# ARIZONA WINTER SCHOOL 2014 COURSE OUTLINE: GEOMETRIC ANALYTIC NUMBER THEORY

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The main theme of the course will be the interplay between

- arithmetic statistics over function fields of curves; and
- algebraic geometry of moduli spaces.

In particular: the regularities (typically in the form of asymptotic formulae) that we are led to expect by theorems and heuristics in arithmetic statistics can be "translated" into regularities (typically in the form of stabilization theorems for cohomology groups) that we are led to expect by theorems and heuristics in (complex) algebraic geomety.

We will explore this idea through a series of examples.

**Configuration spaces and squarefree polynomials.**

- Basic definitions: the set of monic squarefree polynomials over $\mathbb{F}_q$ is $\mathrm{Conf}_n(\mathbb{F}_q)$.
- Grothendieck-Lefschetz trace formula: $|\mathrm{Conf}_n(\mathbb{F}_q)|$ in terms of the etale cohomology of $\mathrm{Conf}_n/\mathbb{F}_q$.
- Comparison: etale cohomology of $\mathrm{Conf}_n/\mathbb{F}_p$ in terms of cohomology of $\mathrm{Conf}_n(\mathbb{C})$ in terms of cohomology of the braid group a la Arnold.
- "The three columns" and the meaning, in general, of the prefix "geometric."

**Variation of class groups, Cohen-Lenstra conjectures, and Hurwitz spaces.**

- Recall: statement of Cohen-Lenstra (depending on where MMW is in her lectures)
- Definition of Hurwitz spaces. Averages of $p$-parts of class groups as point-counts on Hurwitz spaces associated to dihedral groups.
- $\mathbb{F}_q$-points on more general Hurwitz spaces and moments of class groups.
- Trace formula: asymptotics of point-counts on Hurwitz spaces explained by (expected) stable cohomology of Hurwitz spaces. In other words, "Geometric Cohen-Lenstra" as an assertion of "vanishing cohomology for Hurwitz spaces."
- Stabilization of cohomology, whence existence of moments: argument of Ellenberg-Venkatesh-Westerland 1 (plus new argument on compatibility with Frobenius, not yet included in EVW1) *Short* sketch of proof of the main theorem of EVW1, mainly just to give an idea to number theorists about how the argument works and why it's truly topological, not algebraic, in nature.
- Geometric Malle-Bhargava conjectures (also see MMW's talk) are conjectures about vanishing cohomology for (more general) Hurwitz spaces.
- Connected components of Hurwitz spaces, and modification of Cohen-Lenstra when the base field contains roots of unity (work of Derek Garton)
- Mention (briefly) the fact that statements about vanishing cohomology tend to endorse random $\ell$-adic matrix heuristics.

*Date*: 15 Dec 2013.

- Error terms and stable ranges in cohomology (mention Roberts's conjecture and its resolution in various contexts by Bhargava-Shankar-Tsimerman, Thorne-Taniguchi, Zhao)

**Nonconstant coefficients.**

- The cohomology of ordered configuration spaces as an FI-module (work of Church, Ellenberg, Farb)
- The probability that a random degree-$n$ polynomial is prime, and the probability that a random degree-$n$ permutation is irreducible (relate with Granville's work on "anatomy," Tao's blog post about the Poisson-Dirichlet process)
- Aside: what would a "geometric Zhang-Maynard" look like?

**Remarks on other cases.**

- Geometric Hardy-Littlewood and the moduli space of rational curves on a low-degree hypersurface (work of Pigaut, S.A. Lee, Ellenberg-Venkatesh)
- Variation of Selmer groups in a quadratic twist family and relation with cohomology of the orthogonal group
- Geometric Linnik-Duke (equidistribution of Heegner points) and cohomology of theta divisors (work of Shende and Tsimerman)
- Remarks on what happens if you try replacing "geometric" with "motivic" everywhere in this talk. (Work of Vakil and Wood)

**Project.** Somewhat tentatively: I thought it would be fun to study from this point of view the question "How many squarefree polynomials over $\mathbb{F}_q$ of degree $n$ have discriminant a perfect $m$th power?" from this point of view. For some values of $q, n$, and $m$ the answer is easy, but for others it is affected by cohomology classes for certain finite-index subgroups of the braid group, whose cohomology has in some cases been computed in the literature. We'll see what arithmetic facts we can derive from the computations of the topologists, and, in the other direction, see if we can reproduce or even improve on the topological computations by means of counting arguments on the arithmetic side.