Lecture 3 A, B 2 applications

A. Quantum Modular Forms (Zagier)

Kontsevich's Strange Function

$$F(q) = 1 + (1-q) + (1-q)(1-q^2) + \cdots\cdots$$
$$= 1 + \sum_{n=1}^{\infty} \underbrace{(1-q)(1-q^2)\cdots(1-q^n)}_{\to\ n^{\text{th}} \text{ term} \to n}$$

Facts:
1) F does not converge on any open set in $\mathbb{C}$.
2) F is a finite sum for each of root of unity.
3) * Analytic continuation of Feynman $\int$
4) * It turns of that

$q \to e^{-t}$

$$F(e^{-t}) = \sum_{n \geq 0} a(n) t^n \to \text{L-values } L(\chi_{12}, s).$$

Th. (Zagier)

$F(e^{-t})$ is a quantum modular form.

$\Downarrow$

Arises from Dedekind's eta-fcn.

Questions. 1) Is there a better candidate for a quantum modular form which gives the same info as F? ☮

2) How might these strange functions relate to "mock θ-fns" or harmonic Maass forms?

Principle L-values $\longleftrightarrow$ periods
$\updownarrow$
nonholomorphic parts of harmonic Maass forms

Notation: $(a;q)_n := (1-a)(1-aq)\cdots(1-aq^{n-1})$

Theorem [B-O-P-R] Define $U(q) = 1 + \sum_{n\geq 1} (q;q)_n^2 q^{n+1}$.

The following are true:

1) $U(q)$ is well defined on $\mathbb{H}$.
2) $U(q)$ is well defined at roots of unity.
3) If q is a root of unity, then
$$F(q^{-1}) = U(q).$$
4) We have that (gen fcn)
$$e^{t/24}\, U(e^{-t}) = \sum_{n\geq 0} \frac{(2n+1)!}{n!} \cdot L(\chi_{12}, 2n+2) \cdot \left(\frac{-3t}{2\pi^2}\right)^n.$$

Example: $U(-1) = F(-1) = 3$

$U(i) = F(-i) = 8+3i$

$\vdots$

Def. (QMF) A <u>weight k quantum modular form</u> is $f : \mathbb{Q} - S \to \mathbb{C}$ with the property that for each $\gamma \in \Gamma$ there is a "nice" $h_\gamma(x)$ for which

$$h_\gamma(x) := f(x) - (cx+d)^k f(\gamma x).$$

Thm. [BOPR] Define $\phi(x) : \mathbb{H} \to \mathbb{C}$ by:

$$\phi(x) := e^{-\pi i x/12} \cdot U(e^{2\pi i x})$$

1) Then ϕ is a weight $3/2$ quantum modular form.

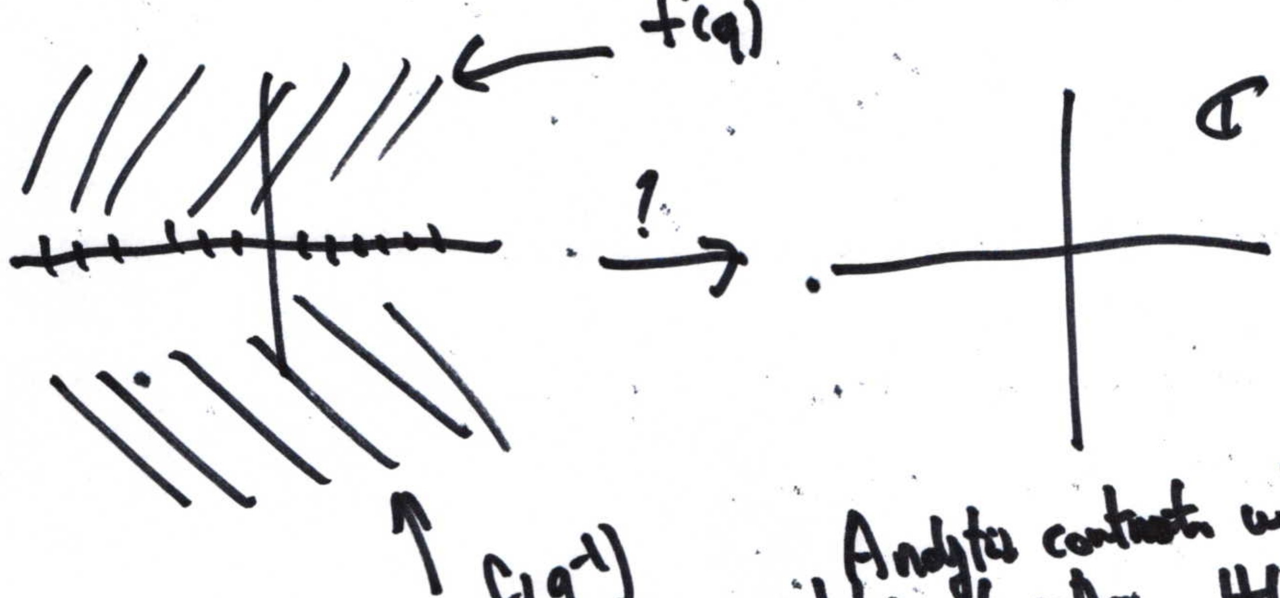
2) Moreover $h_\gamma(x) \text{ "=" }$ sum of functions which are essentially period integrals arising from η, η^3.

