

## Formal Vector Spaces

$$A \in \text{Adic}_{\mathbb{Z}_p}$$

$$A \cong \varprojlim_{p \in \mathbb{I}} A/I^n$$

$$R \in \text{Adic}_A$$

$$R \cong \varprojlim_{I \in \mathcal{J}} R/I^n$$

$$\text{Nil}: \text{Adic}_A \rightarrow \text{Sets}^*$$

$$R \mapsto \text{Nil}(R) = \sqrt{\mathcal{J}}$$

rep'able by  $A \perp T \parallel$

$$\text{Nil}^b: \text{Adic}_A \rightarrow \text{Sets}^*$$

$$R \mapsto \varprojlim_{x \mapsto x^p} \text{Nil}(R)$$

rep'able by

$$\left( \varinjlim_{T \mapsto T^p} A \perp T \parallel \right) \overset{\wedge}{\underset{\substack{\text{w.r.t.} \\ (\mathcal{I}, T)}}{\uparrow}} =: A \perp T^{\wedge/p} \parallel$$

Ex.  $\mathbb{Z}_p \langle T \rangle$

elements are

$$\sum_{\alpha \in \mathbb{Z}[\frac{1}{p}]} c_\alpha T^\alpha$$

$\forall N \geq 0, \{ \alpha \in \mathbb{Z}[\frac{1}{p}] \mid v_p(c_\alpha) \leq N \} < \infty$

$$\cdot T + T^2 + T^3 + \dots$$

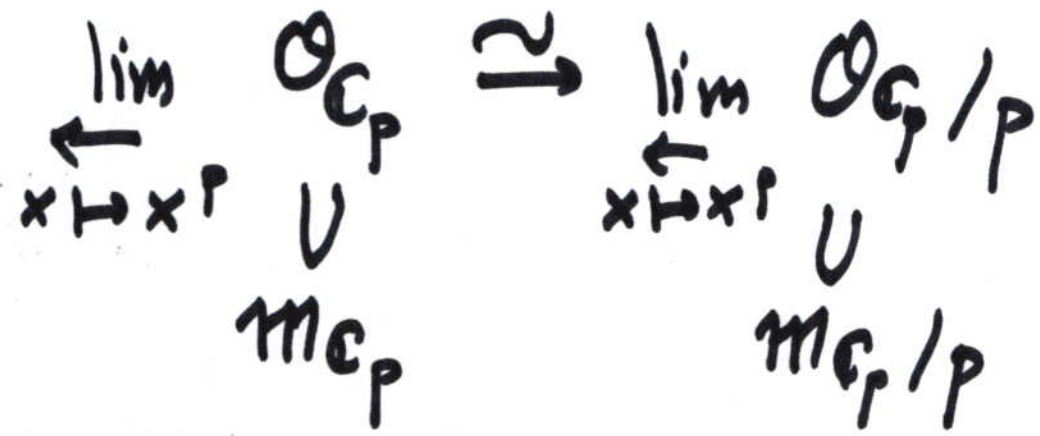
$$\cdot T + pT^{1/p} + p^2T^{1/p^2} + \dots$$

not :  $T + T^{1+\frac{1}{p}} + T^{1+\frac{1}{p^2}} + \dots$

Lemma  $\text{Nil}^b(R) \xrightarrow{\cong} \text{Nil}^b(R/I)$

$A \in \text{Adic}_{\mathbb{Z}_p}$   
 $R \in \text{Adic}_A$

Rmk.



Inverse Map:

if  $(x_0, x_1, \dots) \in \text{Nil}^b(R/I)$

lift  $\downarrow$

$(y_0, y_1, \dots) \in \text{Nil}(R)$

smooth  $\downarrow$

$(z_0, z_1, \dots) \in \text{Nil}^b(R)$

$$z_i = \lim_{n \rightarrow \infty} y_{i+n} p^n$$

- converges
- independent of lift
- compatible
- 2-sided inverse

"crystalline property"

The universal cover of a p-div. gp.

For G an ab. gp., let

$$G \cong \varprojlim_{x \mapsto x^p} G, \text{ a } \mathbb{Z}[\frac{1}{p}] \text{-module}$$

$$\frac{1}{p}(x_0, x_1, \dots) = (x_1, x_2, \dots)$$

Ex.

$$\begin{aligned} \mathbb{Z} &= 0 \\ \mathbb{Q}_p &= \mathbb{Q}_p \\ \mathbb{Q}_p / \mathbb{Z}_p &= \mathbb{Q}_p \\ \varprojlim \mathbb{Z}_p &= \mathbb{R} \times \mathbb{Z}_p \text{ solenoid} \\ \mathbb{Z} & \end{aligned}$$

Let  $G/A$  be a  $p$ -div. formal gp.

