ARITHMETIC OF K3 SURFACES

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Introduction

Being surfaces of intermediate type, i.e., neither geometrically rational or ruled, nor of general type, K3 surfaces have a rich yet accessible arithmetic theory, which has started to come into focus over the last fifteen years or so. These notes, written to accompany a 4-hour lecture series at the 2015 Arizona Winter School, survey some of these developments, with an emphasis on explicit methods and examples. They are mostly expository, though I have included at the end two admittedly optimistic conjectures on uniform boundedness of Brauer groups (modulo constants) for lattice polarized K3 surfaces over number fields, which to my knowledge have not appeared in print before (Conjectures 4.5 and 4.6). The topics treated in these notes are as follows.

Geometry of K3 surfaces. We start with a crash course, light on proofs, on the geometry of K3 surfaces: topological properties, including the lattice structure of $\mathrm{H}^2(X, \mathbb{Z})$ and simple connectivity; the period point of K3 surface, the Torelli theorem and surjectivity of the period map.

Picard groups. Over a number field k, the geometric Picard group $\mathrm{Pic}(\overline{X})$ of a projective K3 surface X/k is a free $\mathbb{Z}$-module of rank $1 \leq \rho(\overline{X}) \leq 20$. Determining $\rho(\overline{X})$ for a given K3 surface is a difficult task; we explain how work of van Luijk, Kloosterman, Elsenhans-Jahnel and Charles [vL07, Klo07, EJ11b, Cha14] solves this problem.

Brauer Groups. The Galois module structure of $\mathrm{Pic}(\overline{X})$ allows one to compute an important piece of the Brauer group $\mathrm{Br}(X) = \mathrm{H}^2(X_{\text{ét}}, \mathbb{G}_m)$ of a locally solvable K3 surface X, consisting of the classes of $\mathrm{Br}(X)$ that are killed by passage to an algebraic closure, modulo Brauer classes coming from the ground field. These algebraic classes can be used to construct counter-examples to the Hasse principle on K3 surfaces via Brauer-Manin obstructions.

For surfaces of negative Kodaira dimension (e.g., cubic surfaces) the Brauer group consists entirely of algebraic classes. In contrast, for K3 surfaces we know that $\mathrm{Br}(X(\mathbb{C})) \cong (\mathbb{Q}/\mathbb{Z})^{22-\rho}$. However, a remarkable theorem of Skorobogatov and Zarhin [SZ08] says that over a number field the quotient of $\mathrm{Br}(X)$ by the subgroup of constant classes is finite. We explain work by several authors on the computation of the transcendental Brauer classes on K3 surfaces, and their impact on the arithmetic of such surfaces [HVAV11, HVA13, MSTVA16].

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Uniform boundedness questions. Finally, we explain in broad strokes an analogy between Brauer classes on K3 surfaces and torsion points on elliptic curves; the later are known to be uniformly bounded over a fixed number field, by work of Merel [Mer96]. It is our hope that analogous statements could be true for K3 surfaces.

Results from AWS. As part of the Arizona Winter School, a number of students were assigned to work on projects related to material of these notes. The experience was successful beyond reasonable expectations, and several members of the resulting three group projects continued working together long after the school. We briefly report on their findings.

I omitted several active research topics due to time constraints, notably rational curves on K3 surfaces, modularity questions, and Mordell-Weil ranks of elliptic K3 surfaces over number fields. I have resisted the temptation to add these topics so that the notes remain a faithful, detailed transcription of the four lectures that gave rise to them.[1]

Prerequisites. The departure point for these notes is working knowledge of the core chapters of Hartshorne's text [Har77, I-III], as well as a certain familiarity with the basic theory of algebraic surfaces, as presented in [Har77, V §§1,3,5] or [Bea96]. I also assume the reader is familiar with basic algebraic number theory (including group cohomology and Brauer groups of fields), and basic algebraic topology, at the level usually covered in first-year graduate courses in the United States. More advanced parts of the notes use étale cohomology as a tool; Milne's excellent book [Mil80] will come in handy as a reference. Many of the topics treated here have not percolated to advanced textbooks yet. For this reason, I provide detailed references throughout for readers seeking more depth on particular topics.

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[1]Videos of the lectures can be found at `http://swc.math.arizona.edu/aws/2015/index.html`

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1. Geometry of K3 surfaces

References: [LP80, Mor88, BHPVdV04, Huy15].
Huybrechts' notes [Huy15] are quite detailed and superbly written, and will soon appear in book form. Our presentation of the material in this section owes a lot to them.

1.1. Examples of K3 surfaces.

By a variety X over an arbitrary field k we mean a separated scheme of finite type over k. Unless otherwise stated, we shall assume varieties to be geometrically integral. For a smooth variety, we write ω_X for the canonical sheaf of X and K_X for its class in $\operatorname{Pic} X$.

Definition 1.1. An algebraic K3 surface is a smooth projective 2-dimensional variety over a field k such that $\omega_X \simeq \mathscr{O}_X$ and $\mathrm{H}^1(X, \mathscr{O}_X) = 0$. A polarized K3 surface is a pair (X, h), where X is an algebraic K3 surface and $h \in \mathrm{H}^2(X, \mathbb{Z})$ is an ample class. The degree of a polarized K3 surface is the self-intersection h^2.

Example 1.2 (K3 surfaces of degrees 4, 6, and 8)**.** Let X be a smooth complete intersection of type $(d_1, \dots, d_r)$ in $\mathbb{P}^n_k$, i.e., $X \subseteq \mathbb{P}^n$ has codimension r and $X = H_1 \cap \cdots \cap H_r$, where H_i is a hypersurface of degree $d_i \geq 1$ for $i = 1, \dots, r$. Then $\omega_X \simeq \mathscr{O}_X(\sum d_i - n - 1)$ [Har77, Exercise II.8.4]. To be a K3 surface, such an X must satisfy $r = n - 2$ and $\sum d_i = n + 1$. It does not hurt to assume that $d_i \geq 2$ for each i. This leaves only a few possibilities for X (check this!):

(1) $n = 3$ and $(d_1) = (4)$, i.e., X is a smooth quartic surface in $\mathbb{P}^3_k$.

(2) $n = 4$ and $(d_1, d_2) = (2, 3)$, i.e., X is a smooth complete intersection of a quadric and a cubic in $\mathbb{P}^4_k$.

(3) $n = 5$ and $(d_1, d_2, d_3) = (2, 2, 2)$, i.e., X is a smooth complete intersection of three quadrics in $\mathbb{P}^5_k$.

In each case, taking h to be the restriction to X of a hyperplane class in the ambient projective space, we obtain a polarized K3 surface whose degree coincides with the degree of X as a variety embedded in projective space.

Exercise 1.3. For each of the three types X of complete intersections in Example 1.2 prove that $\mathrm{H}^1(X, \mathscr{O}_X) = 0$.

Example 1.4 (K3 surfaces of degree 2)**.** Suppose for simplicity that char $k \neq 2$. Let $\pi\colon X \to \mathbb{P}^2_k$ be a double cover branched along a smooth sextic curve $C \subseteq \mathbb{P}^2_k$. Note that X is smooth if and only if C is smooth. By the Hurwitz formula [BHPVdV04, I.17.1], we have $\omega_X \simeq \pi^*(\omega_{\mathbb{P}^2_k} \otimes \mathscr{O}_{\mathbb{P}^2_k}(6)^{\otimes 1/2}) \simeq \mathscr{O}_X$, and since $\pi_*\mathscr{O}_X \simeq \mathscr{O}_{\mathbb{P}^2_k} \oplus \mathscr{O}_{\mathbb{P}^2_k}(-3)$, we deduce that $\mathrm{H}^1(X, \mathscr{O}_X) = 0$; see [CD89, Chapter 0, §1] for details. Hence X is a K3 surface if it is smooth. Letting $h = \pi^*(\ell)$ be the pull-back of a line, we obtain a polarized K3 surface of degree 2.

