4. $p \geq 5$ $\qquad$ $\mathcal{O} = \mathcal{O}_E$

$k \geq 2$. $\qquad$ $\mathbb{F} = \mathcal{O}/\varpi$.

Fix $\bar{\rho} : G_{\mathbb{Q}} \longrightarrow GL_2(\mathbb{F})$

abs. irred., cts, odd, (modular).

Serre's conj.

Also assume: $\bar{\rho}$ unramified outside p.

$\rho : G_{\mathbb{Q}} \longrightarrow GL_2(\mathcal{O})$ lift of $\bar{\rho}$,

- cts, unramified outside p.
- $\rho|_{G_{\mathbb{Q}_p}}$ crystalline, with Hodge-Tate wts $0, k-1$.

[Looks like ρ should come from a modular form of level 1 and weight k.]

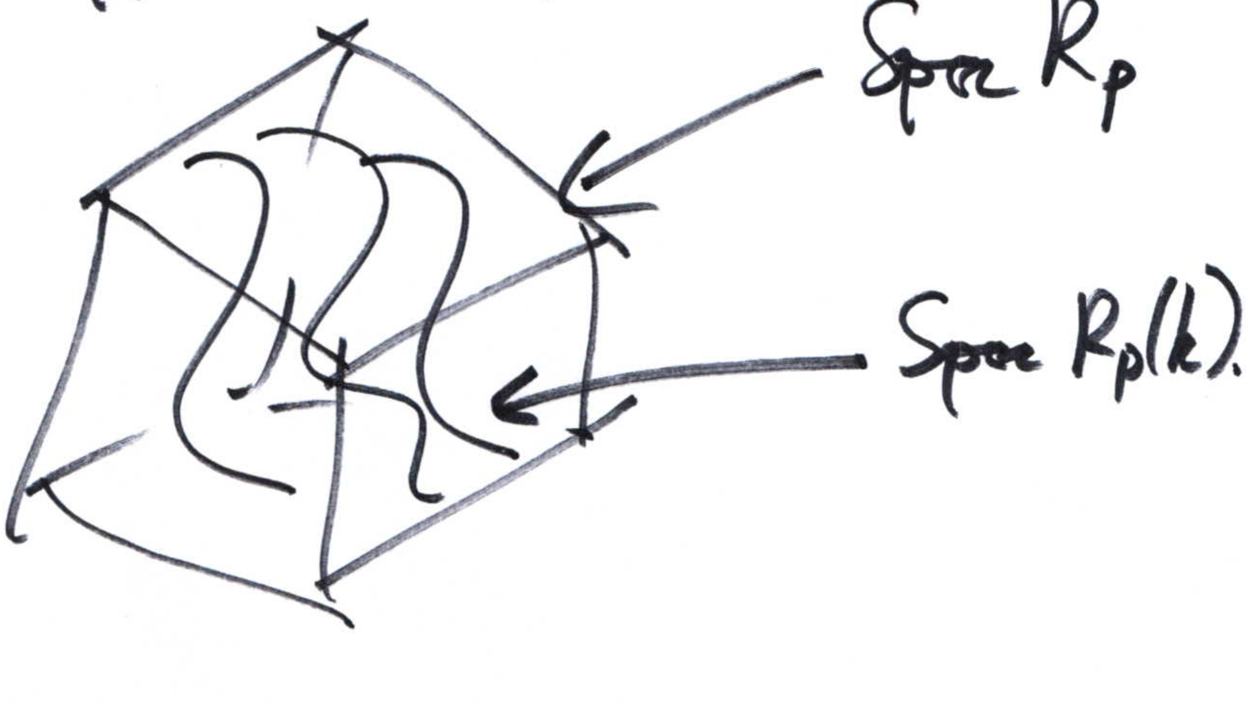
$\bar{\rho} : G_{\mathbb{Q},\{p,\infty\}} \to GL_2(\mathbb{F})$

$\rho^{univ} : G_{\mathbb{Q},\{p,\infty\}} \to GL_2(R^{univ})$

$\bar{\rho}|_{G_{\mathbb{Q}_p}}$ ← assume abs. irred.

$\rho_p : G_{\mathbb{Q}_p} \to GL_2(R_p)$
universal def.

$\rho_p(k) : G_{\mathbb{Q}_p} \to GL_2(R_p(k))$
universal deformation, crystalline
HT wts $0, k-1$.



$$R_k^{\text{univ}} = R^{\text{univ}} \otimes_{R_p} R_p(k)$$

$r: G_{\mathbb{Q},\{p,\infty\}} \longrightarrow GL_2(R_k^{\text{univ}})$

universal deformation to repns unramified outside p, crystalline w/ HT wts $0, k-1$ @ p.