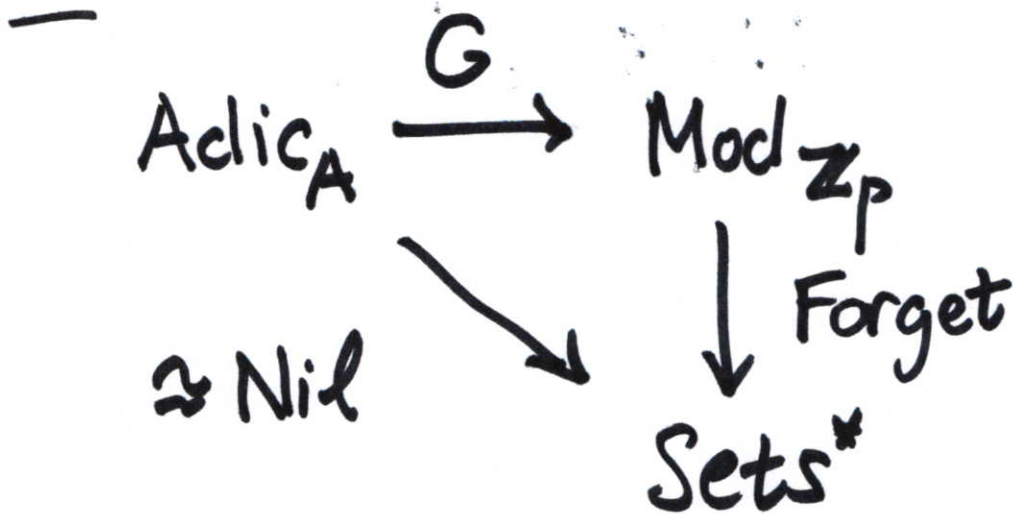
$$[p]_G(X) = \underbrace{X + \dots + X}_p = pX + \dots \in A[[X]]$$

Then  $G(R)$  is a  $\mathbb{Z}_p$ -module.

Ex.  $\hat{G}_m / \mathbb{Z}_p$

for  $a \in \mathbb{Z}_p$

$$\begin{aligned} [a]_{\hat{G}_m}(T) &= (1+T)^a - 1 \\ &= \sum_{n \geq 1} \binom{a}{n} T^n \end{aligned}$$



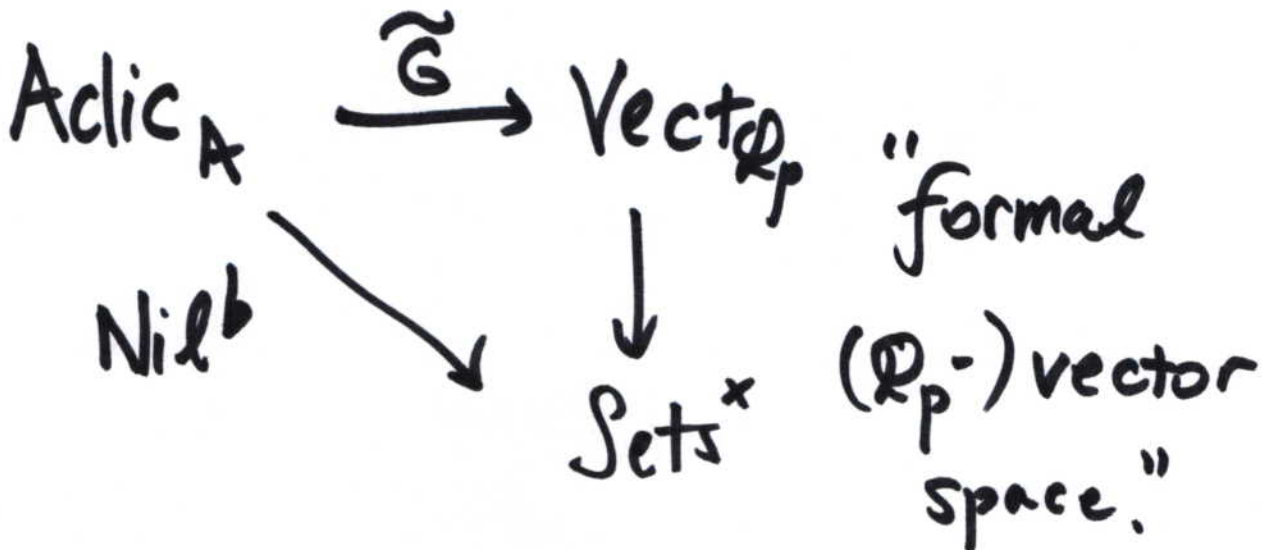
$$\begin{aligned} \tilde{G} : \text{Aclic}_A &\rightarrow \text{Vect}_{\mathbb{Q}_p} \\ R &\mapsto \tilde{G}(R) = \lim_{\leftarrow [P]_G} G(R) \\ &= \widetilde{G(R)} \end{aligned}$$

