2. E elliptic curve $/\mathbb{Q}$

$$\rho_{E,p} : G_{\mathbb{Q}} \longrightarrow GL_2(\mathbb{Z}_p)$$
$$\downarrow$$
$$GL_2(\mathbb{Q}_p).$$

$\{\rho_{E,p}\}$ compatible system

$E \longleftrightarrow f$ newform of wt 2 & level $\Gamma_0(11)$.

<u>Thm</u> (modularity of elliptic curves)

$\exists$ bijection

$$\left\{\begin{array}{c}\text{isogeny classes}\\ \text{of elliptic curves}\\ E/\mathbb{Q}\end{array}\right\} \leftrightarrow \left\{\begin{array}{c}\text{normalised}\\ \text{newform } f\\ \text{of wt 2, level}\\ \Gamma_0(N)\text{, with}\\ \text{integer Fourier}\\ \text{coefficients}\end{array}\right\}$$

In fact $N = N_E$ = conductor of E

Correspondence: $a_p(f) = a_p(E)$ $(p \nmid N_E)$

Proved by passing through compatible systems of Galois repns.

$f \rightsquigarrow E \rightsquigarrow \{\rho_{E,p}\}$

[Eichler-Shimura 60s].

Idea: use the geometric interpretation of f to associate A_f, an abelian variety, to f.

Tate module of A_f gives a compatible system, and if f has integer coefficients, A_f is an elliptic curve.

Keep assumption that f has wt 2, forget the assumption on coefficients.

Let K_f = coefficient field of f.

Then $G_{K_f,\lambda}$ acts naturally on λ-Tate module of A_f, so if λ is a prime of ~~A_f~~ K_f, get

$$\rho_{f,\lambda} : G_{\mathbb{Q}} \longrightarrow GL_2(K_{f,\lambda}).$$

cts, unramified outside $N(N\lambda)$, and irreducible.

$$+\ \operatorname{tr}\rho_{f,\lambda}(\mathrm{Frob}_\ell) = a_\ell(f) \quad \ell \nmid N(N\lambda).$$

Compatible system.

A single $\rho_{f,\lambda}$ is enough to determine f, and so a single $\rho_{f,\lambda}$ is enough to determine the whole compatible system.

If f has wt $k \neq 2$, there is no A_f.

It's still possible to construct the $\rho_{f,\lambda}$:

- if $k > 2$, Deligne used étale cohomology of Kuga-Sato varieties / étale cohomology of modular curves w/ non-const. coefficients.
- $k = 1$, Deligne-Serre used congruences to construct the $\rho_{f,\lambda}$.

In this course, $\underline{k \geq 2}$.

So in general, $f \rightsquigarrow \{\rho_{f,\lambda}\}$.

Want: to go from a compatible system to a modular form.

Question does every compatible system of repns $\rho_\lambda : G_{\mathbb{Q}} \to GL_2(K_\lambda)$ come from a modular form?

[K = # field, λ = finite place of K, ρ_λ cts, ss, unramified outside $N(N\lambda)$, tr $\rho_\lambda(\mathrm{Frob}_\ell)$, $\ell \nmid N(N\lambda)$, independent of λ].

<u>Answer</u> no.

<u>Problems</u>: – given $\rho_{f,\lambda}$, if

set $\rho_\lambda = \varepsilon_\lambda^{\,n} \otimes \rho_{f,\lambda}$, $n \neq 0$, ε_λ = λ-adic cyclotomic character, ρ_λ does not come from a modular form.

– $\exists$ compatible systems of finite image reps which come from Maass forms & not modular forms.

<u>Solutions</u> – understand twists

– understand how to rule out Maass form examples.

Fact / easy calculation:

– If $c \in G_{\mathbb{Q}}$ be complex conjugation, then $\det \rho_{f,\lambda}(c) = -1$.

Say that $\rho_{f,\lambda}$ is odd.

Maass forms ~~examples~~ all have

$$\det \rho_\lambda(c) = +1 \quad [\rho_\lambda \text{ is even}]$$

Can avoid Maass forms by insisting ~~stepping~~ that $\{\rho_\lambda\}$ are odd.

Conj If $\{\rho_\lambda\}$ is a compatible system of odd representations, then $\exists\, n \in \mathbb{Z}$, f modular form s.t. $\rho_\lambda \simeq \varepsilon_\lambda^{-n} \otimes \rho_{f,\lambda}$ $\forall \lambda$.

