Lecture #4

Borcherds Products

$$f = \sum a(n) q^n \in M^!_{\frac{1}{2}}(\Gamma_0(4)) \longrightarrow \left\{ \begin{array}{c} \text{Mod forms} \\ \text{with} \\ \text{Heegner} \\ \text{divisors} \end{array} \right\}$$

*

$f_D = \sum a_D(n) q^n$

$$D \text{ disc.} \longrightarrow \underbrace{q^{h(-D)} \prod_{n=1}^{\infty} (1-q^n)^{a_D(n^2)}}_{q\text{-expansion}}.$$

$H_D(j(z))$

"Hilbert class Polynomial"

Bruinier-Ono

$$H_{\frac{1}{2}} \ni f = f^- + f^+$$

$$f^+ = \sum c^+(n) q^n$$

Plays same role as Borcherds' fns in $M^!_{\frac{1}{2}}(\Gamma_0(4))$.

Given D, $\exists$ $P_D(x)$ rational fcn in x s.t.

$$q^{*_D} \prod_{n=1}^{\infty} P_D(q^n)^{c^+(n^2)}$$

is a mero fcn with a "twisted Heegner divisor"

Note: Borcherds: $P_D(x) = 1 - x$

Example: (Ramanujan's Mock θ)

$$\omega(q) := \sum_{n=0}^{\infty} \frac{q^{2n^2+2n}}{(q;q^2)^2_{n+1}}$$

$$\Rightarrow \sum_{n \in \mathbb{Z}+\frac{2}{3}} a(n) q^n := -2q^{\frac{2}{3}}\left(\omega(q^{\frac{1}{2}}) + \omega(-q^{\frac{1}{2}})\right)$$

Fact (Example $D=-8$)

$$P_{-8}(x) = \frac{1+\sqrt{-2}\,x - x^2}{1-\sqrt{-2}\,x - x^2}$$

$$\Psi(z) := \prod_{n=1}^{\infty} P_{-8}(q^n)^{\left(\frac{n}{3}\right) a(n^2/3)}$$ is a mod fcn on

$\Gamma_0(6)$ with a twisted Heegner divisor of disc -8.

Fact: For every D, $w(q)$

$\Rightarrow$ M.f. on $\Gamma_0(L)$ with alg. int. coeff. with specific divisors!

Fact: For most N, most elts in $H_{\frac{1}{2}}(\Gamma_0(4N))$ have the property that most of coefs of f^+ are transcendental...

Heegner Pts & Heegner Divisors (Special Case)

$$
\begin{tikzcd}
 & & E/\mathbb{Q} \arrow[d, "\text{modularity}"] \\
H_{\frac{1}{2}}(*) \arrow[r, squiggly, "\xi_{\frac{1}{2}}"] & g = \sum b_E(n) q^n \in S_{\frac{3}{2}}(*) \arrow[r, Rightarrow, "\text{Shimura}"] & f_E \in S_2(\Gamma_0(N_E))
\end{tikzcd}
$$

$$h_E = h^- + h^+ \in H_{\frac{1}{2}}(*)$$

What do h^- & h^+ tell us about E?

Standard Facts & Def'ns

- E_D: D-quad. twist of E $\qquad *$
- Kolyvagin's Th. If $\mathrm{ord}_{s=1}(L(E_D, s)) \leq 1$,

then $\mathrm{rk}(E_D/\mathbb{Q}) = \mathrm{ord}_{s=1}(L(E_D,s))$.

• (Waldspurger) If $\mathrm{sfe}(E(E_D,s)) = +1$, then

$$L(E_D,1) = \underset{\text{non-zero}}{*_D} \{ b_E(|D|) \}^2.$$

Fact: Beautiful Gen Fcn.

If $b_E(|D|) \neq 0$, then $\mathrm{rk}(E(D)) = 0$.

By Kolyvagin, we wish to know when $L'(E_D,1) = 0$?

Theorem. [B-O] TFAT:

1) If $\mathrm{sfe}(E_D) = +1$, then $\neq 0 \Rightarrow \mathrm{rk}\,0$.

$$c_h^-(-D) = **_D\, L(E_D,1). \quad [EZ]$$

2) If $\mathrm{sfe}(E_D) = -1$, then

$$c_h^+(D) \in \mathbb{Z} \iff L'(E(D),1) = 0.$$

3) If $\mathrm{sfe}(E_D) = -1$, then $c_h^+(D)$ is trans. $\iff L'(E(D),1) \neq 0$. rk 1

Idea Behind Proof:

- Gross-Zagier $\rightsquigarrow$ L' to heights of Heegner Pts (special Heegner divisors)

- $h_E = h^- + h^+ \in H_{\frac{1}{2}}$

$\downarrow$

Periodic Table for m.f. on $\Gamma_0(N_E)$... with Heegner divisor..

D Disc...

