Formal Vector Spaces

$A \in \mathrm{Adic}_{\mathbb{Z}_p}$ $\qquad A \simeq \varprojlim A/I^n$

$p \in I$

$R \in \mathrm{Adic}_A$ $\qquad R \simeq \varprojlim R/J^n$

$IR \subset J$

$\mathrm{Nil} : \mathrm{Adic}_A \to \mathrm{Sets}^*$

$R \mapsto \mathrm{Nil}(R) = \sqrt{J}.$

rep'able by $A[\![T]\!]$

$\mathrm{Nil}^\flat : \mathrm{Adic}_A \longrightarrow \mathrm{Sets}^*$

$$R \longmapsto \varprojlim_{x \mapsto x^p} \mathrm{Nil}(R)$$

rep'able by

$$\Big(\varinjlim_{T \mapsto T^p} A[\![T]\!]\Big)^{\wedge}_{\text{w.r.t. } (I, T)} =: A[\![T^{1/p^\infty}]\!]$$

Ex. $\mathbb{Z}_p[\![T^{1/p^\infty}]\!]$

elements are

$$\sum_{\alpha \in \mathbb{Z}[\frac{1}{p}]_{\geq 0}} c_\alpha T^\alpha$$

$\forall N \geq 0$, $\{\alpha \leq N \mid v_p(c_\alpha) \leq N\} < \infty$

- $T + T^2 + T^3 + \dots$
- $T + pT^{1/p} + p^2 T^{1/p^2} + \dots$

not : $T + T^{1+\frac{1}{p}} + T^{1+\frac{1}{p}+\frac{1}{p^2}} + \dots$

<u>Lemma</u>. $\mathrm{Nil}^b(R) \xrightarrow{\sim} \mathrm{Nil}^b(R/I)$

$A \in \mathrm{Adic}_{\mathbb{Z}_p}$
$R \in \mathrm{Adic}_A$

<u>Rmk</u>.

$$\begin{array}{ccc} \varprojlim_{x \mapsto x^p} \mathcal{O}_{\mathbb{C}_p} & \xrightarrow{\sim} & \varprojlim_{x \mapsto x^p} \mathcal{O}_{\mathbb{C}_p}/p \\ \cup & & \cup \\ \mathfrak{m}_{\mathbb{C}_p} & & \mathfrak{m}_{\mathbb{C}_p}/p \end{array}$$

Inverse Map:

$$\begin{array}{l} \text{if } (x_0, x_1, \dots) \in \mathrm{Nil}^b(R/I) \\ \quad \downarrow \text{lift} \\ \quad y_0, y_1, \dots \quad \in \mathrm{Nil}(R)). \\ \quad \downarrow \text{smooth} \\ (z_0, z_1, \dots) \in \mathrm{Nil}^b(R) \end{array}$$

$$z_i = \lim_{n \to \infty} y_{i+n}^{p^n}$$

"crystalline property"

- converges
- independent of lifts
- compatible.
- 2-sided inverse

The universal cover of a p-div. gp.

For G an ab. gp., let

$$\widetilde{G} = \varprojlim_{x \mapsto x^p} G \text{ , a } \mathbb{Z}[\tfrac{1}{p}]\text{-module}$$

$$\tfrac{1}{p}(x_0, x_1, \ldots) = (x_1, x_2, \ldots)$$

<u>Ex.</u>

$$\widetilde{\mathbb{Z}} = 0$$

$$\widetilde{\mathbb{Q}_p} = \mathbb{Q}_p$$

$$\widetilde{\mathbb{Q}_p/\mathbb{Z}_p} = \mathbb{Q}_p$$

$$\widetilde{S^1} = \frac{\mathbb{R} \times \mathbb{Z}_p}{\mathbb{Z}} \quad \text{solenoid}$$

Let G/A be a p-div. formal gp.

$$[p]_G(X) = \underbrace{X +_G \cdots +_G X}_{p} = pX + \ldots \in A[\![X]\!]$$

Then $G(R)$ is a $\mathbb{Z}_p$-module.

