3.

Serre's Conjecture

$\bar{\rho} : G_{\mathbb{Q}} \to GL_2(\bar{\mathbb{F}}_p)$ cts, irred, odd [$\det \bar{\rho}(c) = -1$].

Then $\bar{\rho}$ is modular, i.e.

$\bar{\rho} \simeq \bar{\rho}_{f,\lambda}$ some f, some $\lambda | p$.

__Idea__ assume Serre's conjecture, try to deduce Fontaine–Mazur conj.

i.e. take $\rho : G_{\mathbb{Q}} \to GL_2(\bar{\mathbb{Z}}_p)$ lifting $\bar{\rho}$, assume that $\bar{\rho}$ is modular, then deduce that ρ is modular.

Reason that this might be reasonable: there are congruences between modular forms.

$\rho_{f,\lambda} \longleftrightarrow f$

$\bar{\rho}_{f,\lambda}$ does not determine f

In fact, $\exists \infty$ly many g with

$$\bar{\rho}_{f,\lambda} \simeq \bar{\rho}_{g,\lambda}.$$

In particular, there are ∞ly many possibilities for weight / level of g.

e.g. Level: suppose f has level N.
Then $\rho_{f,\lambda}$ is unramified outside primes dividing Np

$\Rightarrow \bar{\rho}_{f,\lambda}$ is unram. outside Np.

If g has level M, then

$\bar{\rho}_{g,\lambda}$ is unram outside Mp.

In fact, if f (resp. g) is a newform of level N (resp. M)

then $\rho_{f,\lambda}$ (resp. $\rho_{g,\lambda}$) are ramified at exactly the primes dividing Np (resp Mp).

So if $\bar{\rho}_{f,\lambda} \simeq \bar{\rho}_{g,\lambda}$, but $M \neq N$, then there there be primes ℓ for which e.g. $\rho_{g,\lambda}|_{G_{\mathbb{Q}_\ell}}$ is ramified, but $\bar{\rho}_{g,\lambda}|_{G_{\mathbb{Q}_\ell}}$ is unramified.

This can certainly happen, and can be understood explicitly [Exercise: think about 1D case].

In fact, if we fix f, then there will be infinitely many ℓ such that $\bar{\rho}_{f,\lambda}|_{G_{\mathbb{Q}_\ell}}$ has ramified lifts.

TG3-4

Question: how do we understand what happens globally?

<u>Philosophy</u> the only obstructions are local.

i.e. if we fix f, and we want to know for which levels M $\exists$ g level M with $\bar{\rho}_{g,\lambda} \simeq \bar{\rho}_{f,\lambda}$, answer can be phrased purely in terms of $\exists$ of ~~local~~ lifts of $\bar{\rho}_{f,\lambda}|_{G_{\mathbb{Q}_\ell}}$ $\ell \mid M$ with particular ramification

<u>e.g.</u> f has level N, $\ell \| N$. Then there should be g of level N/ℓ with $\bar{\rho}_{g,\lambda} \simeq \bar{\rho}_{f,\lambda}$

$\Leftrightarrow$ $\bar{\rho}_{f,\lambda}|_{G_{\mathbb{Q}_\ell}}$ is unramified.

"Level lowering".

In particular, this implies that

$\bar{\rho}_{f,\lambda}$ has a ~~[illegible]~~ lift to char 0. $[\rho_{g,\lambda}]$ which is unramified at ℓ.

From the perspective of group theory this is mysterious.

Similarly, if you think of this as a statement about modular forms, it's even more mysterious: can translate the condition that $\bar{\rho}_{f,\lambda}$ is unramified to a condition on $a_f(\ell)$, and then state a purely "modular form world" conjecture.

Historically, results like this went into the proof of Serre's conjecture, FLT ...

TG3-6

<u>Now</u> can deduce level lowering / level raising from modularity lifting theorems in a more transparent way.

[See project on Weds].