Example 1.5 (Kummer surfaces)**.** Let A be an abelian surface over a field k of characteristic $\neq 2$. The involution $\iota\colon A \to A$ given by $x \mapsto -x$ has sixteen $\bar{k}$-fixed points (the 2-torsion points of A). Let $\widetilde{A} \to A$ be the blow-up of A along the k-scheme defined by these fixed points. The involution ι lifts to an involution $\tilde{\iota}\colon \widetilde{A} \to \widetilde{A}$; the quotient $\pi\colon \widetilde{A} \to \widetilde{A}/\tilde{\iota} =: X$ is a double cover ramified along the geometric components of the exceptional divisors of the blow-up $E_1, \dots, E_{16}$. Let $\overline{E}_i$ be the image of E_i in X, for $i = 1, \dots, 16$.

We have $\omega_{\widetilde{A}} \simeq \mathscr{O}_{\widetilde{A}}(\sum E_i)$, and the Hurwitz formula implies that $\omega_{\widetilde{A}} \simeq \pi^*\omega_X \otimes \mathscr{O}_{\widetilde{A}}(\sum E_i)$. Hence $\mathscr{O}_{\widetilde{A}} \simeq \pi^*\omega_X$. The projection formula [Har77, Exercise II.5.1] then gives

$$\omega_X \otimes \pi_*\mathscr{O}_{\widetilde{A}} \simeq \pi_*\mathscr{O}_{\widetilde{A}}. \tag{1}$$

Since $\pi_*\mathscr{O}_{\widetilde{A}} \simeq \mathscr{O}_X \oplus L^{\otimes -1}$, where L is the square root of $\mathscr{O}_X(\sum \overline{E}_i)$, taking determinants of both sides of (1) gives $\omega_X^{\otimes 2} \simeq \mathscr{O}_X$. We conclude that $K_X \in \operatorname{Pic} X$ is numerically trivial (i.e., its image in $\operatorname{Num} X$ is zero—see §1.3), and thus $h^0(X, \omega_X) = 0$ if $\omega_X \not\simeq \mathscr{O}_X$. Suppose this is the case. Then since $h^0(X, \pi_*\mathscr{O}_{\widetilde{A}}) = 1$, (1) implies that $h^0(X, \omega_X \otimes \pi_*\mathscr{O}_{\widetilde{A}}) = 1$, and hence $h^0(X, \omega_X \otimes L^{\otimes -1}) = 1$. Fix an ample divisor A on X; our discussion above implies that $(A, K_X - [L])_X > 0$, where $(\ ,\ )_X$ denotes the intersection pairing on X. On the other hand, $L \sim \frac{1}{2}\sum \overline{E}_i$, so $(A, [L])_X > 0$. But then $(A, K_X) > 0$, which contradicts the numerical triviality of K_X. Hence we must have $\omega_X \simeq \mathscr{O}_X$.

Exercise 1.6. Prove that $\mathrm{H}^1(X, \mathscr{O}_X) = 0$ for the surfaces in Example 1.5.

1.2. **Euler characteristic.** If X is an algebraic K3 surface, then by definition we have $h^0(X, \mathscr{O}_X) = 1$ and $h^1(X, \mathscr{O}_X) = 0$. Serre duality then gives $h^2(X, \mathscr{O}_X) = h^0(X, \mathscr{O}_X) = 1$, so

$$X \text{ an algebraic K3 surface} \implies \chi(X, \mathscr{O}_X) = 2.$$

1.3. **Linear, algebraic, and numerical equivalence.** Let X be a smooth surface over a field k, and write $\operatorname{Div} X$ for its group of Weil divisors. Let $(\ ,\ )_X\colon \operatorname{Div} X \times \operatorname{Div} X \to \mathbb{Z}$ denote the intersection pairing on X [Har77, § V.1]. Recall three basic equivalence relations one can put on $\operatorname{Div} X$:

(1) **Linear equivalence:** C, $D \in \operatorname{Div} X$ are linearly equivalent if $C = D + \operatorname{div}(f)$ for some $f \in \mathbf{k}(X)$ (the function field of X).

(2) **Algebraic equivalence:** $C,\ D \in \operatorname{Div} X$ are algebraically equivalent if there is a connected curve T, two closed points 0 and $1 \in T$, and a divisor E in $X \times T$, flat over T, such that $E|_{X\times\{0\}} - E|_{X\times\{1\}} = C - D$.

(3) **Numerical equivalence:** $C,\ D \in \operatorname{Div} X$ are numerically equivalent if $(C,E)_X = (D,E)_X$ for all $E \in \operatorname{Div} X$.

These relations obey the following hierarchy:

$$\text{Linear equivalence} \implies \text{Algebraic equivalence} \implies \text{Numerical equivalence.}$$

Briefly, here is why these implications hold. For the first implication: if $C = D + \operatorname{div}(f)$, then we can take $T = \mathbb{P}^1_k = \operatorname{Proj} k[t,u]$ and $E = \operatorname{div}(tf - u)$ in $X \times \mathbb{P}^1_k$ to see that C and D are algebraically equivalent. For the second implication: suppose that an algebraically equivalence between C and D is witnessed by $E \subseteq X \times T$. Let H be a very ample divisor on X, and let $X \hookrightarrow \mathbb{P}^n_k$ be the embedding induced by H. This allows us to embed $X \times T$ (and hence E) in $\mathbb{P}^n_T$. The Hilbert polynomials of the fibers of $E \to T$ above closed points are constant, by flatness (and connectedness of T). Since $(C,H)_X$ is the degree of C in the embedding induced by H, we conclude that $(C,H)_X = (D,H)_X$. Now use the fact that any divisor on X can be written as a difference of ample divisors [Har77, p. 359]—this decomposition need not happen over the ground field of course, but intersection numbers are preserved by base extension of the ground field, so we may work over an algebraically closed field to begin with.

Write, as usual, $\operatorname{Pic} X$ for the quotient of $\operatorname{Div} X$ by the linear equivalence relation; let $\operatorname{Pic}^\tau X \subseteq \operatorname{Pic} X$ be the set of numerically trivial classes, i.e.,

$$\operatorname{Pic}^\tau X = \{L \in \operatorname{Pic} X : (L, L')_X = 0 \text{ for all } L' \in \operatorname{Pic} X\}.$$

Finally, let $\operatorname{Pic}^0 X \subseteq \operatorname{Pic}^\tau X$ be the set of classes algebraically equivalent to zero. Let $\operatorname{NS} X = \operatorname{Pic} X / \operatorname{Pic}^0 X$ be the Néron-Severi group of X, and let $\operatorname{Num} X = \operatorname{Pic} X / \operatorname{Pic}^\tau X$.

Lemma 1.7. *Let X be an algebraic K3 surface, and let $L \in \operatorname{Pic} X$. Then*

$$\chi(X, L) = \frac{L^2}{2} + 2.$$

Proof. This is just the Riemann–Roch theorem for surfaces [Har77, Theorem V.1.6], taking into account that $K_X = 0$ and $\chi(X, \mathscr{O}_X) = 2$. □

Proposition 1.8. *Let X be an algebraic K3 surface over a field. Then the natural surjections*

$$\operatorname{Pic} X \to \operatorname{NS} X \to \operatorname{Num} X$$

are isomorphisms.

