# Hecke Operators.

$$T_p\left(\sum a_n q^n\right) = \sum \left(a_{np} + p^{k-1} a_{n/p}\right) q^n$$

$p \nmid$ level

wt $= k$.

$$U_p\left(\sum a_n q^n\right) = \sum a_{np} q^n$$

level $\Gamma_0(p)$.

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$$T_p F(\Lambda) = \frac{1}{p} \sum_{\Lambda \subseteq_p \Lambda'} F(\Lambda').$$

$$U_p F(\Lambda \subseteq_p \Lambda'') = \frac{1}{p} \sum_{\substack{\Lambda \subseteq_p \Lambda' \\ \Lambda'' \neq \Lambda'}} F(\Lambda').$$

$$T_p f(E,\omega) = \frac{1}{p} \sum_{\phi: E \to D}' f(D, \hat{\phi}^* \omega).$$

$$\phi: E \longrightarrow D \xrightarrow[\hat{\phi}]{} E$$

$(E, P)$, $\quad \underset{\text{order } p}{P} \subseteq E[p]$

$D = E/P.$

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$T(q)$: $\quad T(q)[p] = \{ q^{\frac{1}{p}}, \zeta_p \}.$

$\quad \| \; \mathbb{G}_m / q^{\mathbb{Z}}$

$$T(q)/q^{\frac{1}{p}}\zeta_p^{\,i} = T(q^{\frac{1}{p}}\zeta_p^{\,i})$$

$$T(q)/\zeta_p = T(q^p)$$

## § Hasse Invariant

$A \in M_{p-1}(\Gamma_0(1), \mathbb{F}_p)$

- characterized by the following
- A has a simple zero at the supersingular points.
- the $q$-expansion of A is 1.
- $p \geq 5$ A lifts to $\mathbb{Z}$, ex. $E_{p-1}$.
- $p = 2, 3$ ($A^4 \bmod 8$) lifts to $E_4$
($A^3 \bmod 9$) lifts to $E_6$.

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## § p-adic modular forms

two MF are close if they are cong. mod $p^*$.

let $A \in \mathbb{Z}_p[\![q]\!]$ be a lift of Hasse Inv.