<u>Mazur's idea</u> study the congruences between Galois reps in a systematic way.

$\bar{\rho} : G_{\mathbb{Q}} \to GL_2(\mathbb{F})$ abs. irred.

Fix $\mathcal{O} = \mathcal{O}_E$, $E/\mathbb{Q}_p$ finite,

$\mathcal{O}/\varpi_{\mathcal{O}} = \mathbb{F}$.

Let

$\mathcal{C}_{\mathcal{O}} = \{$complete local noetherian $\mathcal{O}$-algebras with residue field $\mathbb{F}\}$.

e.g. $\mathcal{O}$, $\mathcal{O}[[x_1, \ldots, x_n]]$.

Then a <u>lift</u> of $\bar{\rho}$ is a ~~repn~~ cts repn

$$\rho : G_{\mathbb{Q}} \to GL_2(R),$$

$R \in \mathrm{ob}(\mathcal{C}_{\mathcal{O}})$

s.t. $\rho \bmod m_R = \bar{\rho}$.

A <u>deformation</u> of $\bar{\rho}$ is an equivalence class of lifts,

~~by~~ $\rho \sim A \rho A^{-1}$

$$A \in \ker\left(GL_2(R) \to GL_2(\mathbb{F})\right).$$

<u>Thm</u> (Mazur) $\exists$ a universal deformation:

i.e. $R^{univ} \in \mathrm{ob}(\mathcal{C}_{\mathcal{O}})$

$+ \ \rho^{univ} : G_{\mathbb{Q},S} \to GL_2(R^{univ})$

s.t.
if $\rho: G_{\mathbb{Q},S} \to GL_2(R)$ is any
__deformation__ then $\exists!\ \theta: R^{univ} \to R$
s.t. $\rho \cong \theta \circ \rho^{univ}$.

~~$\mathbb{Q}(S) =$~~ Fix S finite set of
primes, containing p, primes where
$\bar{\rho}$ is ramified.

$\mathbb{Q}(S) =$ max extn of $\mathbb{Q}$ unramified
outside S

$G_{\mathbb{Q},S} = \mathrm{Gal}(\mathbb{Q}(S)/\mathbb{Q})$.

So R^{univ} knows about every lift of
$\bar{\rho}$ to char 0.

__Goal__ understand R^{univ}.

TG39

<u>Problem</u> R^{univ} is too big for our purposes.

<u>FM conj</u> $\rho : G_{\mathbb{Q}} \to GL_2(\mathcal{O})$ cts, ~~well~~ odd, unramified outside a finite set of primes, de Rham at p.

R^{univ} also parametrises repns which are not de Rham at p

i.e. which <u>don't</u> come from modular forms.

<u>Idea</u> cut R^{univ} down to take the de Rham condition into account.

TG 3-10

<u>Assume</u> $\bar{\rho}|_{G_{\mathbb{Q}_p}}$ is abs. irred.