<u>Ex</u>. $\hat{\mathbb{G}}_m / \mathbb{Z}_p$

for $a \in \mathbb{Z}_p$

$$[a]_{\hat{\mathbb{G}}_m}(T) = (1+T)^a - 1$$
$$= \sum_{n \geq 1} \binom{a}{n} T^n$$

—

$$\begin{tikzcd}
\mathrm{Adic}_A \arrow[r, "G"] \arrow[dr] & \mathrm{Mod}_{\mathbb{Z}_p} \arrow[d, "\mathrm{Forget}"] \\
& \mathrm{Sets}^*
\end{tikzcd}$$

$\simeq \mathrm{Nil}$

$$\widetilde{G} : \mathrm{Adic}_A \longrightarrow \mathrm{Vect}_{\mathbb{Q}_p}$$

$$R \longmapsto \widetilde{G}(R) = \varprojlim_{[p]_G} G(R) = \widetilde{G(R)}$$

Simplest case: $A = \overline{\mathbb{F}}_p$

$G = \widehat{\mathbb{G}}_m$, or $\widehat{E}_0$ $\quad E_0/\overline{\mathbb{F}}_p$ s.s.

$[p]_G(T) = T^p$, or T^{p^2}

$$\mathrm{Forget}(\widetilde{G}(R)) = \varprojlim_{x \mapsto x^p} \mathrm{Nil}(R) = \mathrm{Nil}^\flat(R)$$

$$\begin{tikzcd}
\mathrm{Adic}_A \arrow[r, "\widetilde{G}"] \arrow[dr, "\mathrm{Nil}^\flat"'] & \mathrm{Vect}_{\mathbb{Q}_p} \arrow[d] \\
& \mathrm{Sets}^*
\end{tikzcd}$$

"formal ($\mathbb{Q}_p$-) vector space."

Next Case: $A/I = \overline{\mathbb{F}}_p$ $\quad$ ($A \overset{\text{eg.}}{=} W(\overline{\mathbb{F}}_p)$)

G/A, set $G_0 = G \otimes_A \overline{\mathbb{F}}_p$.

Then for $R \in \mathrm{Adic}_A$

$$\widetilde{G}(R) \xrightarrow{\sim} \widetilde{G}_0(R/I) \quad !$$

$$\begin{aligned} \mathrm{Forget} \circ \widetilde{G}(R) &= \mathrm{Forget} \circ \widetilde{G}_0(R/I) \\ &\simeq \mathrm{Nil}^\flat(R/I) \\ &\simeq \mathrm{Nil}^\flat(R) \end{aligned}$$

$\widetilde{G}$ is a formal $\mathbb{Q}_p$ v.s. $/A$

If G, G' are 2 lifts of G_0 to A:

$$\widetilde{G}(R) \simeq \widetilde{G}_0(R/I) \simeq \widetilde{G}'(R)$$
$$\Rightarrow \widetilde{G} \simeq \widetilde{G}'.$$

Formal Linear Algebra.

$A/I = \overline{\mathbb{F}}_p$

$G_0/\overline{\mathbb{F}}_p$, lift G/A

$[p]_{G_0}(T) = T^{p^h}$

$[p]_G(T) = pT + ?T^2 + ?T^3 + \ldots$

$$\begin{tikzcd}
\widetilde{G}(R) \arrow[r, "\simeq"] \arrow[d, "{[p]_G}"'] & \widetilde{G}_0(R/I) \arrow[r, "\simeq"] \arrow[d, "{[p]_{G_0}}"'] & \mathrm{Nil}^b(R/I) \arrow[r, "\simeq"] \arrow[d, "x \mapsto x^{p^h}"'] & \mathrm{Nil}^b(R) \arrow[d, "x \mapsto x^{p^h}"'] \\
\widetilde{G}(R) \arrow[r, "\simeq"] & \widetilde{G}_0(R/I) \arrow[r, "\simeq"] & \mathrm{Nil}^b(R/I) \arrow[r, "\simeq"] & \mathrm{Nil}^b(R)
\end{tikzcd}$$

Exterior Powers: $\Lambda^i \widetilde{G}$?

Assume $G_0 = \hat{E}_0$, $E_0/\overline{\mathbb{F}}_p$ s.s

$\Delta_n : G_0[p^n] \times G_0[p^n] \to \mu_{p^n} = \hat{G}_m[p^n]$

$\mathbb{Z}_p$-alternating.

"promote" to

$\Delta : \widetilde{G} \times \widetilde{G} \to \widetilde{\hat{G}}_m$ $/W = W(\overline{\mathbb{F}}_p)$

$\mathbb{Q}_p$-alt.

If $R \in \mathrm{Adic}_{\overline{\mathbb{F}}_p}$

R/J is discrete, $G_0(R/J)$ is p-power p-torsion.

if $x = (x_0, x_1, \ldots) \in \widetilde{G}_0(R/J)$, $\exists\, n$

$p^n x_0 = 0$, so $p^n x = (0, p^n x_1, \ldots)$

$\in \varprojlim G_0[p^n](R/J)$.

