Lecture 2: Basic Facts

Hyperbolic Laplacian $k \in \mathbb{R}$

$$\Delta_k := -y^2\left(\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2}\right) + iky\left(\frac{\partial}{\partial x} + i\frac{\partial}{\partial y}\right)$$

$$z := x+iy \in \mathbb{H}$$

Def. A real analytic fcn $M: \mathbb{H} \to \mathbb{C}$ is a wgt k harmonic Maass form on Γ if:

1) $M\left(\frac{az+b}{cz+d}\right) \text{ "=" } (cz+d)^k M(z) \quad \forall \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \Gamma$

2) $\Delta_k M = 0$

3) There is a polynomial $P_M(q^{-1}) \in \mathbb{C}[q^{-1}]$ s.t.

$$M(z) - P_M(z) = O(e^{-\varepsilon y})$$

for some $\varepsilon > 0$ as $y \to \infty$. [Principal Part at ∞].

Ex. $j(z) - 744 = q^{-1} + 196884q + \dots$

$P_j(j-744) = q^{-1}$. Principal Parts at ∞

Note: E_2^* is not a HMF in this sense.

Fourier Expansion $1 < K \in \frac{1}{2}\mathbb{Z}$ (simplicity)

Incomplete Γ-function:

$$\Gamma(\alpha; x) := \int_x^\infty e^{-t} t^{\alpha-1} \, dt.$$

Lemma Suppose $f \in H_{2-k}(\Gamma)$, where $\begin{pmatrix}1&1\\0&1\end{pmatrix} \in \Gamma$.

At ∞ f has an expansion of the form

$$f = \sum_{n \gg -\infty} \underbrace{c_f^{+}(n)} q^n + \sum_{n<0} \underbrace{c_f^{-}(n)} \Gamma(k-1, 4\pi|n|y) q^n .$$

Fourier coefficients $\in \mathbb{C}$

$\operatorname{Im} z = y$

Proof. Ex #1. Problem sheet.

Terminology $\quad f \in H_{2-k}(\Gamma)$

$$f^+ = \sum_{n \gg -\infty} c_f^+(n) q^n$$ "Holomorphic part" of f.

$$f^- = \sum_{n<0} c_f^-(n) \Gamma(k-1, 4\pi|n|y) q^n$$ "Nonholomorphic Part of f"

Question: What is the significance of $c_f^+(n)$ + $c_f^-(n)$?

First Question: How is $H_{2-k}(\Gamma)$ related to classical modular forms?

EZ Answer:

$$H_{2-k}(\Gamma) \supsetneq \underbrace{M^!_{2-k}(\Gamma)}_{\text{weakly hol. m.f.s}}$$

(poles allowed but must be supported at cusps)

EZ Answer #2 If $f_1, f_2 \in H_{2-k}(\Gamma)$ s.t.

$f_1^- = f_2^-$

$$\Rightarrow f_1 - f_2 = f_1^+ - f_2^+ \in M^!_{2-k}$$

Deeper Answer (ξ - operator)

$$\xi_w := 2i \cdot y^w \cdot \overline{\frac{d}{d\bar{z}}}$$

Exercise: $\xi_w(f) = \xi_w(f^-)$ ✓

Lemma Suppose $f \in H_{2-k}(\Gamma)$, then

$$\xi_{2-k}(f) \in S_k(\Gamma).$$

Moreover, we have that

$$\xi_{2-k}(f) = \xi_{2-k}(f^-) = -(4\pi)^{k-1} \sum_{n \geq 1} c_f^{-}(-n) n^{k-1} q^n \in S_k(\Gamma).$$

and

$$\xi_{2-k} : H_{2-k} \twoheadrightarrow S_k.$$

Natural Question: Given $F \in S_k(\Gamma)$,

how do we:

1) Find $f \in H_{2-k}(\Gamma)$ s.t.

$$\xi_{2-k}(f) = F.$$

[∞'ly many answers...]

2) How do you find a "good f"?

holomorphic fcn.

$$f = f^{-} + f^{+}$$

Explicit Example

$E/\mathbb{Q}$ conductor 37

$$F_E(z) \in S_2(\Gamma_0(37))$$ "Modularity"

What do we learn about E_D by "finding G_E^+"?

$$\begin{tikzcd}
& & S_2(\Gamma_0(37)) \arrow[d, phantom, "\rotatebox{-90}{$\in$}"] \\
f = f^- + f^+ \arrow[rr, "\iota"] & & F_E \arrow[dd, leftrightarrow, "\text{Shimura}^* \text{ correspondence}"] \\
& G_E^+ \in H_{\frac{1}{2}}(\Gamma_0(4\cdot 37)) & \\
G_E = G_E^- + G_E^+ \arrow[rr, "\iota_{\frac{1}{2}}"] & & g_E \in S_{3/2}(\Gamma_0(4\cdot 37))
\end{tikzcd}$$

* (Waldspurger...) Coefficients of g_E interpolate $L(E_D, 1)$ as D varies over f.d.

