modular form as a function $f(E,\omega)$ $\omega \in H^0(E,\Omega^1)$

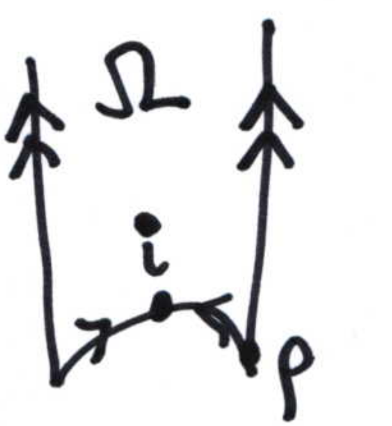
s.t $f(E,\mu\omega) = \mu^{-k} f(E,\omega)$

<u>OR</u>

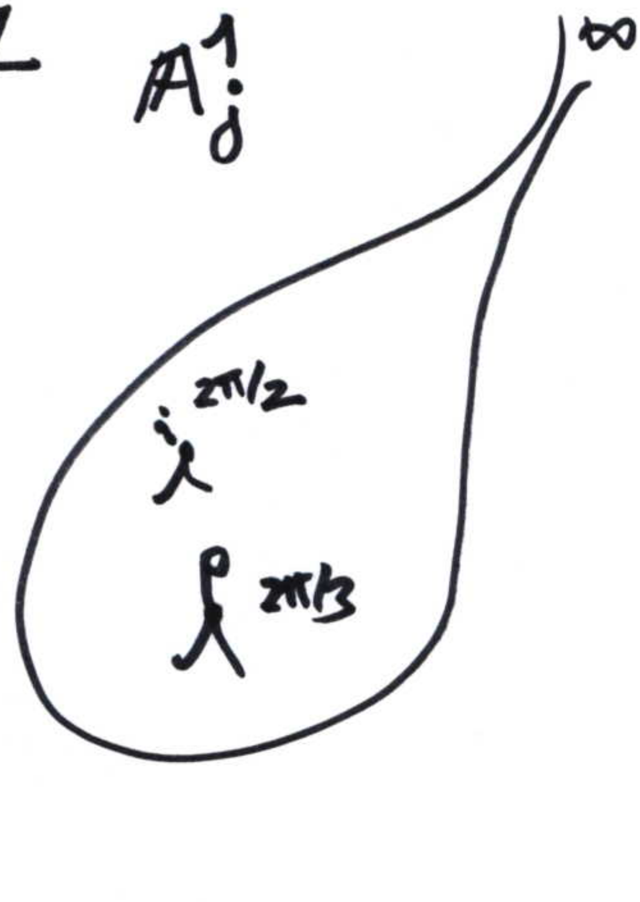
$$f(E,\omega)\,\omega^{\otimes k} = f(E,\mu\omega)\,(\mu\omega)^{\otimes k}.$$

§ Geometry.

$Ell/\mathbb{C}$ $\quad \mathbb{H}/SL_2\mathbb{Z}$ $\quad \mathbb{A}^1_j$



$\rho = e^{2\pi i/6}.$



Level structures

instead of Λ,
consider pairs

$$\left\{ \Lambda \underset{\text{index } p.}{\subseteq} \Lambda' \right\}$$

$$\Big\downarrow (p+1).$$

$$\{ \Lambda \}$$

$$\Lambda \subseteq \Lambda' \subseteq \tfrac{1}{p}\Lambda$$

$$\Lambda' \longrightarrow \tfrac{1}{p}\Lambda/\Lambda \subseteq \mathbb{C}/\Lambda$$

