Modulan Curves at Infinite level

$$\mathcal{Y}(1) \leftarrow \mathcal{Y}(p) \leftarrow \mathcal{Y}(p^2) \leftarrow \cdots$$

$$\mathcal{Y}(p^\infty) = \varprojlim_{n \geq 1} \mathcal{Y}(p^n)$$

Analogue:

$$\varprojlim_{x \mapsto x^p} S^1$$

 . . . solenoid

- connected
- not path. connected
- $\mathbb{Z}[\frac{1}{p}]$ - module

$$\simeq \frac{\mathbb{R} \times \mathbb{Z}_p}{\mathbb{Z}} \longrightarrow \frac{\mathbb{Z}_p}{\mathbb{Z}}$$

p-adic analogue

$$D = \mathrm{Spf}\ \mathbb{Z}_p[[T]]$$

p-adic formal $\underset{\text{open}}{\wedge}$ unit disc

$$\varprojlim_{x \mapsto x^p} D = \mathrm{Spf}\ \mathbb{Z}_p[[T^{1/p^\infty}]]$$

$$\mathbb{Z}_p[[T]] \to \mathbb{Z}_p[[T^{1/p}]] \to \dots$$

$$T + pT^{1/p} + p^2 T^{1/p^2} + \dots$$

$\mathbb{Z}_p[[T^{1/p^\infty}]]$ = completion of union w.r.t (p, T)

$f(T) \in \mathbb{Z}_p[\![T^{1/p^\infty}]\!]$

$f(0) = 1$

$f(T)^p = f(T^p)$

$$f(T) = \lim_{n\to\infty} \left(1 + T^{1/p^n}\right)^{p^n}$$

max. ideal

$(p, T^{1/p}, T^{1/p^2}, \dots)$

Thesis: $\mathcal{Y}(p^\infty)$ admits surprisingly nice description (at least locally analytically)

Modular Curves

$\Gamma \subset SL_2\mathbb{Z}$ arithmetic subgroup

eg $\Gamma = \Gamma(N)$

$Y(N)_{\mathbb{C}} = {}_{\Gamma(N)}\backslash \mathcal{H}$

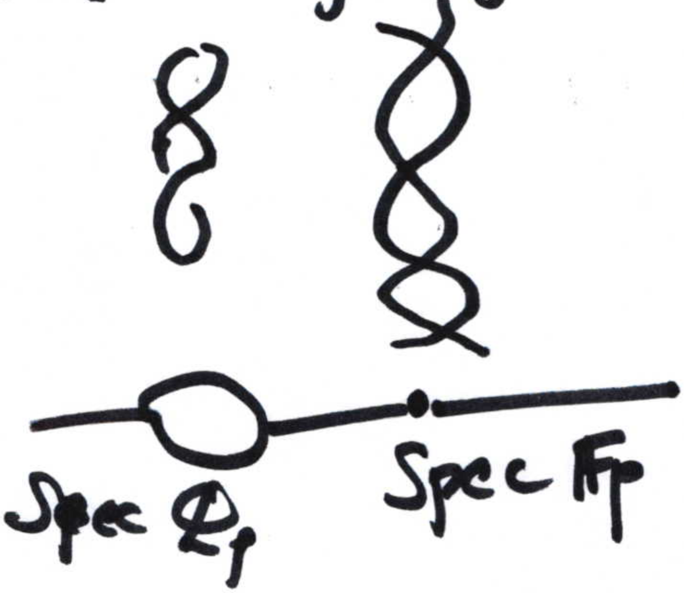
$Y(N)_{\mathbb{Q}}$ not too hard to define.

$Y(N)_{\mathbb{Z}}$ tricky at $p \mid N$

Example of bad reduction

[DR]

$Y(1)$ $Y(1)$



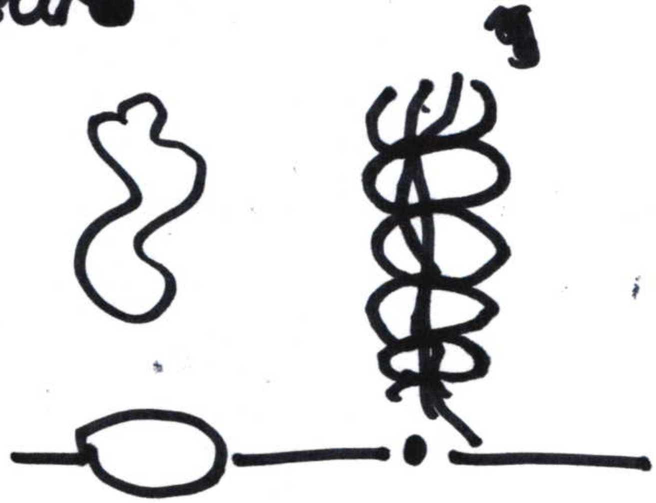
$$\begin{array}{c} Y_0(p) \\ \downarrow \\ \operatorname{Spec} \mathbb{Z}_p \end{array}$$

singularities at supersingular curves

JW1-5

Katz-Mazur

$$\begin{array}{c} Y(p^n) \\ \downarrow \\ \mathrm{Spec}\,\mathbb{Z}_p \end{array}$$



The moduli problem $\Gamma(N)$

Informally: if E/S is an ell. curve, a $\Gamma(N)$-structure is a basis P, Q for $E[N](S)$

Fine if $1/N \in \mathcal{O}_S$.