Then $\exists$ a universal deformation

~~ρ~~ $\rho_p^{\text{univ}} : G_{\mathbb{Q}_p} \longrightarrow GL_2(R_p)$

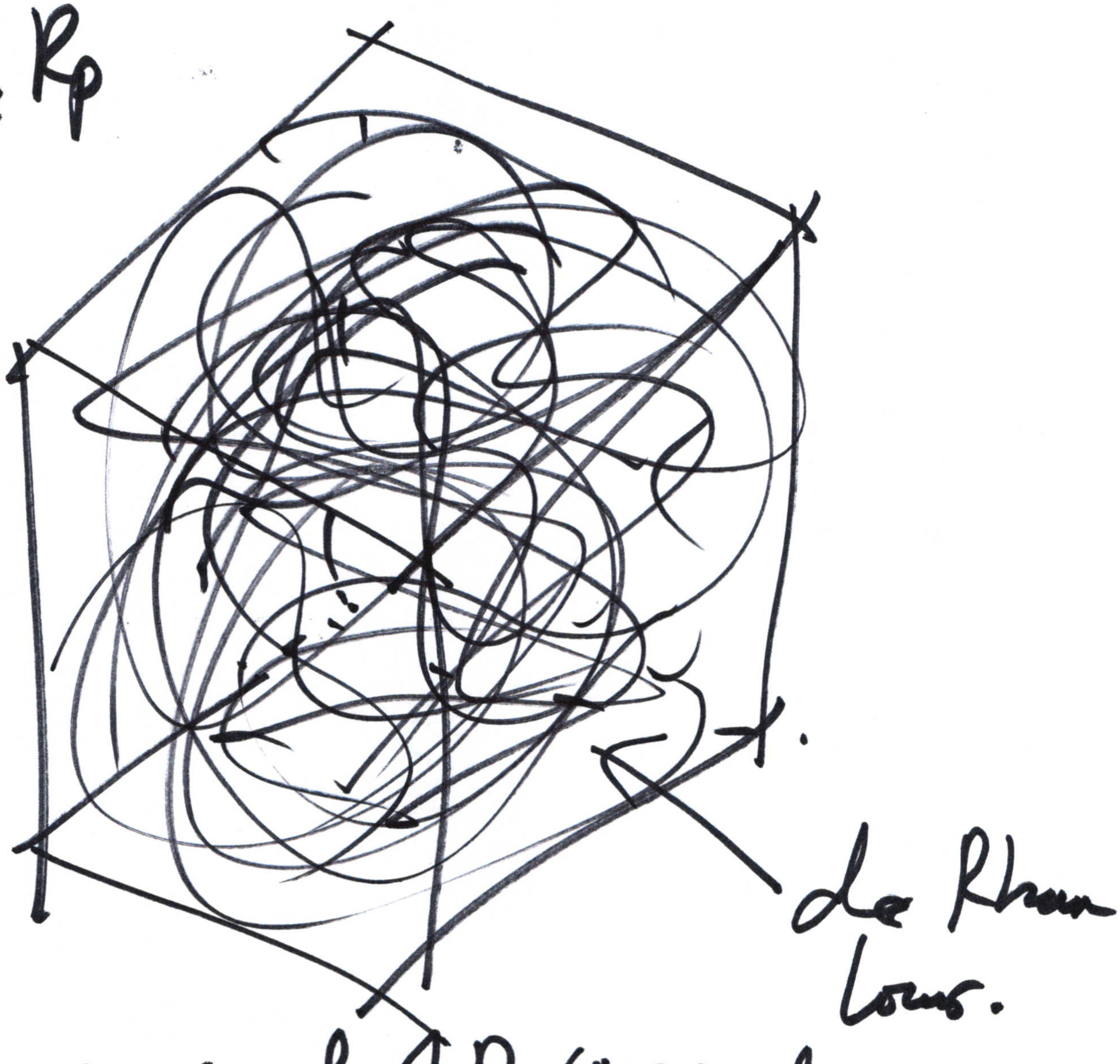
of $\bar{\rho}|_{G_{\mathbb{Q}_p}}$.

<u>Ask</u> what is the locus of ~~dR~~ de Rham representations?