$$P \subseteq E[p] \subseteq E.$$

$$\Lambda = \{\tau\mathbb{Z} + \mathbb{Z}\}$$

$$\cap\!|$$

$$\Lambda' = \{\tau\mathbb{Z} + \frac{\mathbb{Z}}{p}\}.$$

$$\Lambda_\tau \subseteq \Lambda'_\tau.$$

$$\Updownarrow$$

$$\Lambda_{\tau'} \quad \text{if} \quad \tau' = \frac{a\tau+b}{c\tau+d}$$

$$\Lambda_{\tau'} \subseteq \Lambda'_{\tau'}$$

$$\tau'\mathbb{Z}+\mathbb{Z} \qquad \tau'\mathbb{Z} + \frac{\mathbb{Z}}{p}$$

$$\parallel$$

$$\{\frac{a\tau+b}{c\tau+d}\mathbb{Z} + \frac{\mathbb{Z}}{p}\}$$

$$= \{(a\tau+b)\mathbb{Z} + (\frac{c\tau+d}{p})\mathbb{Z}\}$$

$$\{\text{Ellip}/\mathbb{C} + P \underset{p}{\subseteq} E[p]\}/\sim$$

$$\Big\updownarrow$$

$$\mathbb{H}/\Gamma_0(p)$$

$$\Gamma_0(p) := \left\{ \begin{pmatrix} a & b \\ c & d \end{pmatrix} \;\middle|\; c \equiv 0 \bmod p \right\}.$$

$$\Gamma_1(p) := \left\{ \begin{pmatrix} a & b \\ c & d \end{pmatrix} \;\middle|\; \gamma \equiv \begin{pmatrix} 1 & * \\ 0 & 1 \end{pmatrix} \bmod p \right\}.$$

$$\Gamma(p) := \left\{ \begin{pmatrix} a & b \\ c & d \end{pmatrix} \;\middle|\; \gamma \equiv \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \bmod p \right\}.$$

$$Y_0(p) = \mathbb{H}/\Gamma_0(p)$$

$$Y_1(p) = \mathbb{H}/\Gamma_1(p). \text{ etc.}$$

$$Y(\Gamma) = \mathbb{H}/\Gamma.$$

Alg. description of mod forms.

$\underline{k=2}$. $$f\left(\frac{a\tau+b}{c\tau+d}\right) = (c\tau+d)^2 f(\tau)$$

$$d\left(\frac{a\tau+b}{c\tau+d}\right) = \frac{1}{(c\tau+d)^2}\, d\tau$$

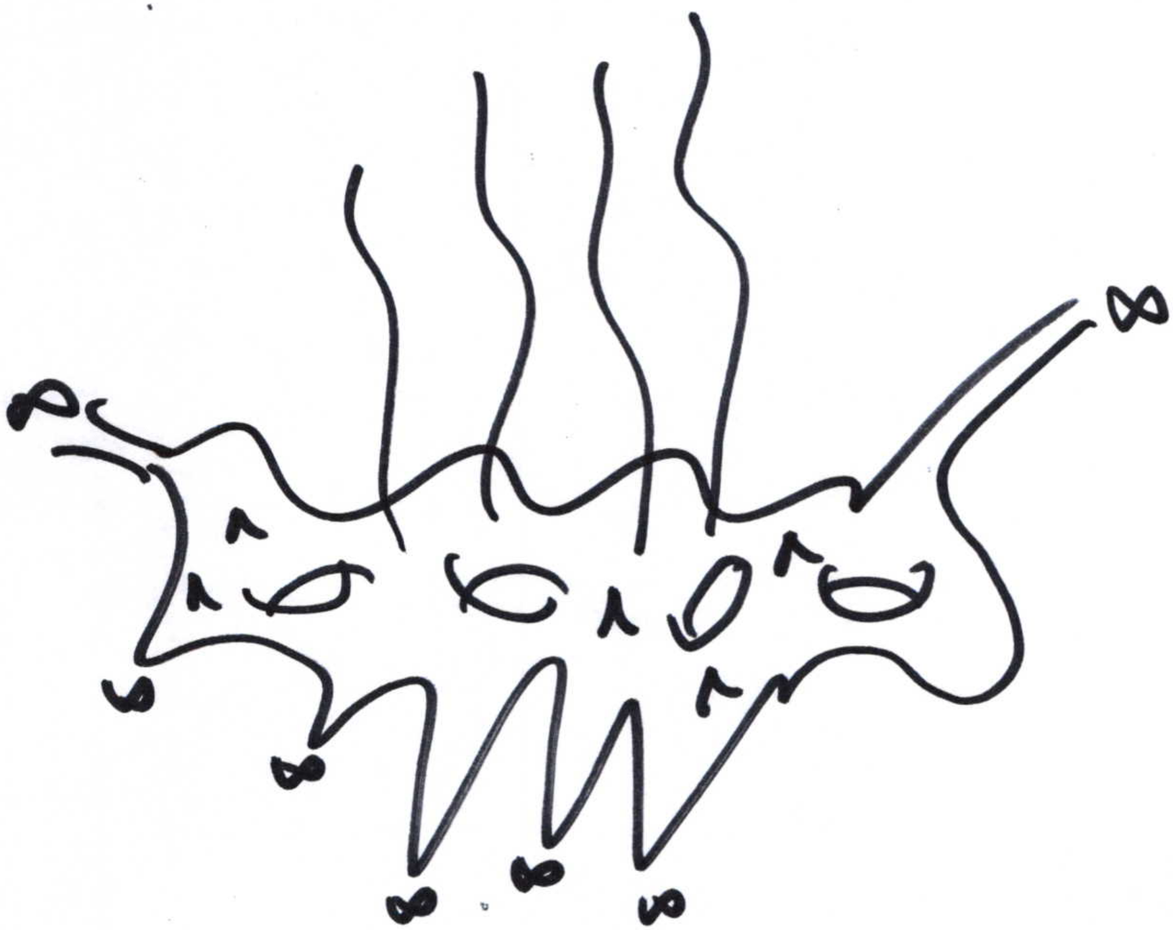
"$f(\tau)\,d\tau$"