Bad otherwise

If $E/\overline{\mathbb{F}}_p$, $N = p$: $\dim_{\mathbb{F}_p} E[p](\overline{\mathbb{F}}_p)$

$$= \begin{cases} 0, & E \text{ is supersingular} \\ 1, & E \text{ is ordinary.} \end{cases}$$

A $\Gamma(N)$ - structure on E/S

is a group hom.

$$\phi : (\mathbb{Z}/N\mathbb{Z})^2 \to E[N](S)$$

s.t.

$$E[N] = \sum_{a,b \in \mathbb{Z}/N\mathbb{Z}} [\phi(a,b)]$$

<u>Ex.</u> $E/\bar{\mathbb{F}}_p$ s.s.

A $\Gamma(p)$ level structure

$$\phi : (\mathbb{Z}/p\mathbb{Z})^2 \to E[p](\bar{\mathbb{F}}_p) = 0.$$

has to be 0.

p prime

$N \geq 5$ prime to p

The moduli problem

$$S \mapsto \{(E/S,\ \Gamma(p^n) \text{ level structure},\ \Gamma_1(N) \text{ level structure})\}$$

is representable by a regular scheme

$$Y_n := Y(\underbrace{\Gamma(p^n) \cap \Gamma_1(N)}_{\Gamma})$$

$Y_n(\mathbb{C})$ not actually $\Gamma \backslash \mathcal{H}$.

$$Y_n^{\zeta}(\mathbb{C}) = \Gamma \backslash \mathcal{H} \qquad \zeta = e^{2\pi i/p^n}.$$

The Weil pairing

For E/S, have a pairing

$$e_{p^n} : E[p^n] \times E[p^n] \to \mu_{p^n}$$

If ϕ is a $\Gamma(p^n)$ level str. on E/S,

$$\phi : (\mathbb{Z}/p^n)^2 \to E[p^n](S)$$

$$e_{p^n}(\phi(1,0), \phi(0,1)) \in \mu_{p^n}(S)$$

If $Y_n = Y(\Gamma(p^n), \Gamma_1(N))$, get

- $Y_n \to \mu_{p^n}$. If $K = \mathbb{Q}_p(\zeta_{p^n})$,

$Y_n^{\zeta} \subset (Y_n)_{\mathcal{O}_K}$ preimage of $\zeta = \zeta_{p^n}$

The special fiber of $(\mathcal{Y}_n^{\zeta_{p^n}})_{\mathbb{Z}_p[\zeta_{p^n}]}$

If $E/S/\overline{\mathbb{F}}_p$, a $\Gamma(p^n)$-level str.

$$\phi : (\mathbb{Z}/p^n)^2 \to E[p^n](S)$$

is never injective.

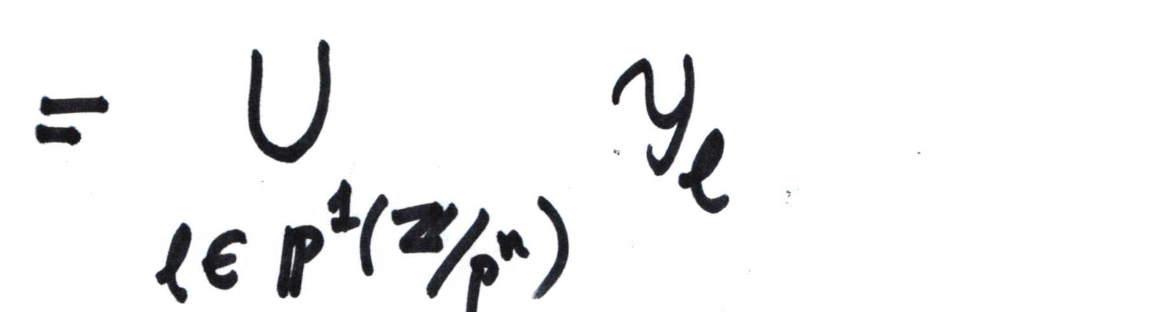
The kernel contains a line $\ell \subset (\mathbb{Z}/p^n)^2$.

