CONGRUENCES FOR CLASSICAL MODULAR FORMS

1. Course Outline

Suppose that $f = \sum_{n=1}^{\infty} a(n) q^n$ is a classical modular form of integral weight. The form f can be written as a linear combination of a finite number of *eigenforms*:

$$f = \sum_{i=1}^{m} \alpha_i f_i.$$

Since the eigenforms f_i may be associated with Galois representations (by results of Shimura and Deligne), this implies that the coefficients $a(n)$ satisfy interesting congruences for *any* modulus, a fact which was first noted and systematically explored by Serre and Swinnerton–Dyer [SD73]. This project concerns what happens if one considers instead certain *meromorphic*[1] modular forms f. For example, one could consider Klein's j-invariant:

$$j = \sum_{n=-1}^{\infty} c(n) q^n = \frac{1}{q} + 744 + 196884q + \ldots$$

One could also consider the inverse of Dedekind's eta function, which is a generating function for the partitions:

$$\eta^{-1} = \frac{1}{q^{1/24} \prod_{n=1}^{\infty} (1 - q^n)} = q^{-1/24} \sum_{n=0}^{\infty} p(n) q^n.$$

There is nothing particularly interesting about these *particular* q-expansions, although they have a certain historical significance in the subject and thus provide a good example as any. Congruences for $c(n)$ and $p(n)$ were first observed by Ramanujan, and the first systematic congruences were established by Watson [Wat38] in 1938. Approximately 50 years later, Gouvêa and Mazur [GM95] made several conjectures concerning p-adic properties of modular forms which imply very general congruences for all suitable f and for all primes p. Recall that U_p is the Hecke operator which acts (on q-expansions) by $U_p \sum a(n) q^n = \sum a(pn) q^n$. Gouvêa and Mazur's conjecture predicts that $U_p f$ can be decomposed as an *infinite* sum of Hecke eigenforms, i.e.,

$$\sum_{n=0}^{\infty} c(pn) q^n = \sum_{i=1}^{\infty} \alpha_i f_i.$$

Here this sum takes place as power series in $\mathbf{C}_p$ (the completion of the algebraic closure $\overline{\mathbf{Q}}_p$ of $\mathbf{Q}_p$), the α_i are p-adic numbers converging to zero, and the f_i are *p-adic modular eigenforms.* The forms f_i are, moreover, eigenforms for the U_p operator, and if $U f_i = \lambda_i f_i$, then

$$|\lambda_1| \geq |\lambda_2| \geq |\lambda_3| \geq \ldots$$

are all non-zero elements of $\mathbf{C}_p$ which converge to zero as i increases. The *slope* of f_i is, by definition, equal to the p-adic valuation of λ_i. As in the classical case, the forms f_i are

[1] Well, at least *some* meromorphic modular forms — the poles of f should either be at the cusps or at points whose mod-p reductions on the special fiber of the modular curve are supersingular.

associated with Galois representations, and thus one obtains congruences (for example) for $c(pn)$ of Cebotarev type for any power of p. There is a similar story with forms of half–integer weight, except now one uses the operator U_{p^2} rather than U_p. Watson's congruence for the partition function modulo powers of 5 (see the exercises below) can be interpreted as computing information concerning the *first* eigenform f_1 in this expansion, and proving that the slope of f_1 (that is, the 5-adic valuation of λ_1) is at least 2.

These expansions may be viewed as a "master congruence" which encodes all the congruences for coefficients of the initial form modulo all powers of p simultaneously. In its strongest form, this "spectral conjecture" conjecture is still open. It follows easily from results of Coleman [Col96, Col97] that there exists an *asymptotic* expansion of this form, which is already enough to deduce very strong consequences for congruences for various meromorphic modular forms. The work of Coleman may appear technical and abstract, but one of the goals of this course is to explain that it has very concrete consequences for congruences which are not usually considered in any detail. Moreover, I claim that Watson's argument and Coleman's argument are closer than they may appear at first examination. One could summarize this course as explaining how:

$$\boxed{\text{Coleman} \;=\; \text{Watson} \;+\text{algebraic geometry} \;+\text{(easy) } p\text{-adic functional analysis}}$$

To be fair to Coleman, I am talking here about the theory of overconvergent modular forms in a *fixed* weight (which is all we shall require). The idea of studying *families* of eigenforms in varying weights is also very important, but it can be ignored for the purposes of this project.

