Formal Groups

R ring

A (1-dim'l) formal group G/R is

- $X \underset{G}{+} y = X + y + \ldots \in R[[X, y]]$
- $i(X) = -X + \ldots \in R[[X]]$

s.t. get structure of ab. group

- $X \underset{G}{+} y = y \underset{G}{+} X$
- $X \underset{G}{+} (y \underset{G}{+} z) = (X \underset{G}{+} y) \underset{G}{+} z.$

<u>Ex.</u> $\hat{G}_a$: $X \underset{\hat{G}_a}{+} y = X + y$

$\hat{G}_m$: $X \underset{\hat{G}_m}{+} y = (X+1)(y+1) - 1$
$= X + y + Xy.$

$\hat{E}$: for E/R e.c.

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$K/\mathbb{Q}_p$ complete

$E/\mathcal{O}_K$ e.c.

$\mathfrak{m}_K \subset \mathcal{O}_K$, $\mathcal{O}_K/\mathfrak{m}_K = k$

$$0 \to \hat{E}(\mathcal{O}_K) \to E(\mathcal{O}_K) \to E(k) \to 0$$

$\hat{E}(\mathcal{O}_K) = \mathfrak{m}_K$, under $+_{\hat{E}}$

If E/k is supersingular:

$E(k)[p^n] = 0.$

$E(\mathcal{O}_K)[p^n] = \hat{E}(\mathcal{O}_K)[p^n]$

An <u>adic ring</u> is a topological ring R, containing ideal I,

$$R \simeq \varprojlim \underbrace{R/I^n}_{\text{discrete}} \quad \Big| \quad I = \text{ideal of definition.}$$

<u>Ex</u>

- $\mathbb{Z}_p$, $I = (p)$
- $\mathbb{Z}[[T]]$ $\quad I = (T), (T^2)$
- $\mathbb{Z}[[x, y]]$ $\quad I = (x, y)$, $I = (x^2, y)$
- $\mathcal{O}_{\mathbb{C}_p}$ $\quad I = (p)$, $I \neq \mathfrak{m}$, $\mathfrak{m}^2 = \mathfrak{m}$

Similarly, can define adic A-alg.

$\mathrm{Adic}_A = \{\text{adic } A\text{-algs}\}$

A ring, G/A formal gp.

For $R \in \mathrm{Adic}_A$, let

$$G(R) = \{\text{top. nilpotent elts of } R\}$$

$$= \sqrt{I}, \quad I \text{ an ideal of defn}$$

$$= \mathrm{Nil}(R). \quad \text{(as a set)}$$

Group law by $+_G$.

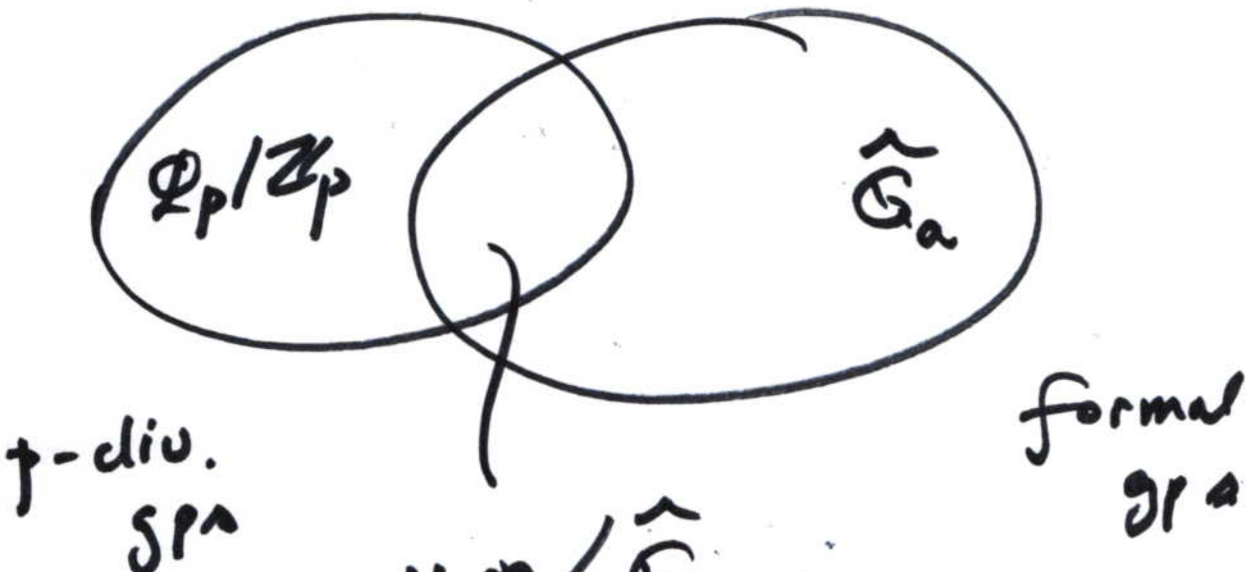
$\mathrm{Nil} : \mathrm{Adic}_A \to \mathrm{Sets}^*$ is representable by $A[[x]]$.