Proof. Since X is projective, there is an ample sheaf L' on X. If $L \in \ker(\operatorname{Pic} X \to \operatorname{Num} X)$, then $(L, L')_X = 0$, and thus if $L \neq \mathscr{O}_X$ then $\mathrm{H}^0(X, L) = 0$. Serre duality implies that

$\mathrm{H}^2(X,L) \simeq \mathrm{H}^0(X,L^{\otimes -1})^\vee = 0$. Hence $\chi(X,L) \leq 0$; on the other hand, by Lemma 1.7 we have $\chi(X,L) = \frac{1}{2}L^2+2$, and hence $L^2 < 0$, which means L cannot be numerically trivial. □

1.4. **Complex K3 surfaces.** Over $k = \mathbb{C}$, there is a notion of K3 surfaces as complex manifolds that includes algebraic K3 surfaces over $\mathbb{C}$, although most complex K3 surfaces are not projective. This more flexible theory is crucial in proving important results for K3 surfaces, such as the Torelli Theorem [PŠŠ71, BR75, LP80]. It also allows us to study K3 surfaces via singular cohomology.

Definition 1.9. A complex K3 surface is a compact connected 2-dimensional complex manifold X such that $\omega_X := \Omega_X^2 \simeq \mathscr{O}_X$ and $\mathrm{H}^1(X,\mathscr{O}_X) = 0$.

Let us explain the sense in which an algebraic K3 surface is also a complex K3 surface. To a separated scheme X locally of finite type over $\mathbb{C}$ one can associate a complex space X^{an}, whose underlying space consists of $X(\mathbb{C})$, and a map $\phi\colon X^{\mathrm{an}} \to X$ of locally ringed spaces in $\mathbb{C}$-algebras. For a ringed space Y, let $\mathfrak{Coh}(Y)$ denote the category of coherent sheaves on Y. To $\mathscr{F} \in \mathfrak{Coh}(X)$ one can then associate $\mathscr{F}^{\mathrm{an}} := \phi^*\mathscr{F} \in \mathfrak{Coh}(X^{\mathrm{an}})$; we have $\Omega^{\mathrm{an}}_{X/\mathbb{C}} \simeq \Omega_{X^{\mathrm{an}}}$. If X is a projective variety, then the functor

$$\Phi\colon \mathfrak{Coh}(X) \to \mathfrak{Coh}(X^{\mathrm{an}}) \qquad \mathscr{F} \to \mathscr{F}^{\mathrm{an}}$$

is an equivalence of abelian categories. This is known as Serre's GAGA principle [Ser55]. In the course of proving this equivalence, Serre shows that for $\mathscr{F} \in \mathfrak{Coh}(X)$, certain functorial maps

$$\epsilon\colon \mathrm{H}^q(X,\mathscr{F}) \to \mathrm{H}^q(X^{\mathrm{an}},\mathscr{F}^{\mathrm{an}})$$

are bijective for all $q \geq 0$ [Ser55, Théorème 1]. Hence:

Proposition 1.10. *Let X be an algebraic K3 surface over $k = \mathbb{C}$. Then X^{an} is a complex K3 surface.* □

1.5. **Singular cohomology of complex K3 surfaces.** In this section X denotes a complex K3 surface, $e(\cdot)$ is the topological Euler characteristic of a space, and $c_i(X)$ is the i-th Chern class of (the tangent bundle of) X for $i = 1$ and 2. As in §1.2, one can show that $\chi(X,\mathscr{O}_X) = 2$. Noether's formula states that

$$\chi(X,\mathscr{O}_X) = \frac{1}{12}(c_1(X)^2 + c_2(X));$$

see [BHPVdV04, Theorem I.5.5] and the references cited therein. Since $\omega_X \simeq \mathscr{O}_X$, we have $c_1(X)^2 = 0$, and hence $e(X) = c_2(X) = 24$.

For the singular cohomology groups of X, we have

$$\mathrm{H}^0(X,\mathbb{Z}) \cong \mathbb{Z} \text{ because } X \text{ is connected, and}$$
$$\mathrm{H}^4(X,\mathbb{Z}) \cong \mathbb{Z} \text{ because } X \text{ is oriented.}$$

The exponential sequence

$$0 \to \mathbb{Z} \to \mathscr{O}_X \to \mathscr{O}_X^\times \to 0$$

gives rise to a long exact sequence in sheaf cohomology

$$
\begin{aligned}
(2)\qquad & 0 \to \mathrm{H}^0(X,\mathbb{Z}) \to \mathrm{H}^0(X,\mathscr{O}_X) \to \mathrm{H}^0(X,\mathscr{O}_X^\times) \to \mathrm{H}^1(X,\mathbb{Z}) \to \mathrm{H}^1(X,\mathscr{O}_X) \to \\
& \to \mathrm{H}^1(X,\mathscr{O}_X^\times) \xrightarrow{c_1} \mathrm{H}^2(X,\mathbb{Z}) \to \mathrm{H}^2(X,\mathscr{O}_X) \to \mathrm{H}^2(X,\mathscr{O}_X^\times) \to \mathrm{H}^3(X,\mathbb{Z})
\end{aligned}
$$

Since $\mathrm{H}^0(X,\mathscr{O}_X) \to \mathrm{H}^0(X,\mathscr{O}_X^\times)$ is surjective and $\mathrm{H}^1(X,\mathscr{O}_X) = 0$, we have $\mathrm{H}^1(X,\mathbb{Z}) = 0$. Poincaré duality then gives

$$0 = \operatorname{rk}\mathrm{H}^1(X,\mathbb{Z}) = \operatorname{rk}\mathrm{H}_1(X,\mathbb{Z}) \overset{\mathrm{PD}}{=} \operatorname{rk}\mathrm{H}^3(X,\mathbb{Z}),$$

so $\mathrm{H}^3(X,\mathbb{Z})$ is a torsion abelian group, and $\mathrm{H}^3(X,\mathbb{Z})_{\mathrm{tors}} \cong \mathrm{H}_1(X,\mathbb{Z})_{\mathrm{tors}}$. The universal coefficients short exact sequence

$$0 \to \mathrm{Ext}^1(\mathrm{H}_1(X,\mathbb{Z}),\mathbb{Z}) \to \mathrm{H}^2(X,\mathbb{Z}) \to \mathrm{Hom}(\mathrm{H}_2(X,\mathbb{Z}),\mathbb{Z}) \to 0$$

shows that $\mathrm{H}_1(X,\mathbb{Z})_{\mathrm{tors}}$ is dual to $\mathrm{H}^2(X,\mathbb{Z})_{\mathrm{tors}}$ (fill in the details!).

Proposition 1.11. *Let X be a complex K3 surface. Then* $\mathrm{H}_1(X,\mathbb{Z})_{\mathrm{tors}} = 0$.