Beautiful Picture! (Hypergeometric harmonic Maass form)

e.g. $$f(q) := 1 + \sum_{n \geq 1} \frac{q^{n^2}}{(1+q)^2(1+q^2)^2 \cdots (1+q^n)^2}$$

hol. part of a wgt $\frac{1}{2}$ harmonic Maass form

1) Exercise: $f(q^{-1})$ is well defined.



Analytic continuation with patching together $H(\pm)$?

B. Singular Moduli.

Classical Theory (CM Elliptic curves)

$j(\rho)=0$

$j(i)=1728$

Classical Modular Polynomials

CM pt.

$$H_{-D}(x) := \prod_{Q \in \mathcal{Q}_{-D}\backslash\Gamma} (x - j(\alpha_Q))$$

Hilbert

Ex $H_{-3}(x)=X$

$H_{-4}(x)=X-1728$

Gross-Zagier $\Rightarrow$

$$H_{-D}(x) = x^{h(-D)} + \cdots\cdots + C_D.$$

Factorization for C_D.

(note. C_D only have small prime factors).

Questions: 1) What all the coef. of $H_{-D}(x)$?

DONE!

2) What can be said about

$$H_{-D}(F; x) := \prod_{Q \in \mathcal{Q}_{-D}\backslash\Gamma} (x - F(\tau_Q))$$

Maass form?

KO 3-8

Test Case (Q #1)

$$\mathrm{Tr}(F;-D) := \sum_{Q \in \mathcal{Q}_{-D}/\Gamma} \frac{f(\alpha_Q)}{w_Q}$$

Eg. $F = j \Rightarrow \mathrm{Tr}(j-744;-3) = -248$

$\Rightarrow \mathrm{Tr}(j-744;-4) = 492$

On the other hand

$$-\frac{\eta(z)^2 E_4(4z)}{\eta(2z)\eta(4z)^6} = -q^{-1} + 2 - 248 q^3 + 492 q^4 + \cdots$$

Th. (Z) If $f \in \mathbb{Z}[j]$, then there is a "principal part" A_f s.t.

$$A_f(z) + \sum \mathrm{Tr}(f;-D) q^D \in M^!_{3/2}(\Gamma_0(4))$$

"Moral" The coef. of $H_{-D}(j;x)$ of fixed degree "below $h(-D)$" is essentially the Fourier expansion of a wgt $3/2$ m.f.

Fact: $$H^{!}_{2-k}(\Gamma) \supseteq M^{!}_{2-k}(\Gamma)$$

$k=2$ this includes j

Q. Is Zagier's Thm. part of a much bigger picture?

A. Yes!

KO 3-10

EZ Case: $E_2^*(z)$ = ghost harmonic Maass Form.

"def'n" $\to$ $\gamma(z) :=$ " $\dfrac{E_4(z)\,E_2^*(z)}{E_6(z)\cdot j(z)}$ wgt 0, on $SL_2(\mathbb{Z})$.

$\gamma(\alpha_Q)$ algebraic.