$H^0(Y(\Gamma), \Omega^1)$

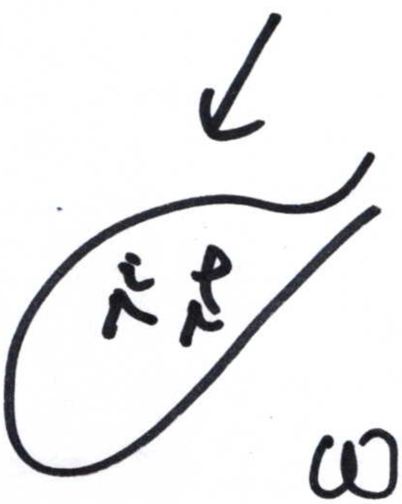
$k=k$. $f(\tau)\, dz^{\otimes k}$.

what is dz?

each point x of $Y(\Gamma)$
have an elliptic curve E_x
$dz \in H^0(E_x, \Omega^1_{E_x})$



$$\begin{tikzcd} \mathcal{E} \arrow[d] \\ Y \end{tikzcd}$$



$\Omega^1_{\mathcal{E}/Y}$ lc on $\mathcal{E}$.

$$\omega := \pi_* \Omega^1_{\mathcal{E}/Y}$$

$$\omega_x := (\pi_* \Omega^1_{\mathcal{E}/Y})_x$$
$$= \pi_* (\Omega^1_{\mathcal{E}/Y} \to x)$$

$$= \pi_* \Omega^1_{E_x/\mathbb{C}}.$$

$$= H^0(E_x, \Omega^1_{E_x}).$$

modular forms of weight k
as sections of
$H^0(Y, \omega^{\otimes k})$

$k=2$ $\quad H^0(Y, \omega^{\otimes 2}), H^0(Y, \Omega^1)$

$$\underbrace{\omega^{\otimes 2} \cong \Omega^1}_{\text{on } Y}.$$

better for $Y = Y_1(p), Y(p)$
rather than $Y(1)$ itself.

$X(1)$:

$H^0(X(1), \omega^{\otimes k})$

$\underbrace{H^0(\mathbb{P}^1, \mathcal{O}(dk))}$

horribly wrong
$d = \frac{1}{12}$.

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§ <u>Cusps</u> $E = \mathbb{C}/\Lambda$

$\tau \longrightarrow i\infty$. $q = e^{2\pi i\tau} \longrightarrow 0$.

$\mathbb{C}/\mathbb{Z}\tau+\mathbb{Z} \xrightarrow{\exp(2\pi i x)} \mathbb{C}^\times/q^{\mathbb{Z}}$.

