1. Examples

Example 1. $p(n) = \#$ partitions of n

$p(4) = 5$ $\quad 4 = 3+1 = 2+2 = 2+1+1 = 1+1+1+1$

Euler

$$\sum_{n\geq 0} p(n) q^n := \prod_{n\geq 1} \frac{1}{1-q^n} \overset{*}{=} \frac{q^{\frac{1}{24}}}{\eta(z)}$$

Dedekind eta
wgt $\frac{1}{2}$ m.f.

Note: $q = e^{2\pi i z}$ $\quad z \in \mathbb{H}$

Less well known:

$$\sum p(n) q^n = 1 + \underbrace{\sum_{n \geq 1} \frac{q^{n^2}}{(1-q)^2(1-q^2)^2 \cdots (1-q^n)^2}}_{}.$$

Modular Prototype of a "basic hypergeometric series".

Ramanujan (Last Letter to Hardy)

$$f(q) = 1 + \sum_{n \geq 1} \frac{q^{n^2}}{(1+q)^2(1+q^2)^2 \cdots (1+q^n)^2}.$$

"3rd order mock Θ-function"

K01-3

"Death bed letter"

Dear Hardy,

"I am extremely sorry for not writing you a single letter up to now. I discovered very interesting functions recently which I call "Mock" ϑ-functions. Unlike the "False" ϑ-functions (partially studied by Rogers), they enter into mathematics as beautifully as the ordinary theta functions. I am sending you with this letter some examples."

Ramanujan, January 12, 1920.

KO 1-4

"...Suppose there is a function in the Eulerian form and suppose that all or an infinity of points $q = e^{2i\pi m/n}$ *are exponential singularities and also suppose that at these points the asymptotic form of the function closes as neatly as in the cases of (A) and (B). The question is: - is the function taken the sum of two functions one of which is an ordinary theta function and the other a (trivial) function which is* $O(1)$ *at all the points* $e^{2i\pi m/n}$*? The answer is it is not necessarily so. When it is not so I call the function Mock* ϑ*-function. I have not proved rigorously that it is not necessarily so. But I have constructed a number of examples in which it is inconceivable to construct a* ϑ*-function to cut out the singularities of the original function."*

Ramanujan, 1920

Near Miss

$$b(q) := \underbrace{(1-q)(1-q^3)(1-q^5)\cdots\left(1-2q+2q^4-2q^9+-\cdots\right)}_{\text{"essentially a wgt } \frac{1}{2} \text{ m.f."}}$$

Crazy Accident If ζ is an even order $2k^{th}$ primitive root of unity, then

$$\lim_{q\to\zeta}\left(f(q)-(-1)^k b(q)\right) = O(1)$$

KOl-6

Ramanujan's question

Question (Ramanujan)

Must Eulerian series with "similar asymptotics" be the sum of a modular form and a function which is $O(1)$ at all roots of unity?

Ramanujan's Answer

The answer is it is not necessarily so.
When it is not so I call the function
Mock ϑ-function. I have not proved
rigorously that it is not necessarily
so. But I have constructed a number
of examples in which it is ~~not~~ in
-conceivable to construct a ϑ func
-tion to cut out the singularities

K01-8

Numerics

As $q \to -1$, we have

$f(-0.994) \sim -1{\cdot}10^{31}$, $f(-0.996) \sim -1{\cdot}10^{46}$, $f(-0.998) \sim -6{\cdot}10^{90}$,

$f(\text{-}0.998185) \sim$ -Googol

K01-9

Numerics continued...

Amazingly, Ramanujan's guess gives:

q	-0.990	-0.992	-0.994	-0.996	-0.998
$f(q)+b(q)$	$3.961\ldots$	$3.969\ldots$	$3.976\ldots$	$3.984\ldots$	$3.992\ldots$

This suggests that

$$\lim_{q\to -1}(f(q)+b(q)) = 4.$$

K01-10

As $q \longrightarrow i$

q	$0.992i$	$0.994i$	$0.996i$
$f(q)$	$2 \cdot 10^6 - 4.6 \cdot 10^6 i$	$2 \cdot 10^8 - 4 \cdot 10^8 i$	$1.0 \cdot 10^{12} - 2 \cdot 10^{12} i$
$f(q) - b(q)$	$\sim 0.05 + 3.85i$	$\sim 0.04 + 3.89i$	$\sim 0.03 + 3.92i$

This suggests that

$$\lim_{q \to i} (f(q) - b(q)) = 4i.$$

Example 2 (Zagier)

$$\mathcal{H}(z) := -\frac{1}{12} + \sum_{n \equiv 0,3(4)} H(-n) q^n$$

$H(-n) =$ "Hurwitz class #" for disc. $-n$

Example 3

$$E_2(z) = 1 - 24 \sum_{n \geq 1} \sigma_1(n) q^n$$

$$\downarrow$$

$$\sum_{d \mid n} d$$

"Nearly modular form"

Example 4 M_{24} Mathieu Group

$$\mathcal{M}(\tau) := -2q^{-\frac{1}{8}}\left[1 - \sum_{n\geq 1} a(n) q^n\right]$$

"Elliptic genus"

n	1	2	3	4	5	....
a(n)	45	231	770	2277	5796	

subexponential growth...