$$A \equiv 1 + p\mathbb{Z}_p[\![q]\!].$$

$$A^{p^n} \equiv 1 \mod p^{n+1}.$$

$$\lim_{n\to\infty} A^{p^n} = 1.$$

$$\lim_{n\to\infty} A^{p^n-1} = \frac{1}{A}.$$

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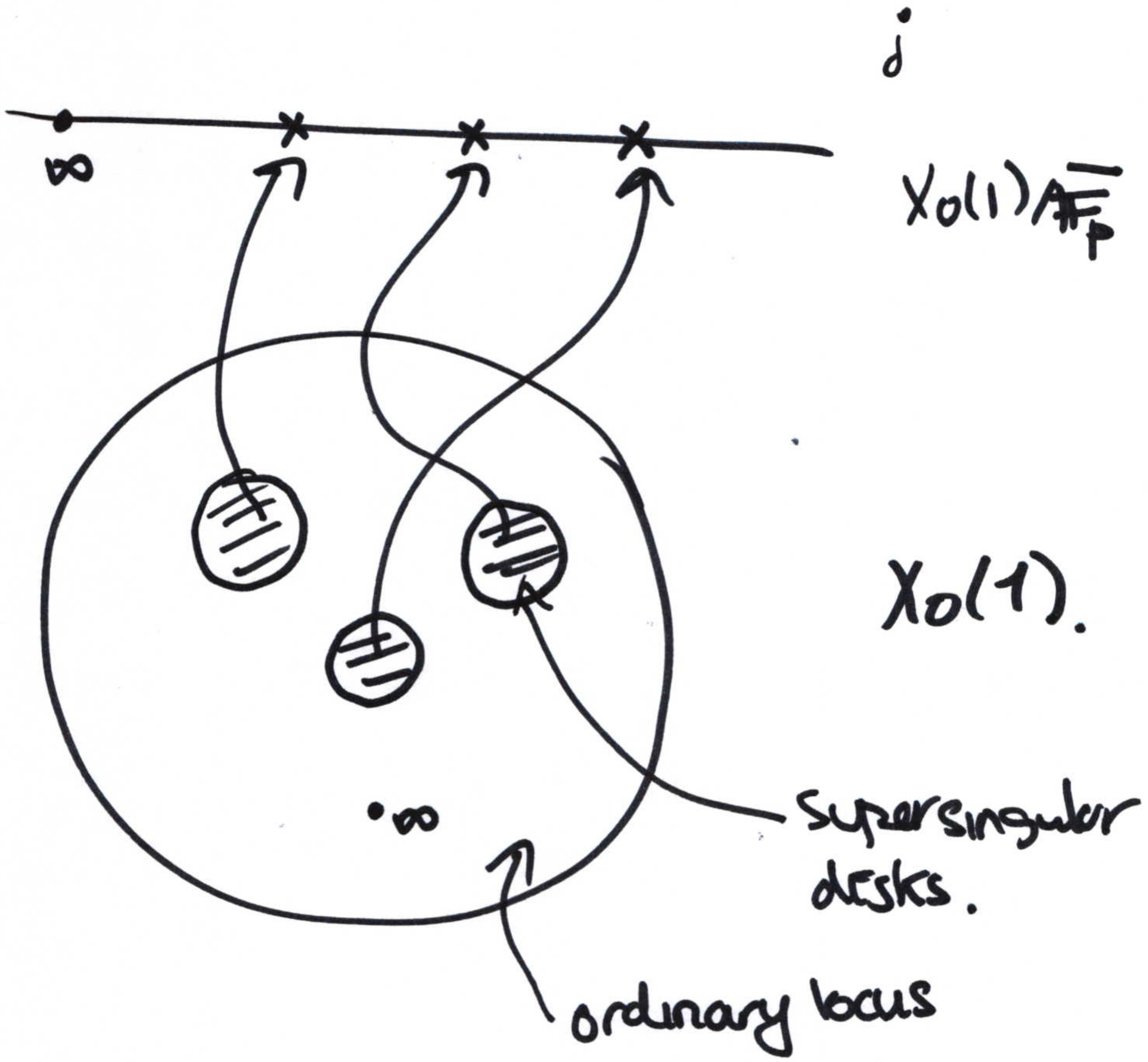
let $B$ be invertible in the top closure of all mod. forms.

$$BC \equiv 1 \bmod p.$$

reduce mod $p$: $BC = 1$.




$$H^0(X_0(1)^{ord}, \omega^{\otimes k})$$

no longer alg.
swiss cheese

<u>Def</u> p-adic modular forms.

$R$ = complete wrt $p$ $= \varprojlim R/p^n$.

$(R = \mathbb{Z}_p)$.

of weight $k$: [defined on pairs $(E/R, \omega_R)$

$\omega_R$ nowhere van $\Omega^1_{E/R}$

s.t $A(E_{R/p}, \omega_{R/p})$ invertible]

$f(E_{/R}, \mu\omega_R) = \mu^{-k} f(E_R, \omega_R)$

$\mu \in R^\times$

$f(T(q)_R, \omega_{can}) \in R[[q]]$.

comp with $R \to S$.

p-adic modular forms over $\overline{\mathbb{F}}_p$

$$
\begin{tikzcd}
H^0(X_{\overline{\mathbb{F}}_p}, \omega^{k'}) \arrow[r, phantom, "\subseteq"] \arrow[drr] & H^0(X_{\overline{\mathbb{F}}_p}-ss, \omega^{\otimes \underbrace{k+(p-1)m}_{k'}}) \arrow[d, "\cong A^{-m}"] \\
& H^0(X_{\overline{\mathbb{F}}_p}-ss, \omega^{\otimes k})
\end{tikzcd}
$$

$\uparrow$ classical mf

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F classical, wt $=k$, level $\Gamma_0(p)$ defined over $\mathbb{Z}_p$.