$y^2 = 4x^3 - 60\,G_4(q)\,x - 140\,G_6(q)$

$$y^2 = 4x^3 - Ax - B$$

$A, B \in \mathbb{Z}[\tfrac{1}{6}][[q]]$.

Elliptic curve over $\mathbb{Z}[\tfrac{1}{6}][[q]]$

$$\Delta = q - 24q^2 + 252q^3 \cdots$$
$$= q \prod_{n=1}^{\infty} (1-q^n)^{24}.$$

Equation defines an elliptic curve

$T(q) \,/\, \mathbb{Z}((q))$

---

modular form of weight k

(E, ω)

$\mathbb{C}/\Lambda$: dz.

$$z \longmapsto e^{2\pi i z} = t$$

$$dt = 2\pi i \, e^{2\pi i z} dz$$

$$\frac{dt}{t} \cdot \frac{1}{2\pi i} = dz.$$

$T(q)$ comes with $\omega_{can} = \frac{dt}{t}$.

$\mathbb{C}^\times / q^{\mathbb{Z}}$. $\qquad \mathbb{G}_m / q^{\mathbb{Z}}$

$$\mathbb{G}_m \longleftrightarrow \mathbb{Z}[t, t^{-1}]$$

Given f weight k

$$f(T(q), \omega_{can}) \in \mathbb{Z}((q)).$$

this is the q-expansion of f.

Define a modular form of weight k over R.

$$f(E_R, \omega) \qquad \omega \in H^0(E, \Omega^1)$$

nowhere vanishing

$$f(E_R, \mu\omega) = f(E_R, \omega) \cdot \mu^{-k}$$

$$\mu \in R^\times$$

$$\phi : R \longrightarrow S$$

$$\phi(f(E_R, \omega_R)) = f(E_S, \omega_S)$$

$$f(T_R(q), \omega_{can}) \in R[\![q]\!].$$

<u>Example</u> given (E, ω)

$$y^2 = x^3 - ax - b.$$

a and b are modular forms of weights 4 and 6 respectively.

$\Delta = h(a,b)$ weight 12.

The q-expansion map is injective.

$$M_k(R) \hookrightarrow R[\![q]\!].$$

what about level structure?

$f(E, \omega, \alpha)$.

level structure

compatible wi the natural way.

~~Deligne–Rapo~~

[DR]

Y, $Y_0(p)$, $Y_1(p)$, etc.

$Y(N)$ • has a natural compactification $X(N)$.

• $X(N)$ has a smooth projective model over $\mathbb{Z}[\frac{1}{N}]$.

modular forms of level Γ, weight k, defined over R ($\Gamma \supseteq \Gamma(N)$; $R = \mathbb{Z}[\frac{1}{N}]$-alg).

describe this by

$H^0(X_R, \omega^{\otimes k})$.] (finie Γ).

- determined by their q-expansion

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map from modular forms over $R = \mathbb{Z}[\frac{1}{N}]$, leve $\Gamma \supseteq \Gamma(N)$ to $R/p = \mathbb{F}_p$ for primes $p \nmid N$.

Lemma: This is surjective if $N \geq 3$, $k \geq 2$.

$$H^0(X_{\mathbb{Z}_p}, \omega^{\otimes k})$$
$$\downarrow$$
$$H^0(X_{\mathbb{F}_p}, \omega^{\otimes k})$$

is this surjective?

yes, if $H^1(X_{\mathbb{F}_p}, \omega^{\otimes k}) = 0$.

$k \geq 2$ just for degree reasons.

$\underline{k=1}$: <u>false</u>

Mestre: $N = 1429$, $p = 2$.

Buzzard, $N = 82$, $p = 199$

Schaefer, ?

$\underline{N=1}$ $\quad (E,\omega) \quad p=2.$

$y^2 + a_1 xy + a_3 y$
$= x^3 + a_2 x^2 + a_4 x + a_6.$

all trans. (fixing ω) fix a_1.

$a_1 \in M_1(\Gamma_0(1), \mathbb{F}_2)$

---

what is the q-expansion of a_1?

1.