$$f = \sum_{n=1}^{\infty}{}' a_n q^n \qquad a_n \in \mathbb{Z}$$

f = modular form.

f is an eigenform.

$$\rho_{f,p} : \mathrm{Gal}(\overline{\mathbb{Q}}/\mathbb{Q}) \longrightarrow GL_2(\mathbb{Z}_p)$$

$\mathrm{tr}(\rho_{f,p}(\mathrm{Frob}_\ell)) = a_\ell$ all $\ell \nmid Np$.

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f no longer an eigenform

$\in S_k(\Gamma)$ finite dimensional

$\exists$ Petersson Inner prod $\langle \, , \rangle$

$$f = \sum{}' \frac{\langle \phi_i, f \rangle}{\langle \phi_i, \phi_i \rangle} \cdot \phi_i = \sum \alpha_i \phi_i$$

ϕ_i eigen.

j-invariant.

$$j = \frac{1}{q} + 744 + 196884q + \cdots$$

$$= \sum c(n) q^n.$$

$$c(n) \sim \frac{e^{4\pi\sqrt{n}}}{\sqrt{2}\, n^{3/4}}.$$

Thm [Lehner]

If $n > 0$, $n \equiv 0 \bmod 2^m$, $m \geq 1$.

then $c(n) \equiv 0 \bmod 2^{3m+8}$.

j not sum of eigenforms

- coef too big
- only eigenforms are const
- q^{-1} problem

replace $\sum c(n)q^n$ by $\sum c(2n)q^n$.

$$q = e^{2\pi i\tau}$$

$$F(\tau) = \sum a_n q^n$$

$$UF(\tau) = \frac{1}{2}\left(F\left(\tfrac{\tau}{2}\right) + F\left(\tfrac{\tau+1}{2}\right)\right)$$

$$= \sum a_{2n} q^n$$

$Uj(\tau) =$ modular function (of level $\Gamma_0(2)$).

$$= \sum_{n=0}^{\infty} c(2n)q^n$$

ϕ_i over convergent 2-adic eigenforms. with associated Galois reps.

$$\stackrel{?}{=} \sum \alpha_i \phi_i$$

2-adically convergent.

$b_n \in \overline{\mathbb{Q}}_2$

$$\phi_i = \sum b_n q^n$$

hope for 2-adic Petersson Inner product,

and that

$$\boxed{u_j \overset{?}{\equiv} \sum{}' \frac{\langle u_j, \phi_i \rangle}{\langle \phi_i, \phi_i \rangle} \cdot \phi_i}$$

MASTER FORMULA.

Known For $N=1, K=0, p=2$.

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§ Weierstrass. $\Lambda \subseteq \mathbb{C}$

$$x = \wp(z;\Lambda) = \frac{1}{z^2} + \sum_{\Lambda \setminus 0} \frac{1}{(z-\lambda)^2} - \frac{1}{\lambda^2}.$$

$$y = \wp'(z:\Lambda) = -\sum_{\Lambda} \frac{2}{(z-\lambda)^3}$$

$$y^2 = 4x^3 - 60\,G_4(\Lambda)x - 140\,G_6(\Lambda)$$

$$G_{2k}(\Lambda) = \sum_{\Lambda} \frac{1}{\lambda^{2k}}.$$

<u>DEF</u> Λ, Λ' homothetic if $\exists\, \mu \in \mathbb{C}^\times$ $\lambda = \mu \Lambda'$.

<u>EX</u> $\mathbb{C}/\Lambda \cong \mathbb{C}/\Lambda'$.

$$\{\text{Lattices}\}/_{\text{hom}} \leftrightsquigarrow \{\text{Ell}/\mathbb{C}\}/_{\sim}$$

All lattices are hom to

$$\{\mathbb{Z}\tau + \mathbb{Z}\} \quad \tau \in \mathbb{H} = \left\{ \begin{matrix} z \in \mathbb{C} \\ \text{Im}(z) > 0 \end{matrix} \right\}$$

$\{\mathbb{Z}\tau + \mathbb{Z}\} \sim \{\mathbb{Z}\tau' + \mathbb{Z}\}$ iff $\tau' = \dfrac{a\tau + b}{c\tau + d}$.

$\begin{pmatrix} a & b \\ c & d \end{pmatrix} \in SL_2\mathbb{Z}$.

$\underline{G_{2k}(\Lambda).}$

$$G_{2k}(\mu\Lambda) = \mu^{-2k} G_{2k}(\Lambda).$$

$\underline{\text{DEF}^{*}\,0}$: Modular Forms: wt $= k$
functions on lattices Λ st
$$F(\mu\Lambda) = \mu^{-k} F(\Lambda)$$

- - - - - - - - - -

$$\begin{aligned} f(\tau) &= F(\tau\mathbb{Z} + \mathbb{Z}). \\ &= F((a\tau+b)\mathbb{Z} + ((c\tau+d))\mathbb{Z}) \\ &= (c\tau+d)^{-k} F(\tau'\mathbb{Z} + \mathbb{Z}) \\ &= (c\tau+d)^{-k} f\left(\frac{a\tau+b}{c\tau+d}\right). \end{aligned}$$

$$\begin{tikzcd}
\{\text{Lattices}\} \arrow[r] \arrow[d, two heads] & ? \arrow[d] \\
\{\text{Lattices}\}/\text{hom} \arrow[r, dash] & \{\text{Ell}/\mathbb{C}\}/\sim
\end{tikzcd}$$

$$\Lambda \longmapsto y^2 = 4x^3 - Ax - B.$$

<u>Lemma</u> The space of holomorphic 1-forms on E, $H^0(E, \Omega^1)$, is $\cong \mathbb{C}$ for an elliptic curve.

$E = \mathbb{C}/\Lambda . \longleftarrow \mathbb{C}$. $\quad f(z)dz$.

$H^0(E, \Omega^1) \cong \mathbb{C} \cdot dz$. $\quad = f(z+\lambda)\, d(z+\lambda)$

$= f(z+\lambda)\, dz$.

$$x \qquad y = \frac{dx}{dz}.$$

$$dz = \frac{dx}{y}.$$

$$\Lambda \longmapsto \mu\Lambda.$$

$$y^2 = 4x^3 - Ax - B$$

$$Y^2 = 4X^3 - \mu^{-4}Ax - \mu^{-6}B.$$

$$X = \mu^{-2}x$$

$$Y = \mu^{-3}y$$

$$\frac{dX}{Y} = \mu\frac{dx}{y}.$$

MF wt=k: $F(\mu\Lambda) = \mu^{-k}F(\Lambda)$.

$$\Lambda \longmapsto (E, \omega \text{ "=" } dz)$$

$$\mu\Lambda \longmapsto (E, \mu\omega).$$

DEF* 1: modular form of weight k function $f(E,\omega)$ s.t

$$f(E, \mu\omega) = \mu^{-k} f(E,\omega).$$

$$\omega \in H^0(E, \Omega^1).$$

$$\underbrace{f(E,\omega)\cdot\omega^{\otimes k}}_{}. \quad \text{well def.}$$