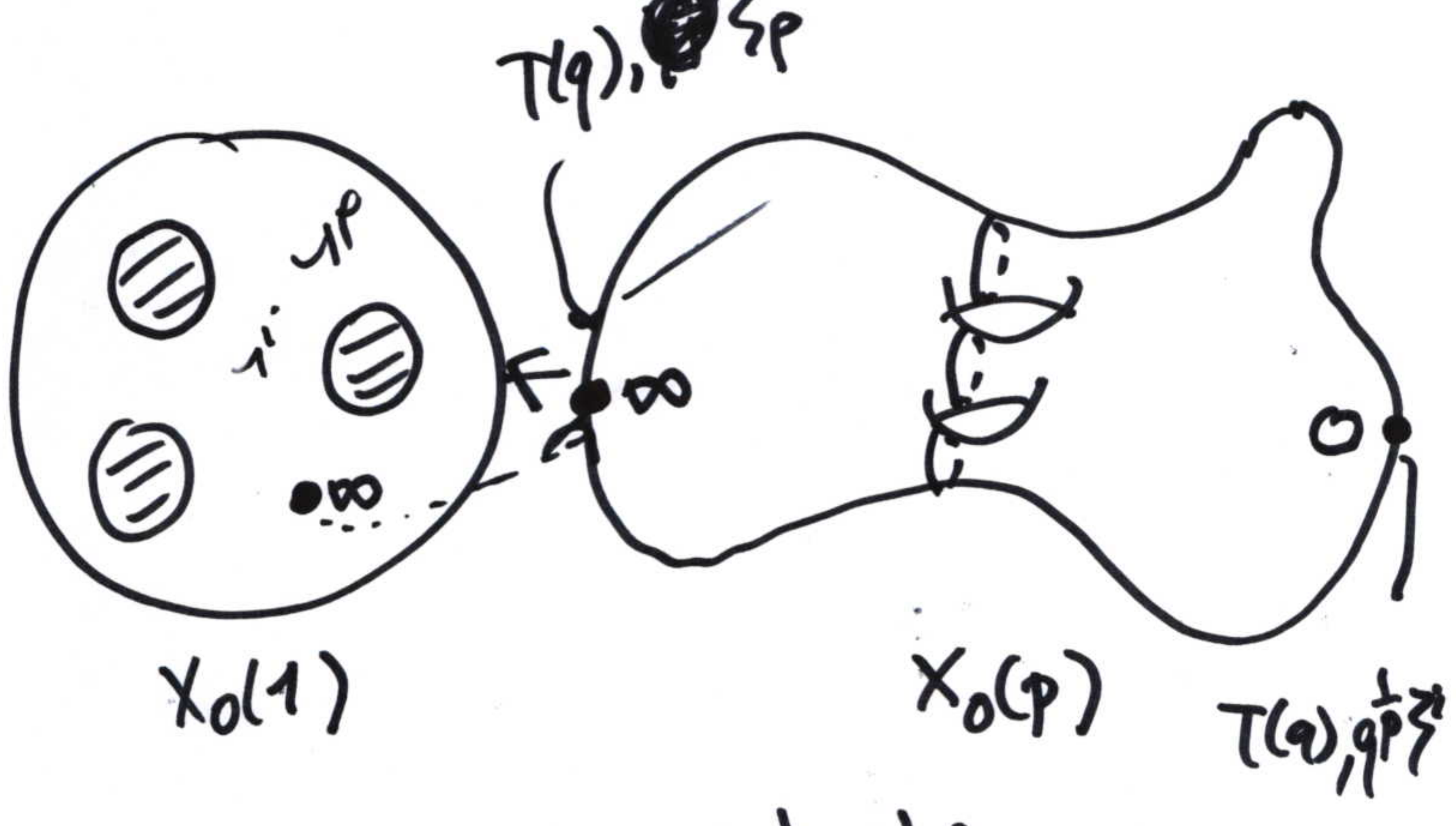
__Lemma__ F is a p-adic mod. form.

F: def on pairs $(E, P \subseteq E[p])$.

p-adic: def on $(E, \text{ordinary})$.

$$E/\overline{\mathbb{F}}_p[p] \cong \mathbb{Z}/p\mathbb{Z}.$$
$$\uparrow \text{ reduction map}$$
$$E_{\overline{\mathbb{Q}}_p}[p] \cong (\mathbb{Z}/p\mathbb{Z})^2$$