Reasonable conj, but probably very hard to prove.

Reason this is hard is that we haven't said anything about $\rho_\lambda|_{G_{\mathbb{Q}_p}}$, $p = N\lambda$.

"Motto: ρ_λ is determined by $\rho_\lambda|_{G_{\mathbb{Q}_p}}$".

Idea: $\rho_\lambda|_{G_{\mathbb{Q}_p}}$ can be very complicated, and we should try to understand it ~~better~~.

The way we understand $\rho_\lambda|_{G_{\mathbb{Q}_p}}$ is via <u>p-adic Hodge theory</u>.

If f has weight k, then

$\rho_{f,\lambda}|_{G_{\mathbb{Q}_p}}$ is <u>de Rham</u>

with <u>Hodge-Tate weights</u>

$0, k-1$.

If we believe the Conj above, should also believe:

<u>Conj'</u> Let $\{\rho_\lambda\}$ be an ~~[illegible]~~ compatible system of odd irred. repns, with the property that $\exists$ integers a, b s.t. $b \geq 0$, and ~~[illegible]~~ for each λ, if $p = N\lambda$, then $\rho_\lambda|_{G_{\mathbb{Q}_p}}$ is de Rham with Hodge-Tate weights $a, a+b$.

Then $\exists$ f a modular form of wt $b+1$ s.t. $\rho_\lambda \otimes \varepsilon_\lambda^{-a} \cong \rho_{f,\lambda}$.

[Conj $\Rightarrow$ Conj' using ε_λ^a has Hodge-Tate weight a].

Advantage of Conj': can actually prove it in a lot of cases.

Conjecture (Fontaine–Mazur).

If $E/\mathbb{Q}_p$ is finite, and

$$\rho : G_{\mathbb{Q}} \to GL_2(E)$$

is cts, odd, irreducible, de Rham at p [$\rho|_{G_{\mathbb{Q}_p}}$ is de Rham], unramified at all but finitely many primes.

Then $\exists\, a, f$ s.t. $\rho \simeq \varepsilon_\lambda^a \otimes \rho_{f,\lambda}$

for some $\lambda | p$.

~~Ex~~ FM conj $\Rightarrow$ Conj′

[Each ρ_λ satisfies hypotheses of FM conj].

Gen Rks.

— If we drop de Rham condition, or the condition that ρ is unramified a.e., then conj. is false.

— This implies that ρ is part of a compatible system.

— f is determined uniquely by ρ.

— If ρ has distinct Hodge-Tate weights, then: — f ~~should~~ will have weight $k \geq 2$

— conjecturally, "odd" should follow from the other hypotheses.

[Proved in many cases by FC, using modularity lifting theorems]

[We will keep the assumption of oddness]

Strategy for proving Conj':

- choose a "nice" λ.
- prove FM conjecture for ρ_λ

  - prove that $\bar{\rho}_\lambda$, the reduction mod p of λ, is modular [Serre's conjecture]
  - deduce that ρ_λ itself is modular [modularity lifting theorems]

$\rho : G_\mathbb{Q} \longrightarrow GL_2(E)$ $E/\mathbb{Q}_p$ finite.

Conjugate: $\rho : G_\mathbb{Q} \to GL_2(\mathcal{O}_E)$.

Reduce modulo m_E

TG 2-12

$\bar{\rho} : G_{\mathbb{Q}} \longrightarrow GL_2(\mathbb{F})$ $\quad \mathbb{F} = \mathcal{O}_E / m_E.$

$$\begin{array}{c} \mathbb{F} \\ | \\ \mathbb{F}_p \end{array} \text{— finite.}$$

This is only well-defined up to semisimplification.

Assume: $\bar{\rho}$ is absolutely irreducible. Then $\bar{\rho}$ is well-defined, depends only on ρ.

<u>Serre's conj</u> [mod p version of FM conjecture:]

<u>Conj</u> If $\bar{\rho} : G_{\mathbb{Q}} \to GL_2(\mathbb{F})$ is cts, odd, absolutely irreducible, then it is modular, i.e. $\bar{\rho} \simeq \bar{\rho}_{f,\lambda}$ some f, λ.