Proof. An element of order n in $\mathrm{H}_1(X,\mathbb{Z})_{\mathrm{tors}}$ gives a surjection $\mathrm{H}_1(X,\mathbb{Z}) \twoheadrightarrow \mathbb{Z}/n\mathbb{Z}$, hence a surjection $\pi_1(X,x) \twoheadrightarrow \mathbb{Z}/n\mathbb{Z}$, which corresponds to an unramified cover $Y \to X$ of degree n, and we must have $e(Y) = ne(X) = 24n$. The Hurwitz formula tells us that $\omega_Y \simeq \pi^*\omega_X$, so $\omega_Y \simeq \mathscr{O}_Y$, which implies $h^2(Y,\mathscr{O}_Y) = h^0(Y,\omega_Y) = 1$. Noether's formula tells us that $\chi(Y,\mathscr{O}_Y) = \frac{1}{12}(c_1(Y)^2 + c_2(Y))$. So $2 - h^1(\mathscr{O}_Y) = \frac{1}{12}\cdot 24n$ and hence $h^1(\mathscr{O}_Y) = 2-2n$. We conclude that $n = 1$. □

Proposition 1.11 and the discussion preceding it shows that $\mathrm{H}^3(X,\mathbb{Z}) = 0$ and $\mathrm{H}^2(X,\mathbb{Z})$ is a free abelian group. Since $e(X) = 24$, we deduce that $\operatorname{rk}\mathrm{H}^2(X,\mathbb{Z}) = 24-1-1 = 22$. Poincaré duality thus tells us that the cup product induces a perfect bilinear pairing:

$$B\colon \mathrm{H}^2(X,\mathbb{Z}) \times \mathrm{H}^2(X,\mathbb{Z}) \to \mathbb{Z}.$$

Proposition 1.12 ([BHPVdV04, VIII.3.1])**.** *The pairing B is even, i.e., $B(x,x) \in 2\mathbb{Z}$ for all $x \in \mathrm{H}^2(X,\mathbb{Z})$.* □

The bilinear form B thus gives rise to an even integral quadratic form

$$q\colon \mathrm{H}^2(X,\mathbb{Z}) \to \mathbb{Z}, \qquad x \mapsto B(x,x).$$

Extend q by $\mathbb{R}$-linearity to a form $q_{\mathbb{R}}\colon \mathrm{H}^2(X,\mathbb{Z}) \otimes \mathbb{R} \to \mathbb{R}$, and let b_+ (resp. b_-) denote the number of positive (resp. negative) eigenvalues of q. The Thom-Hirzebruch index theorem [Hir66, p. 86] says that

$$b_+ - b_- = \frac{1}{3}(c_1(X)^2 - 2c_2(X)) = -16.$$

On the other hand, we know that

$$b_+ + b_- = 22,$$

so we conclude that $b_+ = 3$ and $b_- = 19$. In sum, $\mathrm{H}^2(X,\mathbb{Z})$ equipped with the cup-product is an indefinite even integral lattice of signature $(3,19)$. Perfectness of the pairing B tells us that the lattice $\mathrm{H}^2(X,\mathbb{Z})$ is unimodular, i.e., the absolute value of the determinant of a Gram matrix is 1. This is enough information to pin down the lattice $\mathrm{H}^2(X,\mathbb{Z})$, up to isometry. To state a precise theorem, recall that the hyperbolic plane U is the rank 2 lattice, which under a suitable choice of $\mathbb{Z}$-basis has Gram matrix

$$\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix},$$

and $E_8(-1)$ denotes the rank 8 lattice, which under a suitable choice of $\mathbb{Z}$-basis has Gram matrix

$$\begin{pmatrix} -2 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & -2 & 0 & 1 & 0 & 0 & 0 & 0 \\ 1 & 0 & -2 & 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 1 & -2 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & -2 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & -2 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & -2 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 & -2 \end{pmatrix}.$$

Theorem 1.13 ([Ser73, § V.2.2]). *Let L be a an even indefinite unimodular lattice of signature (r,s), and suppose that $s-r \geq 0$. Then $r \cong s \bmod 8$ and L is isometric to*

$$U^{\oplus r} \oplus E_8(-1)^{\oplus (s-r)/8}. \qquad \square$$

The above discussion can thus be summarized in the following theorem.

Theorem 1.14. *Let X be a complex K3 surface. The singular cohomology group $\mathrm{H}^2(X,\mathbb{Z})$, equipped with the cup-product, is an even indefinite unimodular lattice of signature $(3,19)$, isometric to the K3 lattice*

$$\Lambda_{\mathrm{K3}} := U^{\oplus 3} \oplus E_8(-1)^{\oplus 2}. \qquad \square$$

1.6. **Complex K3 surfaces are simply connected.**

Theorem 1.15. *Every complex K3 surface is simply connected.*

Sketch of the proof: The key ingredient is that all complex K3 surfaces are diffeomorphic to each other [BHPVdV04, VIII Corollary 8.6]; this theorem takes a fair amount of work: first, (complex) Kummer surfaces are diffeomorphic, because any two 2-tori are isomorphic as real Lie groups. Second, there is an open set in the period domain around the period point of a K3 surface where the K3 surface can be deformed. Third, projective Kummer surfaces are dense in the period domain. Putting these three ideas together shows all complex K3 surfaces are diffeomorphic. It thus suffices to compute $\pi_1(X,x)$ for a single K3 surface. We will pick X a smooth quartic in $\mathbb{P}^3_{\mathbb{C}}$ and apply the following proposition.

Proposition 1.16. *Any smooth quartic in $\mathbb{P}^3_{\mathbb{C}}$ is simply connected.*

Proof. Let $\nu : \mathbb{P}^3_{\mathbb{C}} \to \mathbb{P}^{34}_{\mathbb{C}}$ be the 4-uple embedding. Any smooth quartic $X \subset \mathbb{P}^3_{\mathbb{C}}$ is embedded under ν as $\nu(\mathbb{P}^3_{\mathbb{C}}) \cap H$ for some hyperplane $H \subset \mathbb{P}^{34}_{\mathbb{C}}$. By the Lefschetz hyperplane theorem $\pi_1(\nu(\mathbb{P}^3) \cap H)$ is isomorphic to $\pi_1(\nu(\mathbb{P}^3)) = \pi_1(\mathbb{P}^3) = 0$. □

1.7. **Differential geometry of complex K3 surfaces.** A theorem of Siu [Siu83] (see also [BHPVdV04, § IV.3]) asserts that complex K3 surfaces are Kähler; thus there is a Hodge decomposition on $\mathrm{H}^k(X, \mathbb{C}) \simeq \mathrm{H}^n_{\mathrm{dR}}(X)_{\mathbb{R}} \otimes_{\mathbb{R}} \mathbb{C}$ (here $\mathrm{H}^n_{\mathrm{dR}}(X)_{\mathbb{R}}$ denotes de Rham cohomology on the underlying real manifold X):

$$\mathrm{H}^k(X, \mathbb{C}) = \bigoplus_{p+q=k} \mathrm{H}^{p,q}(X), \tag{3}$$

where $\mathrm{H}^{p,q}(X)$ denotes the Dolbeault cohomology group of complex differential forms of type (p, q) (isomorphic by Dolbeaut's theorem to $\mathrm{H}^q(X, \Omega^p_X)$), which satisfy:

$$\mathrm{H}^{p,q}(X) = \overline{\mathrm{H}^{q,p}(X)} \qquad \text{and} \qquad \sum_{p+q=k} h^{p,q}(X) = b_k,$$

where $h^{p,q}(X) = \dim_{\mathbb{C}} \mathrm{H}^{p,q}(X)$, and $b_k = \mathrm{rk}(\mathrm{H}^k(X, \mathbb{Z})) = \dim_{\mathbb{C}} \mathrm{H}^k(X, \mathbb{C})$ denotes the k-th Betti number of X; see [Voi07, Chapter 6].