$\rho: G_{\mathbb{Q},\{p,\infty\}} \longrightarrow GL_2(\mathcal{O})$

So by the universal property, $\exists$ $R_k^{\text{univ}} \xrightarrow{\theta} \mathcal{O}$ s.t.

$$\rho \simeq \theta \circ r.$$

$S_k(SL_2(\mathbb{Z}), \mathcal{O})_{\bar{\rho}}$ ← i.e. modular forms whose Galois repns are congruent to $\bar{\rho}$.

$$\mathbb{T} = \mathcal{O}[T_\ell, S_\ell]_{\ell \neq p}.$$

Then $\mathbb{T}$ acts on $S_k(SL_2(\mathbb{Z}), \mathcal{O})$ via usual Hecke operators.

Write $\mathfrak{m}$ for the maximal ideal of $\mathbb{T}$ ~~given~~ generated by p, $(T_\ell - \operatorname{tr}\bar{\rho}(\mathrm{Frob}_\ell))$, $(S_\ell - \ell \det \bar{\rho}(\mathrm{Frob}_\ell))$.

$$S_k(SL_2(\mathbb{Z}), \mathcal{O})_{\bar{\rho}} := S_k(SL_2(\mathbb{Z}), \mathcal{O})_{\mathfrak{m}}.$$

$$\mathbb{T}_k := \text{image of } \mathbb{T} \text{ in } \operatorname{End}_{\mathcal{O}}(S_k(SL_2(\mathbb{Z}), \mathcal{O}))$$

$$\mathbb{T}_{k,\mathfrak{m}} := (\mathbb{T}_k)_{\mathfrak{m}}.$$

<u>Fact/exercise</u> $\exists$ a deformation of $\bar{\rho}$

$$\rho^{\mathrm{mod}} : G_{\mathbb{Q}, \{p, \infty\}} \to GL_2(\mathbb{T}_{k,\mathfrak{m}}).$$

$$\text{s.t. } \operatorname{tr}\rho(\mathrm{Frob}_\ell) = T_\ell$$

$$\det \rho(\mathrm{Frob}_\ell) = \ell S_\ell.$$

This is unramified outside p, cryst. HT $0, k-1$

$$R_k^{univ} \twoheadrightarrow \mathbb{T}_{k,m}.$$

Now, our original ~~ogsr~~ deformation ρ corresponded to a point of $\operatorname{Spec} R_k^{univ}$

$$\begin{tikzcd} R_k^{univ} \arrow[rr, two heads] \arrow[dr] & & \mathbb{T}_{k,m} \arrow[dl, dotted, "?"] \\ & G^k & \end{tikzcd}$$

ρ is modular $\Leftrightarrow$? exists.

What we would like to prove that $R_k^{univ} \xrightarrow{\sim} \mathbb{T}_{k,m}$; in fact we only need

$$\boxed{(R_k^{univ})^{red} \xrightarrow{\sim} \mathbb{T}_{k,m}.}$$

"$R=\mathbb{T}$ theorem".

example: R_k^{univ}, $\mathbb{T}_{k,m}$ is typically something like

$$\{(r,s)\in\mathbb{Z}_p^2 \mid r\equiv s \pmod p\}.$$

i.e. there are two modular forms $f,g \in S_k(SL_2(\mathbb{Z}),\mathbb{Z}_p)$ with Galois reps congruent to $\bar{\rho}$, so that $f\equiv g \bmod p$, but $f\not\equiv g \bmod p^2$.

<u>Problem</u> get some control on R_k^{univ}.

It's hard to compute R_k^{univ} explicitly.

Easier: compute $R_p(k)$.

Strategy: study the image of $\operatorname{Spec} R_k^{univ}$ in $\operatorname{Spec} R_p(k)$.

In considerable generality, the map

$$\operatorname{Spec} R_k^{\mathrm{univ}} \longrightarrow \operatorname{Spec} R_p(k)$$

is finite.

<u>Assume</u> it's a closed embedding.

[This condition can be checked on tangent spaces, which can be computed.