TC3-11

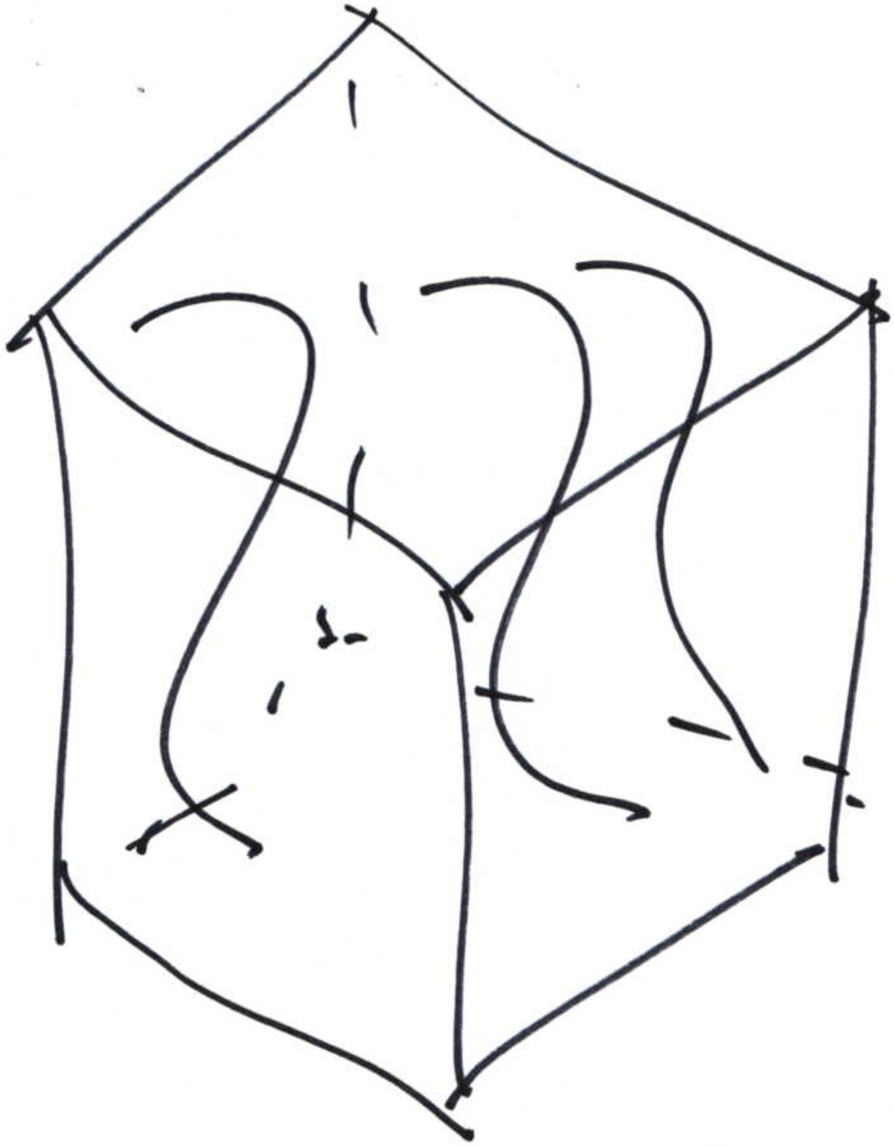
<u>Imagine</u> $R_p \simeq \mathcal{O}[[x_1, x_2, x_3]]$.
[After fixing determinants].

Spec R_p



Dense union of 1D components.