Proposition 1.17. *Let X be a complex K3 surface. The Hodge diamond of X is given by*

$$\begin{matrix} & & h^{0,0} & & \\ & h^{1,0} & & h^{0,1} & \\ h^{2,0} & & h^{1,1} & & h^{0,2} \\ & h^{2,1} & & h^{1,2} & \\ & & h^{2,2} & & \end{matrix} \quad = \quad \begin{matrix} & & 1 & & \\ & 0 & & 0 & \\ 1 & & 20 & & 1 \\ & 0 & & 0 & \\ & & 1 & & \end{matrix}$$

Proof. From $\mathrm{H}^1(X, \mathbb{Z}) = \mathrm{H}^3(X, \mathbb{Z}) = 0$ and the Hodge decomposition (3) applied to the complexification of these groups for $k = 1$ and 3 we get the vanishing of the second and fourth rows. We have $h^{0,0} = h^0(X, \mathscr{O}_X) = 1$, and from $\omega_X \simeq \mathscr{O}_X$ we get $h^{2,0} = 1$. Serre duality and $\omega_X \simeq \mathscr{O}_X$ together give $h^{0,2} = h^{0,0} = h^{2,2}$. Since $b_2 = h^{2,0} + h^{1,1} + h^{0,2} = 22$ we obtain $h^{1,1} = 20$. Finally, the $h^{p,q}$ "outside" this diamond vanish by Serre duality and dimension reasons. □

The lattice $\mathrm{H}^2(X, \mathbb{Z})$ can be endowed with a Hodge structure of weight 2. We review what this means; for more details see [Huy15, Chapter 3] and [Voi07, Chapter 7]

Definition 1.18. Let $\mathrm{H}_{\mathbb{Z}}$ be a free abelian group of finite rank. An integral Hodge structure of weight n on $\mathrm{H}_{\mathbb{Z}}$ is a decomposition, called the Hodge decomposition,

$$\mathrm{H}_{\mathbb{C}} := \mathrm{H}_{\mathbb{Z}} \otimes_{\mathbb{Z}} \mathbb{C} = \bigoplus_{p+q=n} \mathrm{H}^{p,q}$$

such that $\overline{\mathrm{H}^{p,q}} = \mathrm{H}^{q,p}$ and $\mathrm{H}^{p,q} = 0$ for $p < 0$.

When X is a complex K3 surface, the middle cohomology decomposes as

$$\mathrm{H}^2(X,\mathbb{C}) \cong \mathrm{H}^{2,0}(X) \oplus \mathrm{H}^{1,1}(X) \oplus \mathrm{H}^{0,2}(X),$$

and the outer pieces are 1-dimensional. The cup product on $\mathrm{H}^2(X,\mathbb{Z})$ extends to a symmetric bilinear pairing on $\mathrm{H}^2(X,\mathbb{C})$, equal to the bilinear form $(\alpha,\beta) \mapsto \int_X \alpha \wedge \beta$. Write $\mathrm{H}^{2,0}(X) = \mathbb{C}\omega_X$. Then the Hodge–Riemann relations assert that

(1) $(\omega_X, \omega_X) = 0$;

(2) $(\omega_X, \overline{\omega_X}) > 0$;

(3) $V := \mathrm{H}^{2,0}(X) \oplus \mathrm{H}^{0,2}(X)$ is orthogonal to $\mathrm{H}^{1,1}(X)$.

Exercise 1.19. Check the Hodge–Riemann relations above.

Thus $\mathbb{C}\omega_X = \mathrm{H}^{2,0}(X)$ determines the Hodge decomposition on $\mathrm{H}^2(X,\mathbb{C})$. Let

$$V_{\mathbb{R}} = \{v \in V \colon v = \overline{v}\} = \mathbb{R} \cdot \{\omega_X + \overline{\omega_X}, i(\omega_X - \overline{\omega_X})\},$$

so that $V := V_{\mathbb{R}} \otimes_{\mathbb{R}} \mathbb{C}$. The intersection form restricted to $V_{\mathbb{R}}$ is positive definite and diagonal on the basis given above. Hence, the cup product restricted to $\mathrm{H}^{1,1}(X) \cap \mathrm{H}^2(X,\mathbb{R})$ has signature $(1,19)$.

1.8. **The Néron-Severi lattice of a complex K3 surface.** For a complex K3 surface, the long exact sequence (2) associated to the exponential sequence and the vanishing $\mathrm{H}^1(X,\mathscr{O}_X) = 0$ give an injection

$$c_1 : \mathrm{Pic}(X) \cong \mathrm{H}^1(X,\mathscr{O}_X^*) \hookrightarrow \mathrm{H}^2(X,\mathbb{Z}).$$

which is also called the first Chern class. Let $i_* \colon \mathrm{H}^2(X,\mathbb{Z}) \to \mathrm{H}^2(X,\mathbb{C})$ be the canonical map. The Lefschetz (1,1)-theorem says that the image of $i_* \circ c_1$ is $\mathrm{H}^{1,1}(X) \cap i_*\mathrm{H}^2(X,\mathbb{Z})$. It is called the Néron-Severi lattice $\mathrm{NS}\,X$. When X is an algebraic K3 surface, this lattice coincides with the Néron-Severi group previously defined in §1.3 by Proposition 1.8 and the GAGA principle [Ser55, Proposition 18 and the remarks that follow].

In words, the Néron-Severi lattice consists of the integral classes in $\mathrm{H}^2(X,\mathbb{Z})$ that are closed (1,1)-forms. In particular, the Picard number $\rho(X) = \mathrm{rk}\,\mathrm{NS}(X) = \mathrm{rk}\,\mathrm{Pic}(X)$ is at most the dimension of $\mathrm{H}^{1,1}(X)$.

Proposition 1.20. *Let X be a complex K3 surface. Then $0 \le \rho(X) \le 20$. If X is algebraic, then the signature of $\mathrm{NS}\,X \otimes \mathbb{R}$ is $(1, \rho(X)-1)$.* □

1.9. **The Torelli theorem.** A marking on a complex K3 surface X is an isometry, i.e., an isomorphism of lattices,

$$\Phi \colon \mathrm{H}^2(X,\mathbb{Z}) \xrightarrow{\sim} \Lambda_{\mathrm{K3}}.$$

A marked complex K3 surface is a pair (X,Φ) as above. We denote the complexification of Φ by $\Phi_{\mathbb{C}}$. The period point of (X,Φ) is $\Phi_{\mathbb{C}}(\mathbb{C}\omega_X) \in \mathbb{P}(\Lambda_{\mathrm{K3}} \otimes \mathbb{C})$. By the Hodge–Riemann relations, the period point lies in an open subset Ω (in the complex topology) of a 20-dimensional quadric inside $\mathbb{P}(\Lambda_{\mathrm{K3}} \otimes \mathbb{C})$:

$$\Omega = \{x \in \mathbb{P}(\Lambda_{\mathrm{K3}} \otimes \mathbb{C}) : (x, x) = 0, (x, \overline{x}) > 0\};$$

here $(\, , \,)$ denotes the bilinear form on $\Lambda_{\mathrm{K3}} \otimes \mathbb{C}$. We call Ω the period domain of complex K3 surfaces.

Exercise 1.21. Check that Ω is indeed an open subset of a quadric in $\mathbb{P}(\Lambda_{\mathrm{K3}} \otimes \mathbb{C}) \simeq \mathbb{P}^{21}_{\mathbb{C}}$.

Theorem 1.22 (Weak Torelli theorem [PŠŠ71, BR75, LP80])**.** *Two complex K3 surfaces X and X' are isomorphic if and only if there are markings*

$$\Phi \colon \mathrm{H}^2(X, \mathbb{Z}) \xrightarrow{\sim} \Lambda_{\mathrm{K3}} \xleftarrow{\sim} \mathrm{H}^2(X', \mathbb{Z}) \colon \Phi'$$

whose period points in Ω coincide. □

The weak Torelli theorem follows from the strong Torelli theorem. We briefly explain the statement of the latter. Since the intersection form on $\mathrm{H}^{1,1}(X) \cap \mathrm{H}^2(X, \mathbb{R})$ is indefinite, the set $\{x \in \mathrm{H}^{1,1}(X) \cap \mathrm{H}^2(X, \mathbb{R}) : (x, x) > 0\}$ has two connected components. Exactly one of these components contains Kähler classes[2]; we call this component the positive cone. A class $x \in \mathrm{NS}\, X$ is effective if there is an effective divisor D on X such that $x = i_* \circ c_1(\mathscr{O}_X(D))$.