Need to compute

$$\mathrm{Hom}_{\mathcal{O}}\left(R_k^{\mathrm{univ}}, \mathbb{F}[\varepsilon]/(\varepsilon^2)\right)$$

i.e. compute deformations of $\bar{\rho}$ to

$$\mathbb{F}[\varepsilon]/(\varepsilon^2).$$

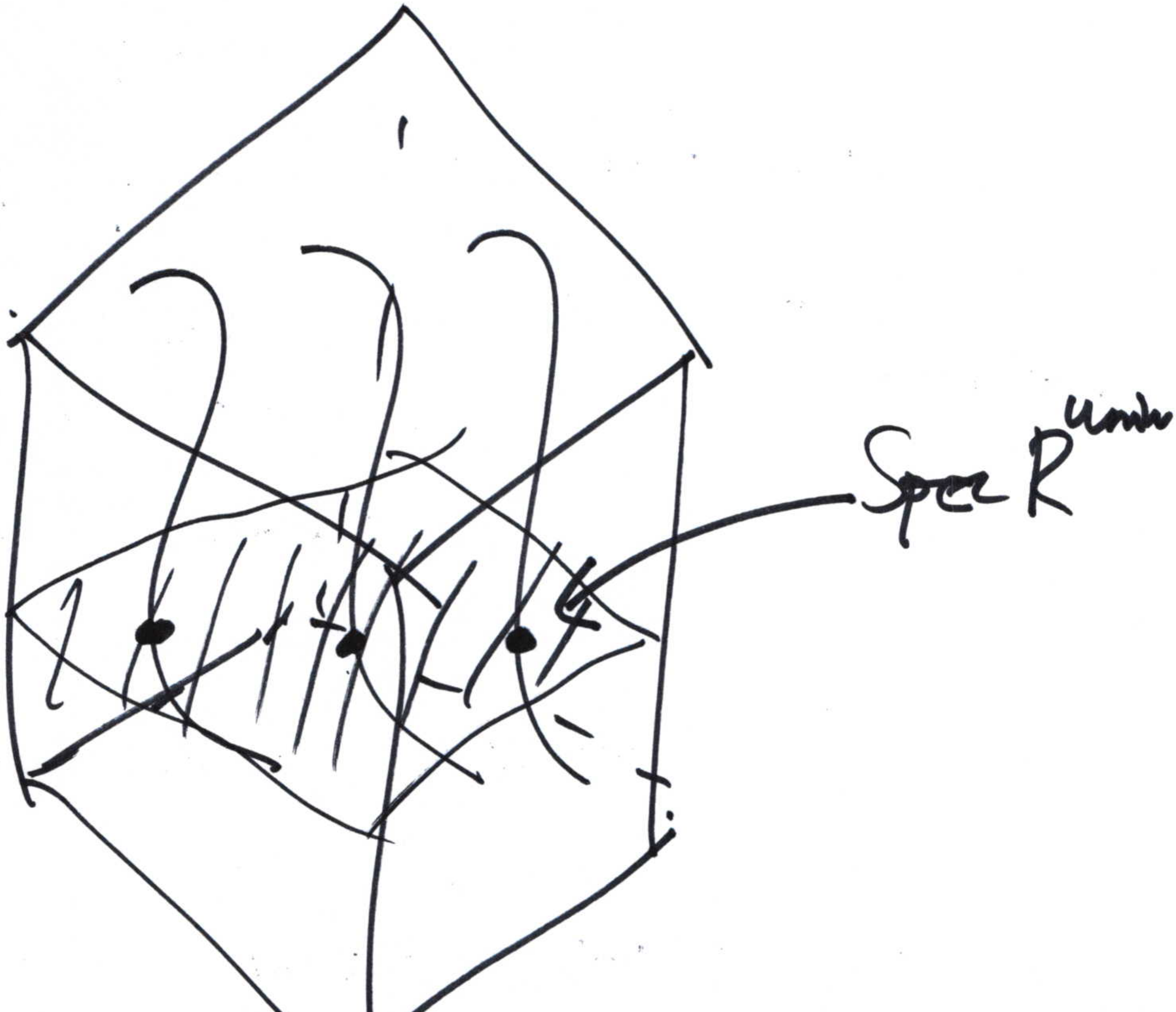
So: can compute in terms of Galois cohomology.

Condition for this to be an immersion is:

$$\ker\left(H^1(G_{\mathbb{Q}, \{p,\infty\}}, \mathrm{ad}^0_{\bar{\rho}}) \to H^1(G_{\mathbb{Q}_p}, \mathrm{ad}^0_{\bar{\rho}})\right) = (0).$$

TG4-8

$ad^0\bar{\rho}$ = trace 0 endos of $Hom(\bar{\rho},\bar{\rho})$.]