BSD...

Periods For Modular Forms

Fact: Suppose $f \in H_{2-k}(\Gamma)$ and $\xi_{2-k}(f) = F \in S_k(\Gamma)$

Then we have $f^- =$ "period integral of F."

$\uparrow$

"Eichler integrals"

$+$

"Period Polynomials"

Lemma.
$$i(2\pi n)^{1-k}\,\Gamma(k-1, 4\pi n y)\, q^{-n}$$
$$\shortparallel$$
$$\int_{-\bar{z}}^{i\infty} \frac{e^{2\pi i n \tau}}{(-i(\tau+z))^{2-k}}\, d\tau.$$

Hecke Eigenforms Suppose $F \in S_k(\Gamma_0(1))$ is a Hecke eigenform. Then the periods of F are, for $0 \leq n \leq k-2$, the numbers

$$L(F, n+1) := \frac{(2\pi)^{n+1}}{n!} \int_0^\infty f(it)\, t^n \, dt.$$

EZ Thm. Suppose that $2 \leq k \in 2\mathbb{Z}$, and suppose $f \in H_{2-k}(\Gamma_0(1))$ s.t. $\xi_{2-k}(f) = F \in S_{2k}$, a normalized Hecke eigenform. Define $P_{F,f}(z)$ by

$$P_{F,f}(z) := \frac{(4\pi)^{k-1}}{\Gamma(k-1)} \cdot \left[f^+(z) - f^+\left(-\tfrac{1}{z}\right) z^{k-2} \right].$$

Then we have $$\overline{P_{F,f}(\bar{z})} = \sum_{n=0}^{k-2} \frac{L(F, n+1)}{(k-2-n)!} \cdot (2\pi i z)^{k-2-n}.$$

$\in \mathbb{C}[z]$

EZ Thm: In integral wgt, the "obstruction to modularity" (i.e. $P_{F,f}$) is the "period polynomials" of Eichler.

Q: How is EZ Thm generalized when $k \in \frac{1}{2}\mathbb{Z} - \mathbb{Z}$?

No such thing as a half-int. weight period polynomial...

Example. $F = \Delta \in S_{12}(\Gamma_0(1))$.

We know

$$\xi_{-10} : H_{-10} \longrightarrow S_{12}.$$
$$f = f^- + f^+ \to \Delta$$

"By the method of Poincaré series" there is a best choice for f.

It turns out

$$!!!\ f^+ = q^{-1} - \frac{65520}{691} - \overbrace{1842.899\ldots}^{c_f^+(1)}\, q - 23274.07\ldots q^2 - \ldots$$

and it is really very ugly!

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Q: Why f^+ is a best choice?

How do we see Δ from f^+?

A: Renormalization

1842.89... ← transcendental

$$\hat{f}^+(z) := 11! \left[\underbrace{f^+(z)}_{*} - c_f^+(1) \sum_{n=1}^{\infty} \tau(n) n^{-11} \right] q^n.$$

$\Rightarrow$ $\hat{f}^+(z)$ still looks messy..

$$\Rightarrow \hat{\hat{f}}^+(z) := \left(q \frac{d}{dq}\right)^{11} \hat{f}^+(z) = \text{integer coeffs.}$$

<u>Exercise</u> If p is an ordinary prime for Δ, then

$$\lim_{n\to\infty} \frac{\hat{\hat{f}}^+ \mid U(p^n)}{\hat{\hat{c}}^+(p^n)} = \Delta. \quad (p\text{-adically})$$

<u>Remarks:</u>

1) I believe $c_f^+(n)$ is transcendental,

2) "In this case"

$c_f^+(1)$ is transcendental $\Longrightarrow$ Lehmer's Conjecture

3) If $F \in S_k(\Gamma_0(N))$ s.t. F has CM, then we know that the "best" $f \in H_{2-k}(\Gamma_0(N))$ s.t. $\xi_{2-k}(f) = F$

has $c_f^+(n)$ are algebraic integers.

Next Time:

- Traces of Singular Moduli
- Borcherds Products
- $L'(E_D,1)$, $L(E_D,1)$ for $E/\mathbb{Q}$.

by making use of the surjectivity of ξ_{2-k}.