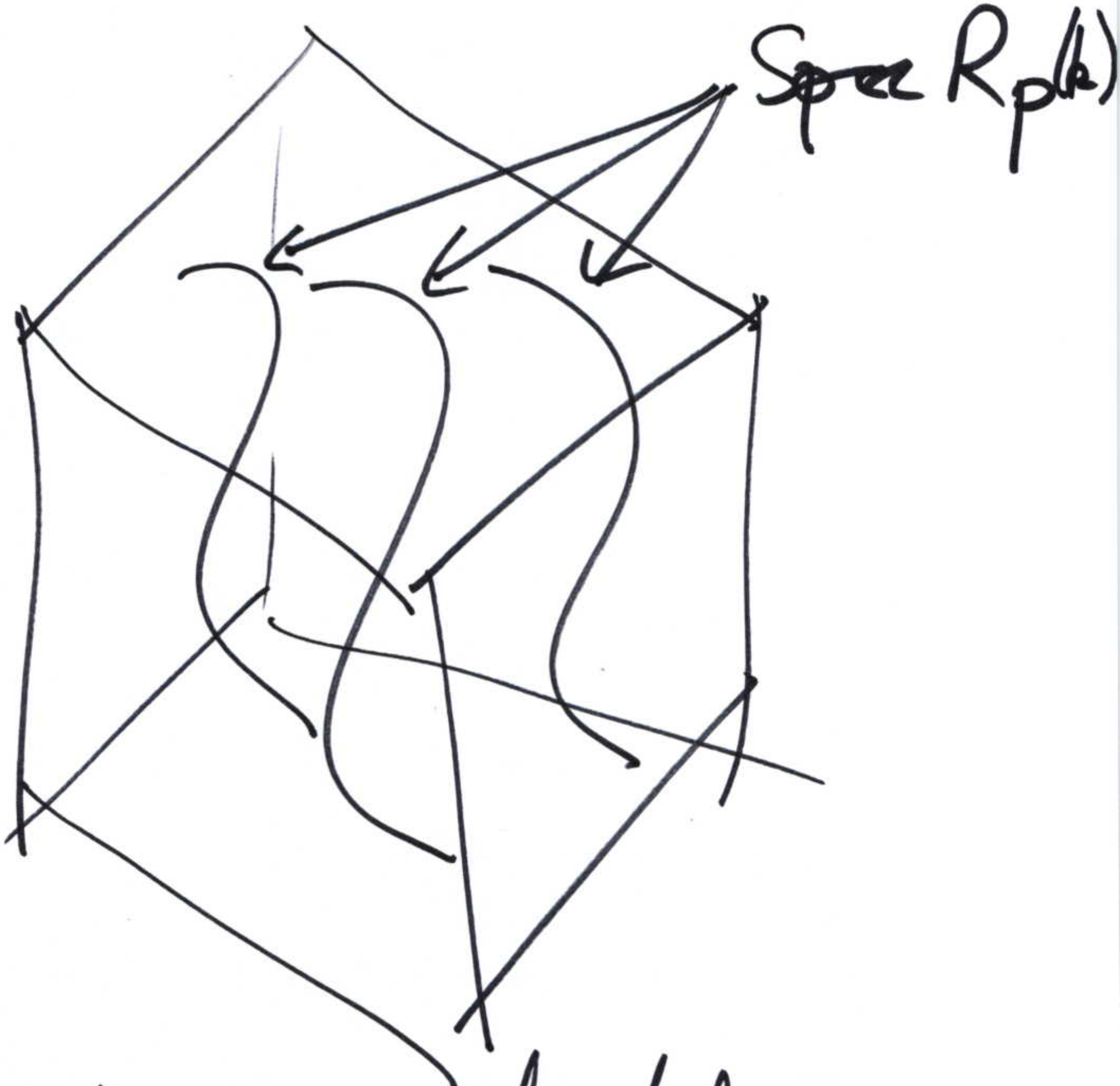
Idea: fix some finite set of components.



↳ Impose a stronger condition than de Rham.

e.g. crystalline, fixed Hodge–Tate weights.

This condition translates to:
modular forms of fixed weight,
and level not divisible by p.



$R_p(k)$ = universal deformation
ring for crystalline repns

with HT wts 0, k-1
[$\leftrightarrow$ modular forms of wt k].

Given a global repn, obtain a local one by restriction to $G_{\mathbb{Q}_p}$.

$\rightsquigarrow$ $\operatorname{Spec} R^{\mathrm{univ}} \longrightarrow \operatorname{Spec} R_p$.

Define $\operatorname{Spec} R^{\mathrm{univ}}(k)$ to be the inverse image of $\operatorname{Spec} R_p(k)$

$$\left[R^{\mathrm{univ}}(k) = R^{\mathrm{univ}} \otimes_{R_p} R_p(k) \right].$$

Then $\underline{R^{\mathrm{univ}}(k)}$ should really be parametrizing Galois repns corresponding to modular forms of wt k, level prime to p, and level | S.