2014 ARIZONA WINTER SCHOOL: ARITHMETIC STATISTICS
STATISTICS FOR ZEROES OF L-FUNCTIONS OF CURVES OVER FINITE FIELDS

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The lectures will be concerned with statistics for the zeroes of L-functions in natural families. This will include discussions of the number field and the function field case (the latter case being the study of zeta functions and L-functions of curves over finite fields), and comparing the two worlds. Those comparisons are usually through analogy, but we will also discuss some surprising recent work [ERGR13] where the results for L-functions of curves over finite fields (obtained from the deep equidistribution theorems of Katz and Sarnak) can be used to prove results for L-functions over number fields.

Since the work of Montgomery [Mon73] on the pair correlation of the zeroes of the Riemann zeta function, it is known that there are many striking similarities between the statistics attached to zeroes of L-functions and the statistics attached to eigenvalues of random matrices. The work of Montgomery was extended and generalised in many directions, in particular to the study of statistics of zeroes in *families of L-functions*. It is predicted by the Katz and Sarnak philosophy that the statistics for the zeroes in families of L-functions, in the limit when the conductor of the L-functions gets large, follow the distribution laws of classical random matrices. Those conjectures are based on the fact that for L-functions of curves over finite fields, those distribution laws can be proven at the q-limit (when the size of the finite field $\mathbb{F}_q$ tends to infinity). In this case, the zeroes of the L-functions have a spectral interpretation: the zeroes are the reciprocal of the eigenvalues of Frobenius acting on the first cohomology (with ℓ-adic coefficients) of the curve. In their seminal work, Katz and Sarnak [KS99] used this spectral interpretation, and some deep equidistribution results due to Deligne, to prove that the pair correlation of zeroes of zeta functions of curves of large genus over large finite fields satisfy the Montgomery law (i.e. their result holds averaging over curves of genus g at the limit when q tends to infinity).

Katz and Sarnak then conjectured that the corresponding statistics for the zeroes of L-functions *over number fields* should also be given by the random matrix model as the limit for the large conductor. The statistics are then given by the scaling density associated to the random matrix measures when the size of the matrices goes to infinity. There is a vast literature of results investigating those conjectures over number fields, and obtaining partial results confirming the Katz and Sarnak philosophy.

In the last few years, a new approach to study statistics for zeroes of curves over finite fields emerged from the work of Rudnick and his collaborators, which is to obtain statistics for families of curves *for q fixed*, and when the genus of the curves (which is the analogue of the conductor for this case) tends to infinity. Then, one cannot use the powerful equidistribution theorems of Katz and Sarnak, and there are many similarities between the number field and the function field case. Some natural families of curves over finite fields that were studied in the recent years include hyperelliptic curves [KR09, FR10, Rud10, RG12], cyclic ℓ-covers [BDFL10b, Xio10], smooth plane curves [BDFL10a] and Artin-Schreier curves [BDF$^+$12, BDFL, Ent12, Ent13], and we will concentrate in the lectures on those families. We will give a picture of this body of recent work, stressing among others the similarities and differences between the number fields and function fields, the compatibility with the q-limit results of Katz and Sarnak, and the particular geometry of each family which influences the counting and the statistics. Some statistics are very robust, while some others are influenced by the geometry of each family.

The lecture series will include some specific projects for graduate students, investigating some statistics for zeroes of zeta functions, in particular, the study of low-lying zeroes in families of zeta functions. This is obtained by studying the one-level density, or in general the n-level density of the zeroes in the family. In

the case of the pair correlation (or in general the n-correlation) mentioned above, the statistics are universal and independent of the family, but the low-lying zeroes of natural families of L-functions can be described by several symmetry types (unitary, symplectic, orthogonal, even orthogonal and odd orthogonal).

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