$$\mathrm{Hom}_{\mathrm{Adic}_A}(A[[x]], R) = \mathrm{Nil}(R)$$

$$\begin{tikzcd}
\mathrm{Adic}_A \arrow[r, "G"] \arrow[dr, "\cong \mathrm{Nil}"'] & \mathrm{Ab.Grp} \arrow[d, "\mathrm{Forget}"] \\
 & \mathrm{Sets}^*
\end{tikzcd}$$

alternate def'n of formal group

p-divisible formal groups $/\mathbb{Z}_p$



$\mu_{p^\infty} / \hat{\mathbb{G}}_m$

$\hat{E}[p^\infty] / \hat{E}$, $E \otimes \mathbb{F}_p$ s.s.

G/A formal gp, $G \simeq \mathrm{Spf}\, A[\![T]\!]$

$p : G \to G \quad \rightsquigarrow [p] : A[\![T]\!] \to A[\![T]\!]$

$T \mapsto [p]_G(T)$

$= T +_G \cdots +_G T$

$= pT + \cdots$

G is p-divisible if $[p]$ makes $A[\![T]\!]$ a loc. free module over itself.

$\hat{\mathbb{G}}_m / \mathbb{Z}_p$. $\quad [p]_{\hat{\mathbb{G}}_m}(T) = (1+T)^p - 1 = pT + \cdots + T^p$

If G is a p-divisible formal gp.,

$$G[p^n] = \operatorname{Spec} \frac{A[\![T]\!]}{[p^n]_G(T)}.$$

is a connected finite flat gp scheme.

A = complete local Noeth. ring
res. field char p.

$$G[p^\infty] = \varinjlim G[p^n]$$

p-div. gp $/A$ <u>connected</u>.

<u>Thm</u> (Tate) $G \mapsto G[p^\infty]$

is an equivalence

$$\left\{\begin{matrix} p\text{-div} \\ \text{formal} \\ \text{gps} \end{matrix}\right\} \xrightarrow{\sim} \left\{\begin{matrix} \text{Conn.} \\ p\text{-div gps} \end{matrix}\right\}$$

$p \nmid N \geq 5$

$x_0 \in Y_1(N)(\overline{\mathbb{F}}_p) \leftrightarrow E_0/\overline{\mathbb{F}}_p$ s.s.

$E_0[p^\infty]$ ht 2, dim 1, connected

$\hat{E}_0 = G_0$

$[p]_{\hat{E}_0}(T) = T^{p^2}$

$$A_0 := \hat{\mathcal{O}}_{Y_1(N), x_0} \simeq W[[t]]$$

classifies deformations of $\hat{E}_0$

There is a universal deformation

G^{univ}/A_0

$$[p]_{G^{univ}}(X) = pX + tX^p + X^{p^2}$$

(approx.) in $A_0[[X]]$

Add level structure.

$$E^{univ} \longrightarrow Y_1(N) \qquad A_0 = \hat{\mathcal{O}}_{Y_1(N), x_0}$$

$E^{univ}_{A_0} / A_0$

$E^{univ}_{A_0} \otimes_{A_0} \overline{\mathbb{F}}_p = E_0$. s.s. curve.

$$G^{univ} = \hat{E}^{univ}_{A_0}$$

$$E^{univ}_{A_0}[p^n] = \underset{\vec{G}}{\hat{E}^{univ}_{A_0}}[p^n] \text{ as gp schemes.}$$

Weil pairing.

$$\begin{array}{ccc} G^{univ}[p^n] \times G^{univ}[p^n] & \xrightarrow{\Delta_n} & \\ \| & & \\ E^{univ}[p^n] \times E^{univ}[p^n] & \xrightarrow[e_{p^n}]{} & \mu_{p^n} \end{array}$$

Over $Y(\Gamma_1(N) \cap \Gamma(p^n)) =: Y_n$,

E^{univ} has a universal p^n-level structure $P_n, Q_n \in E^{univ}(Y_n)[p^n]$

$$A_0 = \hat{\theta}_{\frac{y_1(N)}{y_0}, x_0}$$

$$A_n = \hat{\theta}_{y_n, x_{0n}}$$

$$E^{univ}[p^n](A_n) \ni P_n, Q_n$$

$$\wr\wr$$

$$G^{univ}_{[p^n]}(A_n) \ni X_n, y_n.$$

$$\cap$$

$$Nil(A_n)$$

$$\cap$$

$$A_n$$

$$\boxed{\begin{aligned} A_n^{\zeta_{p^n}} &:= \hat{\theta}^{\zeta_{p^n}}_{y_n, x_n} \ni X_n, y_n \\ \Delta_n(X_n, y_n) &= \zeta_{p^n} \end{aligned}}$$

$$(x_1, x_2, \dots) \in \varprojlim G^{\mathrm{univ}}[p^n](A^S)$$

$$A^S = \left(\varinjlim A_n^{S p^n} \right)^\wedge$$

$$= \hat{\mathcal{O}}^S_{Y_\infty, x}$$