$-D$	$H_{-D}(\gamma;X)$
-3	$X-\dfrac{23}{2^{11}\cdot 3^3}$
-4	X
-7	$X-\dfrac{181}{3^6\cdot 5^3\cdot 7}$
$\vdots$	
-15	$X^2+\dfrac{313}{3^4\cdot 5\cdot 11^3}\cdot X-\dfrac{1045769}{3^8\cdot 5^3\cdot 7^4\cdot 11^5}$

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* Conj. (O-S) $H_{-D}(\gamma; x)$ is p-integral for all $p > |-D|$ and p which split in $\mathbb{Q}(\sqrt{-D})$.

Maass Examples

Theorem. On any $\Gamma_0(N)$, there is a canonical sequence of weak Maass fcns on $\Gamma_0(N)$ of wgt 0 and Δ_0-eigenvalue λ. Then we have

$$\mathrm{Tr}_{(-1)^{\lambda+1}m}(\mathcal{J}_\lambda; n) = \text{"coef. of } q^{|m|n}\text{" of a fixed elt. in } M^!_{\lambda+\frac{1}{2}}(\Gamma_0(4N)).$$

Note: $\lambda = 1 \Longleftrightarrow \{\mathcal{J}_\lambda\} = \mathbb{Z}[j]$ & $N = 1$.

Zagier's

KD 3-12

Proof: 1) Explicit constructions via the method of Poincaré series

2) "Kloostermania"

$\Downarrow$

Finite Fourier transform

$\Downarrow$

Formulas which are "traces of CM pts"

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• [Katok-Sarnak, '80; Isr. J. Math.]

Example!

$$g \in H_{2-k} \xrightarrow{\xi_{2-k}} f \in S_k(\Gamma_0(N))$$

"nice" $\qquad$ Hecke eigenform

$\partial_{2-k} \leftarrow$ Shimura

$$m(z) \in \mathrm{Maass}_0(\Gamma_0(N)).$$

Thm. Traces of sing. for $\partial_{2-k}(g)$ are coeff. of modular forms!

Ex. $f \in S_4(\Gamma_0(6))$ unique $\rightarrow$ Rigid Calabi-Yau

What info does $m(z)$ + its singular values at CM points reveal about f?

$\Longrightarrow$ Ramanujan-Euler partition fcn.

Point: Somehow (Kloostermania or theta lifts)

$\swarrow$ Kudla-Millson theta fcn.

$\Rightarrow$ Q: Suppose you have a divisor on $X_0(N)$ which corresponds to a mod. func. f.

How do you construct f?

How do you "bound" the ring of def. of its Fourier coefs?

This is the Gross-Zagier problem!

KO 3-15

Generalized Borcherds Products

Borcherds (90's)

$$M^!_{\frac{1}{2}}(\Gamma_0(4)) \xrightarrow{\;\cong\;} \left\{ \begin{array}{l} \text{Mod. Fcns on } \Gamma(1) \\ \text{with Heegner divisors} \end{array} \right\}.$$

$$f_D = \sum a(n) q^n \xrightarrow{\;D\;} q^{h(-D)} \prod_{n=1}^{\infty} (1-q^n)^{a(n^2)}.$$

Divisor supp $(-D)$

Easiest Examples

$$f_1 = \Theta = 1 + 2q + 2q^4 + \cdots \longrightarrow$$

$$12 f_1 = 12 + 24q + 24q^4 + \cdots \longrightarrow \Delta = q \prod_{n=1}^{\infty} (1-q^n)^{24}.$$

Generalization:

$$\begin{tikzcd} H_{\frac{1}{2}} \arrow[r] & \left\{ \begin{array}{c} \text{Mod. fns. with} \\ \text{CM divisors} \\ \text{``Twisted''} \end{array} \right\} \\ + \arrow[dr, bend right] & \times \\ & \text{Products} \end{tikzcd}$$

Field of def. of Products arise exactly from the individual field of def. of fns in $H_{\frac{1}{2}}$

Remark. Generic elt. in $H_{\frac{1}{2}}$ has "mostly" transcendental coef. but some sparse alg. coefs.

Q: What is the geometric content of alg. coefs?

Next time:

1) What happens for $f \in H_{\frac{1}{2}} \cap \mathbb{Z}[[q]]$?

2) What is the general case?