Theorem 1.23 (Strong Torelli Theorem)**.** *Let (X, Φ) and (X', Φ') be marked complex K3 surfaces whose period points on Ω coincide. Suppose that*

$$f^* = (\Phi')^{-1} \circ \Phi \colon \mathrm{H}^2(X, \mathbb{Z}) \to \mathrm{H}^2(X', \mathbb{Z})$$

takes the positive cone of X to the positive cone of X', and induces a bijection between the respective sets of effective classes. Then there is a unique isomorphism $f \colon X' \to X$ inducing f^*. □

1.10. **Surjectivity of the period map.** A point $\omega \in \mathbb{P}(\Lambda_{\mathrm{K3}} \otimes \mathbb{C})$ gives a 1-dimensional $\mathbb{C}$-linear subspace $\mathrm{H}^{2,0} \subseteq \Lambda_{\mathrm{K3}} \otimes \mathbb{C}$. Let $\mathrm{H}^{0,2} = \overline{\mathrm{H}^{2,0}} \subseteq \Lambda_{\mathrm{K3}} \otimes \mathbb{C}$ be the conjugate linear subspace, and let $\mathrm{H}^{1,1}$ be the orthogonal complement of $\mathrm{H}^{2,0} \oplus \mathrm{H}^{0,2}$, with respect to the $\mathbb{C}$-linear extension of the bilinear form on Λ_{K3}. We say $\mathrm{H}^{2,0} \oplus \mathrm{H}^{1,1} \oplus \mathrm{H}^{0,2}$ is a decomposition of K3 type for $\Lambda_{\mathrm{K3}} \otimes \mathbb{C}$.

Theorem 1.24 (Surjectivity of the period map [Tod80])**.** *Given a point $\omega \in \Omega$ inducing a decomposition $\Lambda_{\mathrm{K3}} \otimes \mathbb{C} = \mathrm{H}^{2,0} \oplus \mathrm{H}^{1,1} \oplus \mathrm{H}^{0,2}$ of K3 type there exists a complex K3 surface X and a marking $\Phi : \mathrm{H}^2(X, \mathbb{Z}) \xrightarrow{\sim} \Lambda_{\mathrm{K3}}$ whose $\mathbb{C}$-linear extension preserves Hodge decompositions.* □

[2] A Kähler class $h \in \mathrm{H}^2(X, \mathbb{R})$ is a class that can be represented by a real $(1, 1)$-form which in local coordinates (z_1, z_2) can be written as $i \sum \alpha_{ij} z_i \wedge \overline{z}_j$, where the hermitian matrix $(\alpha_{ij}(p))$ is positive definite for every $p \in X$.

1.11. **Lattices and discriminant groups.** To give an application of the above results, we need a few facts about lattices; the objects introduced here will also play a decisive role in identifying nontrivial elements of the Brauer group of a complex K3 surface.

Although we have already been using the concept of lattice in previous sections, we start here from scratch, for the sake of clarity and completeness. A lattice L is a free abelian group of finite rank endowed with a symmetric nondegenerate integral bilinear form

$$\langle\ ,\ \rangle\colon L \times L \to \mathbb{Z}.$$

We say L is even if $\langle x, x\rangle \in 2\mathbb{Z}$ for all $x \in L$. We may extend $\langle\ ,\ \rangle$ $\mathbb{Q}$-linearly to $L \otimes \mathbb{Q}$, and define the dual abelian group

$$L^\vee := \operatorname{Hom}(L, \mathbb{Z}) \simeq \{x \in L \otimes \mathbb{Q} : \langle x, y\rangle \in \mathbb{Z} \text{ for all } y \in L\}.$$

There is an injective map of abelian groups $L \to L^\vee$ sending x to $\phi_x\colon y \mapsto \langle x, y\rangle$. The discriminant group is $L^\vee/L$, which is finite since $\langle\ ,\ \rangle$ is nondegenerate. Its order is the absolute value of the discriminant of L. For an even lattice L we define the discriminant form by

$$q_L\colon L^\vee/L \to \mathbb{Q}/2\mathbb{Z} \qquad x + L \mapsto \langle x, x\rangle \bmod 2\mathbb{Z}.$$

Let $\ell(L)$ be the minimal number of generators of $L^\vee/L$ as an abelian group.

Theorem 1.25 ([Nik79, Corollary 1.13.3]). *If a lattice L is even and indefinite (when tensored with $\mathbb{R}$), and $\operatorname{rk} L \geq \ell(L)+2$ then L is determined up to isometry by its rank, signature and its discriminant form.* □

An embedding of lattices $L \hookrightarrow M$ is primitive if it has saturated image, i.e., if the cokernel M/L is torsion-free.

Exercise 1.26. Let $L \hookrightarrow M$ be an embedding of lattices, and write let $L^\perp = \{x \in M : \langle x, y\rangle = 0 \text{ for all } y \in L\}$.

(1) Show that $L^\perp$ is a primitive sublattice of M.

(2) Show that if L is primitive, then $(L^\perp)^\perp = L$.

Theorem 1.27 ([Nik79, Corollary 1.12.3]). *There exists a primitive embedding $L \hookrightarrow \Lambda_{\mathrm{K3}}$ of an even lattice L of rank r and signature $(p, r-p)$ into the K3 lattice Λ_{K3} if $p \leq 3$, $r - p \leq 19$, and $\ell(L) \leq 22 - r$.* □

1.12. **K3 surfaces out of lattices.** We conclude our discussion of the geometry of complex K3 surfaces with an application of the foregoing results, in the spirit of [Mor88, §12].

Question: Is there a complex K3 surface X with $\operatorname{Pic} X$ a rank 2 lattice with the following intersection form?

	H	C
H	4	8
C	8	4

(A better question would be: does there exist a smooth quartic surface $X \subset \mathbb{P}^3$ containing a smooth curve C of genus 3 and degree 8? Such a surface would contain the above lattice in its Picard group. The answer to this question is yes, but it would take a little more technology than we've developed to answer this better question.)

Let $L = \mathbb{Z}H + \mathbb{Z}C$, with an intersection pairing given by the above Gram matrix. By Theorem 1.27, we know there is a primitive embedding $L \hookrightarrow \Lambda_{\mathrm{K3}}$; fix such an embedding. Our next move is to construct a Hodge structure of weight two on Λ_{K3}

$$\Lambda_{\mathrm{K3}} \otimes \mathbb{C} = \mathrm{H}^{2,0} \oplus \mathrm{H}^{1,1} \oplus \mathrm{H}^{0,2}$$

such that $\mathrm{H}^{1,1} \cap \Lambda_{\mathrm{K3}} = L$. For this, choose $\omega \in \Lambda_{\mathrm{K3}} \otimes \mathbb{C}$ satisfying $(\omega, \omega) = 0$, $(\omega, \overline{\omega}) > 0$, in such a way that $L^\perp \otimes \mathbb{Q}$ is the smallest $\mathbb{Q}$-vector space of $\Lambda_{\mathrm{K3}} \otimes \mathbb{Q}$ whose complexification contains ω. Essentially, this means that we want to set $\omega = \sum \alpha_i x_i$ where $\{x_i\}$ is a basis for $L^\perp \otimes \mathbb{Q}$ and the α_i are algebraically independent transcendental numbers except for the conditions imposed by the relation $(\omega, \omega) = 0$. Then:

$$\mathrm{H}^{1,1} \cap (\Lambda_{\mathrm{K3}} \otimes \mathbb{Q}) = (L^\perp)^\perp \otimes \mathbb{Q} = L \otimes \mathbb{Q},$$

which by the saturatedness of L implies that $\mathrm{H}^{1,1} \cap \Lambda_{\mathrm{K3}} = L$. By Theorem 1.24, there exists a K3 surface X and a marking $\Phi \colon \mathrm{H}^2(X, \mathbb{Z}) \xrightarrow{\sim} \Lambda_{\mathrm{K3}}$ such that $\mathrm{NS}(X) \cong L$. Using stronger versions of Theorem 1.24 (e.g. [Mor88, p. 70]), one can show that $h = \Phi^{-1}(H)$ is ample. Furthermore, Reider's method can be used to show that h is very ample.