KO 1-13

n	1	2	3	4	5	6	7	8	$\cdots$
A_n	45	231	770	2277	5796	13915	30843	65550	$\cdots$

$$1, 23, 45, 45, 231, 231, 252, 253, 483, 770, 770, 990, 990, 1035, 1035, 1035,$$
$$1265, 1771, 2024, 2277, 3312, 3520, 5313, 5544, 5796, 10395.$$

One sees that A_1, A_2, A_3, A_4 and A_5 are dimensions, while

$$A_6 = 3520 + 10395 \qquad \text{and} \qquad A_7 = 10395 + 5796 + 5544 + 5313 + 2024 + 1771.$$

Conjecture (Moonshine). *The Fourier coefficients A_n of $H(\tau)$ are given as special sums*[2] *of dimensions of irreducible representations of the simple sporadic group M_{24}.*

Example 5. Bruinier - I constructed

$$g(z) = q^{-1} + \underbrace{1.0267q^{3} + 1.2205q^{4} + 1.6509q^{7}}_{\text{Transcendental}} + \cdots + 6q^{139}$$

$$- \underbrace{0.8313q^{151} + \cdots - 121.1999q^{815}}_{\text{Transcendental}} + 312q^{823} + \cdots$$

Question. What is the significance of 139, 823?

Fact: $E/\mathbb{Q}$: $y^2 = x^3 + 10x^2 - 20x + 8$

$N_E = 37$

quad twist $\downarrow$

$\mathrm{rk}(E(-139)) = 3.$

Fact: $g(z) = q^{-1} + \sum_{n \geq 1} a(n) q^n$

Thm. If $-n$ is a f.d. ~~of~~ and

$\mathrm{sfc}\left(E(-n)\right) = -1$, then

$L'(E(-n), 1) = 0 \iff a(n) \in \mathbb{Z}$.

Question. What unifies these 5 examples?

Answer: Harmonic Weak Maass Forms..

All of these 5 examples give rise to harmonic weak Maass forms by some method of "completion".

Easiest Example $E_2(z) = 1 - 24 \sum_{n \geq 1} \sigma_1(n) q^n$.

Fix = completion

$$\Longrightarrow \quad E_2^*(z) := -\frac{3}{\pi \operatorname{Im} z} + E_2(z)$$

Exercise

$$E_2^*(-1/z) = z^2 E_2^*(z)$$

$$E_2^*(z+1) = E_2^*(z)$$

$\Longrightarrow$ nonholomorphic wgt 2 m.f.

<u>Hurwitz Example:</u> $\mathcal{H}(z) = -\frac{1}{12} + \sum_{n \geq 1} H(-n) q^n.$

Zagier

$$G(z) := \mathcal{H}(z) + \frac{1}{16\pi\sqrt{\operatorname{Im} z}} \sum_{n \in \mathbb{Z}} \beta(4\pi n^2 \operatorname{Im} z) q^{-n^2}$$

where

$$\beta(s) := \int_1^\infty t^{-3/2} e^{-st}\, dt$$

??

<u>Fact:</u> $G(z)$ is a nonholomorphic m.f. of wgt $3/2$ on $\Gamma_0(4)$.

?? = "Period Integral of the Jacobi Θ fcn"

Natural Questions.

1) Starting with Jacobi's Θ how would we know that info about its period integrals

$\Longrightarrow$ gen fcn. for class #'s.

2) More generally, if $f \in M_k$, what info is naturally revealed from "period info" for f?

The 5 examples are explicit examples of this question.

KO 1-19

Defn's

$\mathbb{H} := \operatorname{Im} z > 0$

$z = x+iy \quad x, y \in \mathbb{R}$

Hyperbolic Laplacian $\quad k \in \mathbb{R}$

$$\Delta_k := -y^2\left(\frac{\partial^2}{\partial x^2} + \frac{\partial}{\partial y^2}\right) + iky\left(\frac{\partial}{\partial x} + i\frac{\partial}{\partial y}\right)$$

"Def" If $k \in \frac{1}{2}\mathbb{Z}$ and $\Gamma \subseteq SL_2(\mathbb{Z})$ is a congruence subgroup, then a fcn $M: \mathbb{H} \to \mathbb{C}$ is a wgt k harmonic Maass form on Γ if

1) $M(\gamma z) = (cz+d)^k M(z) \quad \forall \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \Gamma$.

2) $\Delta_k M = 0$.

These lectures will introduce :

- Fourier expansions and Hecke operators
- ξ_K - operator + Relationship between H_{2-K} + S_K
- Periods of modular forms
- Examples : Zwegers μ, Poincaré series , ...

Applications :

- Ramanujan's mock θ - functions
- Maass form singular moduli + class polynomials
- Borcherds Products
- Gross-Zagier + Waldspurger formulae