$\varprojlim \underbrace{G_0[p^n]}_{\text{has a Weil pairing}}(R/J) \otimes \mathbb{Q}_p \xrightarrow{\sim} \widetilde{G}(R/J) = \widetilde{G}(R)$

so does this

G lift of G_0 to $W = W(\overline{\mathbb{F}}_p)$

$R \in \mathrm{Adic}_W$

$$\begin{tikzcd}
\widetilde{G}(R) \times \widetilde{G}(R) \arrow[d, "\cong"'] \arrow[dd, bend left=60, "\Delta"] \\
\widetilde{G}_0(R/p) \times \widetilde{G}_0(R/p) \arrow[d] \\
\widetilde{\widehat{\mathbb{G}}}_m(R/p) = \widetilde{\widehat{\mathbb{G}}}_m(R).
\end{tikzcd}$$

$$\begin{tikzcd}
\Delta : \widetilde{G} \times \widetilde{G} \arrow[r] \arrow[d, "\wr"'] & \widetilde{\widehat{\mathbb{G}}}_m \\
\mathrm{Nil}^b \times \mathrm{Nil}^b \arrow[ur] &
\end{tikzcd}$$

$\mathbb{Z}_p$-alt. $/W$

$(x, y) \mapsto \Delta(x, y).$

$$\Delta(X, Y) \in \cancel{\ } W[\![X^{1/p^\infty}, Y^{1/p^\infty}]\!]$$

(also get $\Delta^{1/p}, \Delta^{1/p^2}, \ldots$)

$$\Delta(\underset{[p]_G}{X^{p^2}}, Y) = \Delta(X, Y)^{\underset{[p]_{\widehat{\mathbb{G}}_m}}{p}}, \quad \Delta(Y, X) = \Delta(X, Y)^{-1}!$$

The deformation ring at ∞ level

$\mathcal{Y}_n = \mathcal{Y}(\Gamma_1(N) \cap \Gamma(p^n)) / W$

$\mathcal{Y}_0 \leftarrow \mathcal{Y}_1 \leftarrow \mathcal{Y}_2 \cdots$

$\mathcal{Y}_0 \Leftarrow \mathcal{Y}_1^{S_p} \Leftarrow \mathcal{Y}_2^{S_{p^2}} \Leftarrow \cdots \qquad \mathcal{Y}_n^{S_{p^n}} / W[\mu_{p^n}]$

$x_0 \Leftarrow x_1 \leftarrow x_2 \leftarrow \cdots \qquad x_i \in \mathcal{Y}_n^{S_{p^n}}(\overline{\mathbb{F}}_p)$.

$\updownarrow$ $E_0/\overline{\mathbb{F}}_p$ s.s. , $G_0 = \widehat{E}_0$ ht 2.

$A_n^{S_{p^n}} = \widehat{\mathcal{O}}_{\mathcal{Y}_n^{S_{p^n}}, x_n}$, $A_0 \to A_1 \to \cdots$

$A_0 \cong W[[t]]$

$A^S = (\varinjlim A_n^{S_{p^n}})^\wedge$.

$\cup \qquad \cup$

$\mathcal{O}_K = (\varinjlim W[\mu_{p^n}])^\wedge = W[\mu_{p^\infty}]^\wedge$

A_0 = deformation ring of G_0

G^{univ}/A_0. $\quad X_n, Y_n \in G^{\mathrm{univ}}_{[p^n]}(A_n^{S_{p^n}})$

$\Delta_n(X_n, Y_n) = \zeta_{p^n}$.

[Let G/W be ARBITRARY lift of G_0]

$G_{A_0} = G \underset{W}{\otimes} A_0$ and G^{univ} are both lifts of G_0 to A_0. Thus

$\widetilde{G}_{A_0} \simeq \widetilde{G}^{univ}$.

The X_n, Y_n give 2 elts in

$$\varprojlim G^{univ}[p^n](A^\flat) \subset \widetilde{G}^{univ}(A^\flat)$$

$$\simeq \widetilde{G}(A^\flat)$$

$$\simeq Nil^b(A^\flat).$$

$X, X^{1/p}, \ldots \in A^\flat$

$Y, Y^{1/p}, \ldots \in A^\flat$.

$$\Delta(X, Y)^{1/p^r} = \zeta_{p^r}$$

$$\frac{\mathcal{O}_K[[x^{1/p^\infty}, y^{1/p^\infty}]]}{(\Delta(x,y)^{1/p^r} - \zeta_{p^r})_{r \geq 1}} \xrightarrow{\sim} A^{\flat}$$

influences:

- Fargues
- $\widetilde{G}$ Faltings
  Fargues - Fontaine
- Perfectoid Spaces
  Scholze.