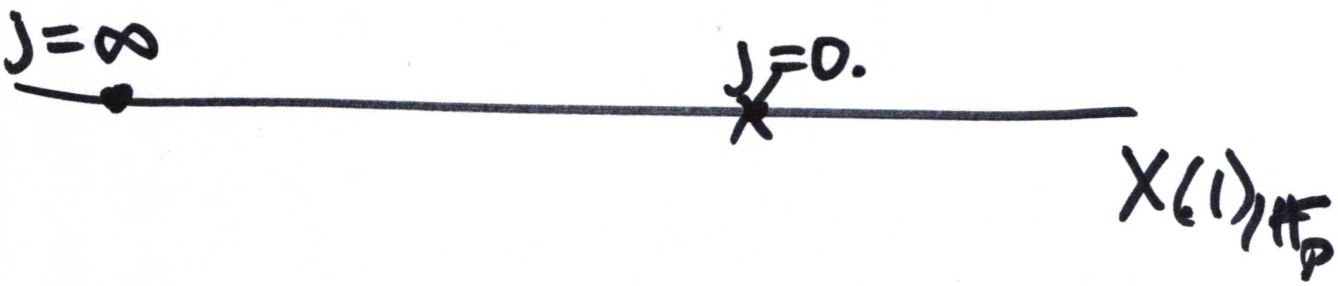
canonically, $K$ = kernel of reduction

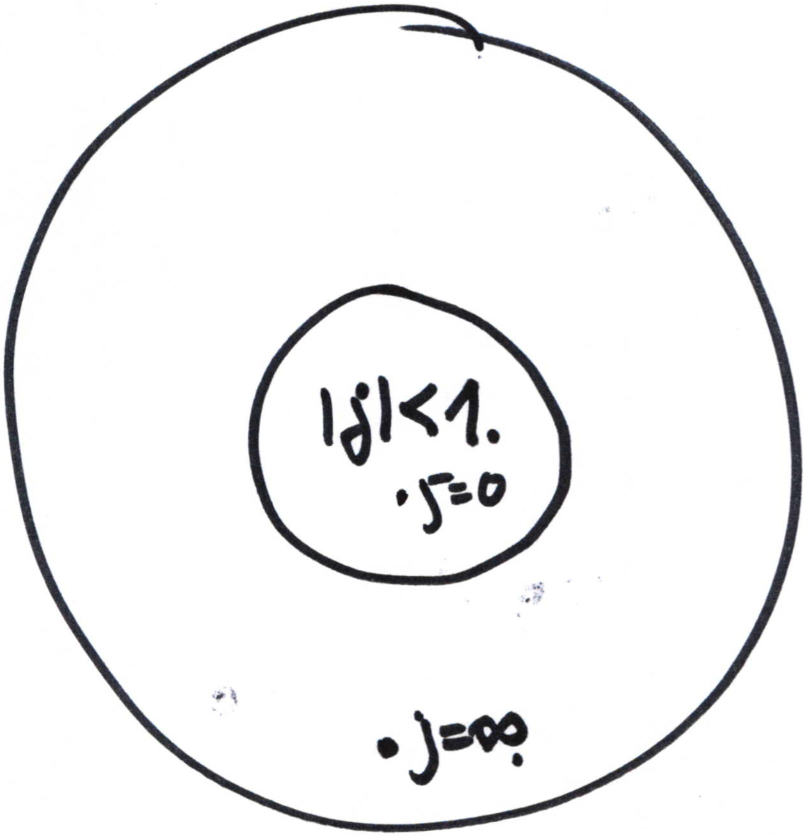



$T(q)$, $E[p] = \{ q^{\frac{1}{p}}, \zeta^{\frac{1}{p}} \}$.

$K = \{ \zeta_p \}$.

$N=1,\ p=2.$




$X(1)$

$X(1)^{ord} = \| j^{-1} \| \leq 1.$

compute p-adic modular functions.

$$H^0(X^{ord}, \mathcal{O}_X)$$

$$= \mathbb{Q}_2\langle\langle j^{-1}\rangle\rangle$$

$$= \sum_{n=0}^{\infty} a_n j^{-n}$$

$$\lim_{n\to\infty} |a_n| = 0.$$

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$$E_2 = 1 - 24\sum_{n=1}^{\infty} \sigma_1(n) q^n$$

$$\frac{E_2^6}{\Delta} = j - 864 - 191808 j^{-1} - 1042770592 j^{-2} \cdots$$