{\overline{\eta}} \setminus \mathfrak{X}_{\overline{\eta}}$ of the universal compactification, a valuation field $\kappa(x)$ of rank

[1] Precisely speaking, Huber considered a locally constant torsion sheaf $\mathcal{F}$.

2 is naturally attached, and a local ramification invariant $\mathrm{Sw}_x \mathcal{F}$ at x, an analogue of the Swan conductor, is defined by using the natural ramification filtration on the Galois group of $\kappa(x)$. This formula provides a powerful method to study the cohomology of some arithmetic curves, such as the Lubin-Tate tower for GL(2) (*cf.* [Wew05]).

In this project, we will try to understand the formula above in the style of Kato-Saito's ramification theory, and generalize it to a higher-dimensional case. I am planning to tackle the following problems:

i) First recall the classical GOS formula for curves over an algebraically closed field [SGA5, Exposé X] and its higher-dimensional generalization [KS08]. Apply them to some concrete cases.

ii) Understand Huber's GOS formula by finding some examples of it.

iii) Reformulate Huber's GOS formula in the framework of algebraic geometry. One key is the Lefschetz trace formula for stable curves; see [Sai87] and [Fal94]. [Kat87] and [AS10, Appendix A] also include related topics. Another task is to find a scheme-theoretic interpretation of the universal compactification $\mathfrak{X}^c_{\overline{\eta}}$, which will be related to Fujiwara's Zariski-Riemann space [Fuj95], [FK06] (see also [KS10, §2.3, §5.2]).

iv) Applying Kato-Saito's log Lefschetz trace formula [KS10, §1] to semistable curves, obtain another GOS formula on the Euler characteristic. Compare it with the original one. This might give us an intersection-theoretic interpretation of Huber's local ramification invariant.

v) Try to generalize the formula developed above to a higher-dimensional case by using a similar method as in [KS08]. It is hopeful because Kato-Saito's log Lefschetz trace formula is valid for higher-dimensional schemes.

vi) This is a slightly different problem. Our cohomology group $H^i_c(X_s, R\psi\mathcal{F})$ naturally carries an action of the Galois group $\mathrm{Gal}(\overline{\eta}/\eta)$. Therefore, it may also be interesting to study its Swan conductor $\mathrm{Sw}\, H^i_c(X_s, R\psi\mathcal{F})$. One expects to have a conductor formula similar to [KS10, (0.1)]. It would be a good idea to start with the one-dimensional case.

The first two are easier and attackable without much knowledge. Students who are familiar with arithmetic curves will be able to challenge the problems iii) and iv). The last two are for more advanced students.

Prerequisites for this project are almost the same as those for Professor Saito's lecture; elementary knowledge about algebraic geometry and étale cohomology theory. Some knowledge of rigid geometry will help you, but it is not required because we will mainly take a purely scheme-theoretic way.

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