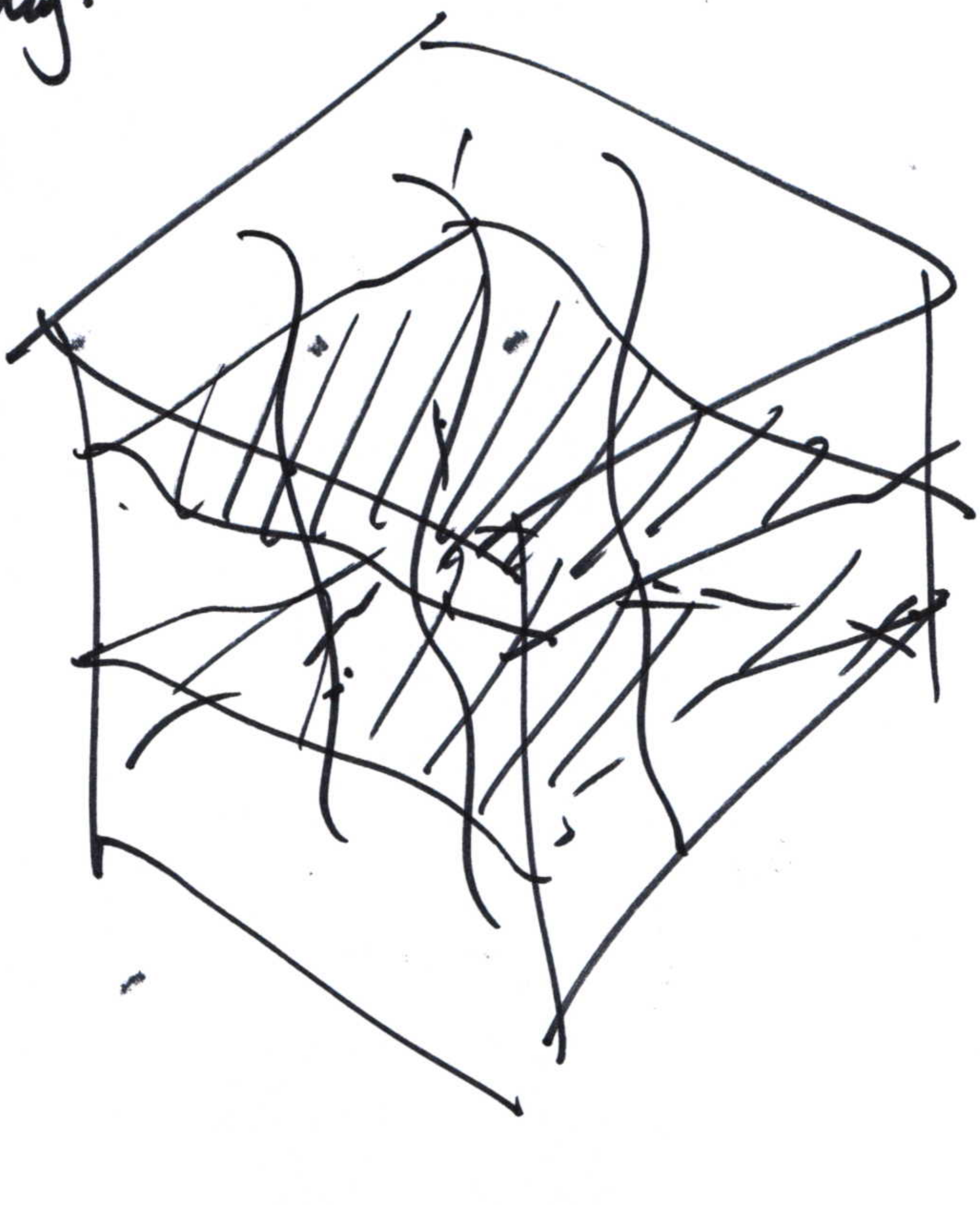


$\bullet$ = Spec $R_{\wp}^{univ}$

Idea of Taylor-Wiles method: allow the level away from p to vary.

Choose a prime $q \neq p$, and repeat the above constructions, allowing my deformation to ramify at q.

Can choose q so the map is still an embedding.