Borcherds Product...

$$q^* \prod_{n=1}^{\infty} P_D(q^n)^{c_h^+(|D|n^2)} \rightarrow$$ m.f. on $\Gamma_0(N_E)$ with twisted Heegner divisor associated to D

• **Idea:** $L'(E_D, 1) = 0$ when

all $\{c_h^+(Dn^2)\} \in \mathbb{Q}$.

• Tricks...

$$\{c_h^+(Dn^2)\} \in \mathbb{Q} \Longleftarrow c_h^+(D) \in \mathbb{Q}.$$

$\square$

Moral: If $f \in S_k$ is interesting, then by the surjectivity of ξ_{2-k}, there is a "cool"

$$g \in H_{2-k} \xrightarrow{\xi_{2-k}} f$$

$g^- + (g^+) \rightarrow$ Tells us something new.

Constructions (Standard)

Ramanujan's Examples

$$f(q) = 1 + \sum_{n \geq 1} \frac{q^{n^2}}{(1+q)^2(1+q^2)^2 \cdots (1+q^n)^2}$$

Seem weird!

Question: How are these "strange" expressions related to recognizable modular objects?

Bilateral sum...

Example:

$$f(q) = \frac{2}{q^{1/24}\,\eta(z)} \sum_{n \in \mathbb{Z}} \frac{(-1)^n q^{\frac{3n^2+n}{2}}}{1+q^n}$$

Dedekind

KO 4-10

Auxiliary Factor ☆ ~ "Eisenstein Series"

Theorem (Zwegers 2002)

If $\tau \in \mathbb{H}$, $u, v \in \mathbb{C} \setminus (\mathbb{Z}\tau + \mathbb{Z})$, and define

- $$\mu(u, v; \tau) := \frac{z^{1/2}}{\vartheta(v;\tau)} \cdot \sum_{n \in \mathbb{Z}} \frac{(-w)^n q^{n(n+1)/2}}{1 - zq^n}$$

where $z := e^{2\pi i u}$, $w := e^{2\pi i v}$, $q := e^{2\pi i \tau}$.

$\vartheta(v;\tau)$ = Jacobi ϑ-fcn.

"Few studied Vignéras"

- Defines

$R(u;\tau)$ = "period integral of a wgt $3/2$ unary theta fcn."

Then we have: for $\begin{pmatrix} \alpha & \beta \\ \gamma & \delta \end{pmatrix} = A \in SL_2(\mathbb{Z})$, the

$$\hat{\mu}\left(\frac{u}{\gamma\tau+\delta}, \frac{v}{\gamma\tau+\delta}; \frac{\alpha\tau+\beta}{\gamma\tau+\delta}\right) = (\gamma\tau+\delta)^{\frac{1}{2}}\hat{\mu}(u,v;\tau)$$

wgt $\frac{1}{2}$.

where $\hat{\mu} := \mu + R$.

nice specializations

$\Downarrow$

$\hat{\mu} \rightarrow$ harmonic Maass form of wgt $1/2$.

Numerics continued...

Amazingly, Ramanujan's guess gives:

q	-0.990	-0.992	-0.994	-0.996	-0.998
$f(q)+b(q)$	$3.961\ldots$	$3.969\ldots$	$3.976\ldots$	$3.984\ldots$	$3.992\ldots$

This suggests that

$$\lim_{q\to -1}(f(q)+b(q)) = 4.$$

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As $q \longrightarrow i$

q	$0.992i$	$0.994i$	$0.996i$
$f(q)$	$2 \cdot 10^6 - 4.6 \cdot 10^6 i$	$2 \cdot 10^8 - 4 \cdot 10^8 i$	$1.0 \cdot 10^{12} - 2 \cdot 10^{12} i$
$f(q) - b(q)$	$\sim 0.05 + 3.85i$	$\sim 0.04 + 3.89i$	$\sim 0.03 + 3.92i$

This suggests that

$$\lim_{q \to i} (f(q) - b(q)) = 4i.$$

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Crazy Formulas (Ramanujan's Claim)

$$\lim_{q \to \zeta} \Big(f(q) - (-1)^k \underbrace{b(q)}_{\text{wt } \frac{1}{2} \text{ m.f.}} \Big) = \underbrace{O(1)}_{\text{mysterious } O(1) \text{ \#'s...}}.$$

ζ: $2k^{th}$ prim. root of unity

Thm. (F-O-R) If ζ is $2k^{th}$ primitive root of unity, then

$$\lim_{q \to \zeta} \Big(f(q) - (-1)^k b(q) \Big) = -4 \underbrace{\sum_{n=0}^{k-1} (1+\zeta)^2 (1+\zeta^2)^2 \cdots (1+\zeta^n)^2 \zeta^{n+1}}_{\in\, \mathbb{Z}[\zeta]}.$$

RHS: $U(q) \rightsquigarrow$ "quantum m.f."