2. Picard Numbers of K3 surfaces

References: [Ter85, Ell04, Klo07, vL07, Sch09, EJ08, EJ11a, EJ11b, EJ12, Sch13, HKT13, Cha14, PTvL15]

In this section, all K3 surfaces considered are algebraic. Let X be a K3 surface over a field K. Fix an algebraic closure $\overline{K}$ of K, and let $\overline{X} = X \times_K \overline{K}$. Let $\rho(\overline{X})$ denote the rank of the Néron-Severi group $\mathrm{NS}\,\overline{X}$ of $\overline{X}$. The goal of this section is to give an account of the explicit computation of $\rho(\overline{X})$ in the case when K is a number field. One of the key tools is reduction modulo a finite prime $\mathfrak{p}$ of K. We will see that whenever X has good reduction at $\mathfrak{p}$, there is an injective specialization homomorphism $\mathrm{NS}\,\overline{X} \hookrightarrow \mathrm{NS}\,\overline{X}_{\mathfrak{p}}$. For a prime ℓ different from the residue characteristic of $\mathfrak{p}$ there is in turn an injective cycle class map $\mathrm{NS}\,\overline{X}_{\mathfrak{p}} \otimes \mathbb{Q}_\ell \hookrightarrow \mathrm{H}^2_{\text{ét}}(\overline{X}_{\mathfrak{p}}, \mathbb{Q}_\ell(1))$ of Galois modules. The basic idea is to use the composition of these two maps (after tensoring the first one by $\mathbb{Q}_\ell$) for *several* finite primes $\mathfrak{p}$ to establish tight upper bounds on $\rho(\overline{X})$. We begin by explaining what good reduction means, and where the two maps above come from.

2.1. **Good reduction.**

Definition 2.1. Let R be a Dedekind domain, set $K = \operatorname{Frac} R$, and let $\mathfrak{p} \subseteq R$ be a nonzero prime ideal. Let X be a smooth proper K-variety. We say X has good reduction at $\mathfrak{p}$ if X

has a smooth proper $R_{\mathfrak{p}}$-model, i.e., if there exists a smooth proper morphism $\mathcal{X} \to \operatorname{Spec} R_{\mathfrak{p}}$, such that $\mathcal{X} \times_{R_{\mathfrak{p}}} K \simeq X$ as K-schemes.

Remark 2.2. Let $k = R_{\mathfrak{p}}/\mathfrak{p}R_{\mathfrak{p}}$ be the residue field at $\mathfrak{p}$. The special fiber $\mathcal{X} \times_{R_{\mathfrak{p}}} k$ is a smooth proper k-scheme.

Remark 2.3. The ring $R_{\mathfrak{p}}$ is always a discrete valuation ring [AM69, Theorem 9.3].

Example 2.4. Let p be a rational prime and let

$$R = \mathbb{Z}_{(p)} = \{m/n \in \mathbb{Q} : m \in \mathbb{Z}, n \in \mathbb{Z} \setminus \{0\} \text{ and } p \nmid n\}.$$

Set $\mathfrak{p} = p\mathbb{Z}_{(p)}$. In this case $K = \mathbb{Q}$ and $R_{\mathfrak{p}} = R$. Let $X \subseteq \mathbb{P}^3 = \operatorname{Proj} \mathbb{Q}[x, y, z, w]$ be the K3 surface over $\mathbb{Q}$ given by

$$x^4 + 2y^4 = z^4 + 4w^4.$$

Let $\mathcal{X} = \operatorname{Proj} \mathbb{Z}_{(p)}[x, y, z, w]/(x^4 + 2y^4 - z^4 - 4w^4)$. Note that if $p \neq 2$, then $\mathcal{X}$ is smooth and proper over R, and $\mathcal{X} \times_R \mathbb{Q} \simeq X$. Hence X has good reduction at primes $p \neq 2$.

Exercise 2.5. Prove that the conic $X := \operatorname{Proj} \mathbb{Q}[x, y, z]/(xy - 19z^2)$ has good reduction at $p = 19$. Naively, we might think that p is not a prime of good reduction if reducing the equations of X mod p gives a singular variety over the residue field. This example is meant to illustrate that this intuition can be wrong.

2.2. **Specialization.** In this section, we follow the exposition in [MP12, §3]; the reader is urged to consult this paper and the references contained therein for a more in-depth treatment of specialization of Néron-Severi groups.

Let R be a discrete valuation ring with fraction field K and residue field k. Fix an algebraic closure $\overline{K}$ of K, and let $\overline{R}$ be the integral closure of R in $\overline{K}$. Choose a nonzero prime $\mathfrak{p} \in \overline{R}$ so that $\bar{k} = \overline{R}/\mathfrak{p}$ is an algebraic closure of k. For each finite extension L/K contained in $\overline{K}$, we let R_L be the integral closure of R in L. This is a Dedekind domain, and thus the localization of R_L at $\mathfrak{p} \cap R_L$ is a discrete valuation ring R'_L; call its residue field k'.

Let $\mathcal{X}$ be a smooth proper R-scheme. Restriction of Weil divisors, for example, gives natural group homomorphisms

$$\operatorname{Pic} \mathcal{X}_L \leftarrow \operatorname{Pic} \mathcal{X}_{R'_L} \to \operatorname{Pic} \mathcal{X}_{k'}, \tag{4}$$

and the map $\operatorname{Pic} \mathcal{X}_{R'_L} \to \operatorname{Pic} \mathcal{X}_L$ is an isomorphism (see the proof of [BLR90, §8.4 Theorem 3]). If $\mathcal{X} \to \operatorname{Spec} R$ has relative dimension 2, then the induced map[3] $\operatorname{Pic} \mathcal{X}_L \to \operatorname{Pic} \mathcal{X}_{k'}$ preserves the intersection product on surfaces [Ful98, Corollary 20.3]. Taking the direct limit over L of the maps (4) gives a homomorphism

$$\operatorname{Pic} \mathcal{X}_{\overline{K}} \to \operatorname{Pic} \mathcal{X}_{\bar{k}}$$

that preserves intersection products of surfaces when $\mathcal{X} \to \operatorname{Spec} R$ has relative dimension 2.

[3]This map has a simple description at the level of cycles: given a prime divisor on $\mathcal{X}_L$, take its Zariski closure in $\mathcal{X}_{R'_L}$ and restrict to $\mathcal{X}_{k'}$. This operation respects linear equivalence and can be linearly extended to $\operatorname{Pic} \mathcal{X}_L$.

Proposition 2.6. *With notation as above, if* $\mathcal{X} \to \operatorname{Spec} R$ *is a proper, smooth morphism of relative dimension* 2*, then* $\rho(\mathcal{X}_{\overline{K}}) \le \rho(\mathcal{X}_{\bar{k}})$*.*

Proof. Since the map $\operatorname{Pic} \mathcal{X}_{\overline{K}} \to \operatorname{Pic} \mathcal{X}_{\bar{k}}$ preserves intersection products, it induces an injection

$$\operatorname{Pic} \mathcal{X}_{\overline{K}} / \operatorname{Pic}^\tau \mathcal{X}_{\overline{K}} \hookrightarrow \operatorname{Pic} \mathcal{X}_{\bar{k}} / \operatorname{Pic}^\tau \mathcal{X}_{\bar{k}}.$$

The claim now follows from the isomorphism $\operatorname{Pic} \overline{Y} / \operatorname{Pic}^\tau \overline{Y} \simeq \operatorname{NS} \overline{Y} / (\operatorname{NS} \overline{Y})_{\mathrm{tors}}$ [Tat65, p. 98], applied to $Y = \mathcal{X}_K$ and $\mathcal{X}_k$. □

Remark 2.7. The hypothesis that $\mathcal{X} \to \operatorname{Spec} R$ has relative dimension 2 in Proposition 2.6 is not necessary, but it simplifies the exposition. See [Ful98, Example 20.3.6].