Simplest case:  $A = \overline{\mathbb{F}_p}$

$$G = \hat{G}_m, \text{ or } \hat{E}_0 \quad E_0 / \overline{\mathbb{F}_p} \text{ s.s.}$$

$$[P]_G(T) = T^p, \text{ or } T^{p^2}$$

$$\text{Forget}(\tilde{G}(R)) = \lim_{\leftarrow x \mapsto x^p} \text{Nil}(R) = \text{Nil}^b(R)$$



Next Case:  $A/\mathfrak{I} = \overline{\mathbb{F}}_p$  ( $A \stackrel{\mathfrak{G}}{=} W(\overline{\mathbb{F}}_p)$ ) JW4-7

$G/A$ , set  $G_0 = G \otimes_A \overline{\mathbb{F}}_p$ .

Then for  $R \in \text{Adic}_A$

$$\tilde{G}(R) \xrightarrow{\sim} \tilde{G}_0(R/\mathfrak{I}) \quad !$$

$$\text{Forget} \cdot \tilde{G}(R) = \text{Forget} \cdot \tilde{G}_0(R/\mathfrak{I})$$

$$\simeq \text{Nil}^b(R/\mathfrak{I})$$

$$\simeq \text{Nil}^b(R)$$

$\tilde{G}$  is a formal, v.s.  $/A$   
 $\mathbb{Z}_p$

If  $G, G'$  are 2 lifts of  $G_0$  to  $A$ :

$$\tilde{G}(R) \simeq \tilde{G}_0(R/\mathfrak{I}) \simeq \tilde{G}'(R)$$

$$\Rightarrow \tilde{G} \simeq \tilde{G}'$$

## Formal Linear Algebra.

$$A/I = \overline{\mathbb{F}_p}$$

$$G_0 / \overline{\mathbb{F}_p}, \text{ lift } G/A$$

$$[p]_{G_0}(T) = TP^h$$

$$[p]_G(T) = pT + ?T^2 + ?T^3 + \dots$$

$$\tilde{G}(R) \simeq \tilde{G}_0(R/I) \simeq \text{Nil}^b(R/I) \simeq \text{Nil}^b(R)$$

$$[p]_G \downarrow$$

$$[p]_{G_0} \downarrow$$

$$x \mapsto x^{p^h} \downarrow$$

$$x \mapsto x^{p^h} \downarrow$$

$$\tilde{G}(R) \simeq \tilde{G}_0(R/I) \simeq \text{Nil}^b(R/I) \simeq \text{Nil}^b(R)$$



Exterior Powers  $\wedge^i \tilde{G}$  ? Jw49

Assume  $G_0 = \hat{E}_0$ ,  $E_0 / \bar{\mathbb{F}}_p$  s.s

$\Delta_n: G_0[p^n] \times G_0[p^n] \rightarrow \mu_{p^n} = \hat{G}_m[p^n]$   
 $\mathbb{Z}_p$ -alternating.

"promote" to

$\Delta: \tilde{G} \times \tilde{G} \rightarrow \hat{\tilde{G}}_m / W = W(\bar{\mathbb{F}}_p)$   
 $\mathbb{Q}_p$ -alt.

If  $R \in \text{Adic } \bar{\mathbb{F}}_p$

$R/J$  is discrete,  $G_0(R/J)$  is  
 $p$ -power  $p$ -torsion.

if  $x = (x_0, x_1, \dots) \in \tilde{G}_0(R/J)$ ,  $\exists n$   
 $p^n x_0 = 0$ , so  $p^n x = (0, p^n x_1, \dots)$

$\in \varprojlim G_0[p^n](R/J)$ .

$\varprojlim G_0[p^n](R/J) \otimes_{\mathbb{Z}_p} \mathbb{Q}_p \xrightarrow{\sim} \tilde{G}(R/J) = \tilde{G}(R)$   
has a Weil pairing so does this