$$\ell \in \mathbb{P}^1(\mathbb{Z}/p^n).$$

$$\mathcal{Y}^{\zeta}_{n,\mathbb{F}_p} = \bigcup_{\ell \in \mathbb{P}^1(\mathbb{Z}/p^n)} \mathcal{Y}_\ell$$

These $\mathcal{Y}_\ell$ intersect at s.s. pts

$\mathcal{Y}^{\zeta}_{n,\mathbb{F}_p}$



$Y(\Gamma)$ consider over

$$W = W(\overline{\mathbb{F}}_p) \text{ DVR.}$$
$$W/p = \overline{\mathbb{F}}_p.$$

if $x \in Y(\Gamma)(\overline{\mathbb{F}}_p)$

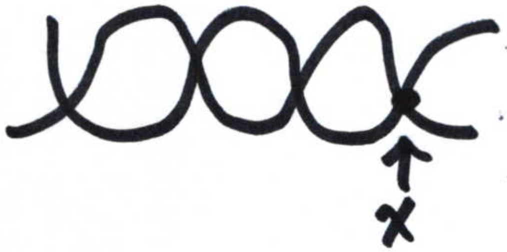
$\rightsquigarrow$ $\mathcal{O}_{Y(\Gamma),x} \supset \mathfrak{m}_x.$

$\hat{\mathcal{O}}_{Y(\Gamma),x} = \mathfrak{m}_x$-adic completion of $\mathcal{O}_{Y(\Gamma),x}$.

if x is ordinary or if Γ has prime-to-p level:

$$\hat{\mathcal{O}}_{Y(\Gamma),x} \simeq W[\![t]\!]$$

In the [DR] model:



$$\hat{\mathcal{O}}_{Y_0(p),x} \simeq \frac{W[[X,Y]]}{XY=p} = A$$

$$\dim_{\mathbb{F}_p} m/m^2 = \dim A = 2$$

$$\hat{\mathcal{O}}_{Y_n,x} \simeq ???$$

$$GL_2\,\mathbb{Z}/p^n\mathbb{Z} \circlearrowright$$

$$\hat{\mathcal{O}}_{Y_n^s,x} \simeq ???$$

$$SL_2\,\mathbb{Z}/p^n\mathbb{Z} \circlearrowright$$

Choose compatible systems

$$\begin{array}{ccccccccc}
\zeta_p & \leftarrow & \zeta_{p^2} & \longleftarrow & \zeta_{p^3} & \leftarrow & \cdots & & \zeta \\
y_1^{\zeta_p} & \leftarrow & y_2^{\zeta_{p^2}} & \leftarrow & y_3^{\zeta_{p^3}} & \leftarrow & \cdots & & y_\infty^{\zeta} \\
\downarrow & & \downarrow & & \downarrow & & & & \downarrow \\
x_1 & \leftarrow & x_2 & \leftarrow & x_3 & \leftarrow & \cdots & & x
\end{array}$$

$$\hat{\mathcal{O}}_{y_1^{\zeta_p}, x_1} \hookrightarrow \hat{\mathcal{O}}_{y_2^{\zeta_{p^2}}, x_2} \hookrightarrow \cdots$$

$\hat{\mathcal{O}}_{y_\infty^{\zeta}, x} :=$ completion of union w.r.t. $\mathfrak{m}_{x_1}$.

not Noeth.

$\circlearrowleft SL_2 \mathbb{Z}_p$

$$\overset{W}{\cancel{\mathbb{Z}_p}}[\zeta_p] \to \overset{W}{\cancel{\mathbb{Z}_p}}[\zeta_{p^2}] \to \cdots$$

$$\mathcal{O}_K = \overset{W}{\cancel{\mathbb{Z}_p}}[\zeta_{p^\infty}]^\wedge$$

<u>Thm</u>.

x ordinary:

$$\hat{\mathcal{O}}_{\mathcal{Y}^{\zeta}_{\infty}, x} \simeq \mathcal{O}_K [\![T^{1/p^\infty}]\!]$$

x supersingular:

$$\hat{\mathcal{O}}_{\mathcal{Y}^{\zeta}_{\infty}, x} \simeq \frac{\mathcal{O}_K [\![X^{1/p^\infty}, Y^{1/p^\infty}]\!]}{\left(\Delta(X,Y)^{1/p^r} - \zeta_{p^r}\right)_{r \geq 1}}$$

where $\Delta \in \mathbb{Z}_p[[x^{1/p^\infty}, y^{1/p^\infty}]]$

is a certain (explicit) series

satisfying

- $\Delta(x^p, y^p) = \Delta(x, y)^{-p}$
- $\Delta(x, y^{p^2}) = \Delta(x, y)^p$
- $\Delta(y, x) = \Delta(x, y)^{-1}$.