2. Projects

The project will comprise of several problems, of increasing levels of difficultly. There are also various other directions one can go even within the problems listed here. More details will be forthcoming on how to do these problems both before the workshop (for students in my group) and during the workshop. The level of difficulty of the problems is taking into account you know some of this extra information, although if you can answer these problems directly, great!

- (Child's play) Prove that $p(5n+4) \equiv 0 \mod 5$. Prove it once by proving the identity
$$\sum_{n=0}^{\infty} p(5n+4)q^n = 5\prod_{n=1}^{\infty} \frac{(1-q^{5n})^5}{(1-q^n)^6},$$
and then prove it a completely different way. Remind yourself why any particular congruence reduces to a finite computation. Compute $p(14974)$ using Euler's pentagonal number formula, compute $p(374349)$ in the same way, and compute their ratio in $\mathbf{Z}_5$. (A computer might be useful for this problem.)

- (Accessible but tricky) Let $p(n)$ denote the partition function. Watson [Wat38] proves that, for all n, there is a congruence:
$$p\left(\frac{25^k n+1}{24}\right) \equiv 0 \mod 5^{2k}.$$
Prove that there exists a 5-adic integer
$\lambda = 4\cdot 5^2 + 2\cdot 5^3 + 3\cdot 5^4 + 5^5 + 2\cdot 5^7 + 3\cdot 5^9 + 5^{11} + 5^{12} + 5^{14} + 4\cdot 5^{15} + \ldots$

and a q-expansion $\theta \in \mathbf{Z}_5[[q]]$ which is an eigenform for U_{25} as well as the Hecke operators T_{ℓ^2} for $(\ell, 60) = 1$ such that

$$\sum_{n=0}^{\infty} p\left(\frac{25^k n + 1}{24}\right) q^n \equiv \lambda^k \cdot \theta \mod 5^{7k}.$$

- (Easier variant) Answer the previous problem, with the congruence 5^{7k} replaced by the weaker congruence 5^{7k-m} for some fixed constant m.

- (Moderately difficult) Fix a prime p, and consider the ordinary projection θ of η^{-1} (explicitly, the limit $\theta := \lim U_{p^2}^{n!}\eta^{-1}$). The form θ can be written as a finite sum of generalized eigenforms. Prove that
$$U_{p^2}^n \eta^{-1} \equiv U_{p^2}^n \theta \mod p^n.$$

- (Linear algebra) Determine whether their exists an $\infty \times \infty$ matrix M with coefficients in $\mathbf{Z}_p$ such that:
  (1) M is symmetric. ($m_{ij} = m_{ji}$.)
  (2) If B is the Banach space of convergent sequences in $\mathbf{Q}_p$, then M acts compactly on B in the natural way. ($\lim m_{ij} = 0$.)
  (3) The kernel of M is trivial.
  (4) The characteristic power series of M is trivial; equivalently, M is topologically nilpotent; equivalently, the trace of M^n is zero for all $n > 0$.

- (Very tricky) Prove the spectral conjecture of Gouvêa and Mazur in weight 0.

3. Reading and Homework

It might be worthwhile to take a look at the paper of Gouvêa and Mazur [GM95] — it's a very easy read. For the classical take on these congruences, look at Waton's paper [Wat38] (For a later, similar approach, see the paper by Atkin and O'Brien [AO67].) All the technical fact concerning modular forms we will need are mostly in the first chapter of Katz's Antwerp paper [Kat73]. Remind yourself what the spectral theorem for compact operators is. Feel free to look at Coleman's papers, although note that we won't require the full machinery he uses because we will be working in fixed weight. Buzzard wrote down a conjecture here [Buz05] which has associated `pari.gp/magma` scripts; play around with those programs if you can.

References

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