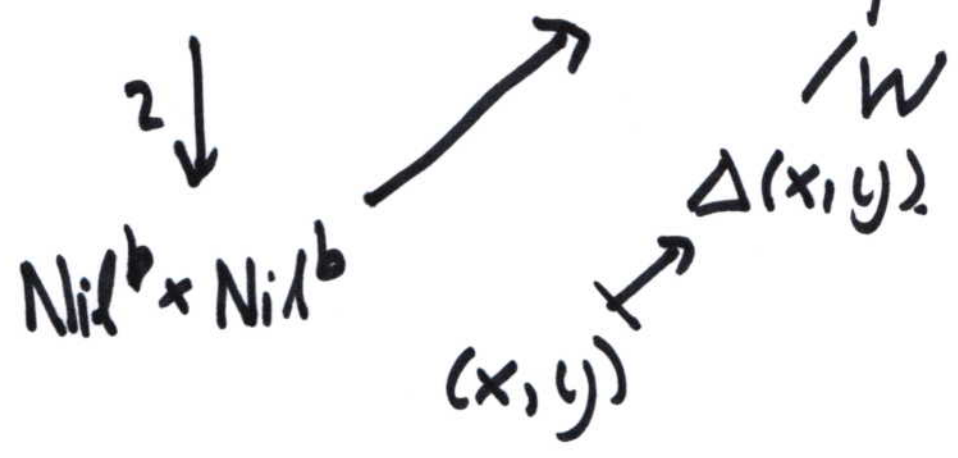
$G$  lift of  $G_0$  to  $W = W(\mathbb{F}_p)$   
 $R \in \text{Adic}_W$

$\widehat{G}(R) \times \widetilde{G}(R)$

$\cong \widetilde{G}_0(R/p) \times \widetilde{G}_0(R/p)$

$\rightarrow \widehat{\widetilde{G}}_m(R/p) = \widehat{\widetilde{G}}_m(R)$

$\Delta: \widetilde{G} \times \widetilde{G} \rightarrow \widehat{\widetilde{G}}_m$   $\mathbb{F}_p$ -alt.



$\Delta(x,y) \in \mathbb{F}_p \llbracket x^{1/p^\infty}, y^{1/p^\infty} \rrbracket$

(also get  $\Delta^4/p, \Delta^8/p^2, \dots$ )

$\Delta(x^{p^2}, y) = \Delta(x, y)^p - \Delta(y, x)$   
 $\uparrow \quad \uparrow$   
 $(p)_G \quad (p)_{\widehat{G}_m} = \Delta(x, y)!$

The deformation ring at  $\infty$  level

$$Y_n = \gamma(\Gamma_1(N) \cap \Gamma(p^n)) / W$$

$$Y_0 \leftarrow Y_1 \leftarrow Y_2 \dots$$

$$Y_0 \leftarrow Y_1^{Sp} \leftarrow Y_2^{Sp^2} \leftarrow \dots \quad Y_n^{Sp^n} / W[\mu_{p^n}]$$

$$x_0 \leftarrow x_1 \leftarrow x_2 \leftarrow \dots \quad x_i \in Y_n^{Sp^n}(\overline{\mathbb{F}}_p)$$

$$\updownarrow E_0 / \overline{\mathbb{F}}_p \text{ s.s.}, G_0 = \widehat{E}_0 \text{ ht } 2.$$

$$A_n^{Sp^n} = \widehat{\mathcal{O}}_{Y_n^{Sp^n}, x_n}, \quad A_0 \rightarrow A_1 \rightarrow \dots$$

|  
W[ $\mu_{p^n}$ ]

$$A^\natural = \varinjlim_U A_n^{Sp^n}^\wedge.$$

$$U$$

$$\mathcal{O}_K = \left( \varinjlim_U W[\mu_{p^n}] \right)^\wedge = W[\mu_{p^\infty}]^\wedge$$

$A_0 =$  deformation ring of  $G_0$

$$G^{\text{univ}} / A_0. \quad X_n, Y_n \in G^{\text{univ}}_{\langle p^n \rangle} (A_n^{Sp^n})$$

$$\Delta_n(X_n, Y_n) = \sum p^n.$$

[Let  $G/W$  be ARBITRARY lift  
of  $G_0$ ]

$G_{A_0} = G \otimes_{W} A_0$  and  $G^{\text{univ}}$  are both  
lifts of  $G_0$  to  $A_0$ . Thus

$$\tilde{G}_{A_0} \simeq \tilde{G}^{\text{univ}}.$$

The  $X_n, Y_n$  give 2 elts in

$$\lim_{\leftarrow} G^{\text{univ}}[p^n](A^S) \subset \tilde{G}^{\text{univ}}(A^S)$$

$$\simeq \tilde{G}(A^S)$$

$$X, X^{1/p}, \dots \in A^S \simeq \text{Nil}^b(A^S)$$

$$Y, Y^{1/p}, \dots \in A^S$$

$$\Delta(X, Y)^{1/p^r} = \sum p^r$$

$$\frac{\mathcal{O}_K \langle X^{1/p^\infty}, Y^{1/p^\infty} \rangle}{(\Delta(X, Y)^{1/p^r} - s_{p^r})_{r \gg 1}} \xrightarrow{\sim} A^{\mathfrak{s}}$$

influences:

- Fargues

- $\tilde{G}$  Faltings

Fargues - Fontaine

- Perfectoid Spaces

Scholze.