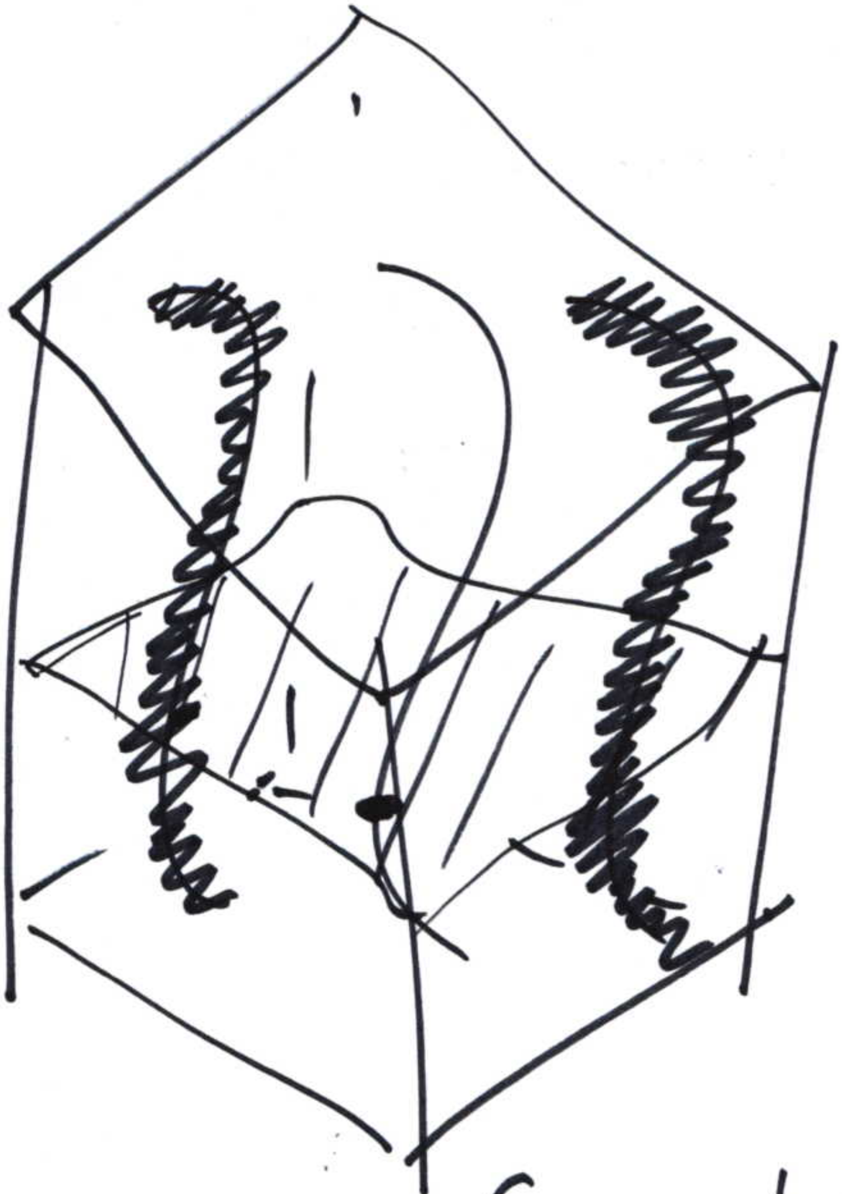
Letting q vary, we fill out Spec R_p with the images of the global deformations.

Now patch ~~these~~ everything together as q varies. In particular, patch together the spaces of modular forms

$$S_k(\Gamma_1(q), \mathcal{O})_{\bar\rho} \;\circlearrowleft\; R_p.$$

<u>Output</u>: M_∞^k, a module over $R_p^{(k)}$, i.e. a sheaf over Spec $R_p(k)$, built out of spaces of modular forms.

Commutative algebra $\Rightarrow$ support of this sheaf is a union of components of ~~R_p~~ Spec $R_p(k)$.



Now return to finite level: formally deduce that each point of Spec R_k^{univ} that lies on one of the supported components is modular.

TG 4-12

Now want to prove that ~~the~~ M_∞^k is supported on every component.

Cheat: Assume $k \leq p-1$. Then Fontaine–Laffaille theory $\Rightarrow$ $\operatorname{Spec} R_p(k)$ has only one component.

$\Rightarrow$ modularity of ρ under these assumptions.

How do we relax extra conditions?

- If the Selmer group doesn't vanish, replace $\operatorname{Spec} R_p$ ~~(k)~~ with

$$\operatorname{Spec} R_p[[x_1, \ldots, x_r]]$$

& use multiple primes $q_1, \ldots, q_s$.

- If we allow ramification away from p, TG 4-13

then we also consider local deformation spaces at ramified primes.