We can do a little better than Proposition 2.6. Indeed, without any assumption on the relative dimension of $\mathcal{X} \to \operatorname{Spec} R$, the map $\operatorname{Pic} \mathcal{X}_{\overline{K}} \to \operatorname{Pic} \mathcal{X}_{\bar{k}}$ gives rise to a specialization homomorphism

$$\operatorname{sp}_{\overline{K},\bar{k}} \colon \operatorname{NS} \mathcal{X}_{\overline{K}} \to \operatorname{NS} \mathcal{X}_{\bar{k}};$$

see [MP12, Proposition 3.3].

Theorem 2.8. *With notation as above, if* $\operatorname{char} k = p > 0$*, then the map*

$$\operatorname{sp}_{\overline{K},\bar{k}} \otimes_{\mathbb{Z}} \mathrm{id}_{\mathbb{Z}[1/p]} \colon \operatorname{NS} \mathcal{X}_{\overline{K}} \otimes_{\mathbb{Z}} \mathbb{Z}[1/p] \to \operatorname{NS} \mathcal{X}_{\bar{k}} \otimes_{\mathbb{Z}} \mathbb{Z}[1/p]$$

is injective and has torsion-free cokernel.

Proof. See [MP12, Proposition 3.6]. □

Remark 2.9. If Y is a K3 surface over a field then $\operatorname{NS} \overline{Y} \simeq \operatorname{Pic} \overline{Y}$ (Proposition 1.8), so $\operatorname{sp}_{\overline{K},\bar{k}}$ is the map we already know, and it is already injective before tensoring with $\mathbb{Z}[1/p]$.

The moral of the story so far (Proposition 2.6) is that if X is a smooth projective surface over a number field, then we can use information at a prime of good reduction for X to bound $\rho(\overline{X})$. The key tool is the cycle class map, which we turn to next; this map is the algebraic version of the connecting homomorphism in the long exact sequence in cohomology associated to the exponential sequence.

2.3. **The cycle class map.** In this section we let X be a smooth projective geometrically integral variety over a finite field $\mathbb{F}_q$ with $q = p^r$ elements (p prime). Write $\overline{\mathbb{F}}_q$ for a fixed algebraic closure of $\mathbb{F}_q$, and let $\sigma \in \operatorname{Gal}(\overline{\mathbb{F}}_q/\mathbb{F}_q)$ denote the Frobenius automorphism $x \mapsto x^q$. Let $\overline{X}_{\text{ét}}$ denote the (small) étale site of $\overline{X} := X \times_{\mathbb{F}_q} \overline{\mathbb{F}}_q$, and let $\ell \neq p$ be a prime. For an integer $m \geq 1$, the Tate twist $(\mathbb{Z}/\ell^n\mathbb{Z})(m)$ is the sheaf $\mu_{\ell^n}^{\otimes m}$ on $\overline{X}_{\text{ét}}$. For a fixed m there is a natural surjection $(\mathbb{Z}/\ell^{n+1}\mathbb{Z})(m) \to (\mathbb{Z}/\ell^n\mathbb{Z})(m)$; putting these maps together, we define

$$\mathrm{H}^2_{\text{ét}}(\overline{X}, \mathbb{Z}_\ell(m)) := \varprojlim_n \mathrm{H}^2_{\text{ét}}(\overline{X}, (\mathbb{Z}/\ell^n\mathbb{Z})(m)),$$

$$\mathrm{H}^2_{\text{ét}}(\overline{X}, \mathbb{Q}_\ell(m)) := \mathrm{H}^2_{\text{ét}}(\overline{X}, \mathbb{Z}_\ell(m)) \otimes_{\mathbb{Z}_\ell} \mathbb{Q}_\ell.$$

Since $\ell \neq p$, the Kummer sequence

$$0 \to \mu_{\ell^n} \to \mathbb{G}_m \xrightarrow{[\ell^n]} \mathbb{G}_m \to 0$$

is an exact sequence of sheaves on $\overline{X}_{\text{ét}}$ [Mil80, p. 66], so the long exact sequence in étale cohomology gives a boundary map

$$\delta_n \colon \mathrm{H}^1_{\text{ét}}(\overline{X}, \mathbb{G}_m) \to \mathrm{H}^2_{\text{ét}}(\overline{X}, \mu_{\ell^n}). \tag{5}$$

Since $\mathrm{H}^1_{\text{ét}}(\overline{X}, \mathbb{G}_m) \simeq \operatorname{Pic} \overline{X}$ [Mil80, III.4.9], taking the inverse limit of (5) with respect to the ℓ-th power maps $\{\mu_{\ell^{n+1}} \to \mu_{\ell^n}\}$ we obtain a homomorphism

$$\operatorname{Pic} \overline{X} \to \mathrm{H}^2_{\text{ét}}(\overline{X}, \mathbb{Z}_\ell(1)). \tag{6}$$

The kernel of this map is the group $\operatorname{Pic}^\tau \overline{X}$ of divisors numerically equivalent to zero [Tat65, pp. 97–98], and since $\operatorname{Pic} \overline{X} / \operatorname{Pic}^\tau \overline{X} \simeq \operatorname{NS} \overline{X} / (\operatorname{NS} \overline{X})_{\text{tors}}$, tensoring (6) with $\mathbb{Q}_\ell$ gives an injection

$$c \colon \operatorname{NS} \overline{X} \otimes \mathbb{Q}_\ell \hookrightarrow \mathrm{H}^2_{\text{ét}}(\overline{X}, \mathbb{Q}_\ell(1)). \tag{7}$$

The map c is compatible with the action of $\operatorname{Gal}(\overline{\mathbb{F}}_q/\mathbb{F}_q)$, and moreover, there is an isomorphism of $\operatorname{Gal}(\overline{\mathbb{F}}_q/\mathbb{F}_q)$-modules

$$\mathrm{H}^2_{\text{ét}}(\overline{X}, \mathbb{Q}_\ell(1)) \simeq \underbrace{\left(\varprojlim_n \mathrm{H}^2_{\text{ét}}(\overline{X}, \mathbb{Z}/\ell^n\mathbb{Z}) \otimes_{\mathbb{Z}_\ell} \mathbb{Q}_\ell \right)}_{=:\, \mathrm{H}^2_{\text{ét}}(\overline{X}, \mathbb{Q}_\ell)} \otimes_{\mathbb{Z}_\ell} \left(\mathbb{Q}_\ell \otimes_{\mathbb{Z}_\ell} \varprojlim \mu_{\ell^n} \right), \tag{8}$$

where $\operatorname{Gal}(\overline{\mathbb{F}}_q/\mathbb{F}_q)$ acts on $\mathbb{Q}_\ell \otimes_{\mathbb{Z}_\ell} \varprojlim \mu_{\ell^n}$ according to the usual action of $\operatorname{Gal}(\overline{\mathbb{F}}_q/\mathbb{F}_q)$ on $\mu_{\ell^n} \subset \overline{\mathbb{F}}_q$. In particular, the Frobenius automorphism σ acts as multiplication by q on $\mathbb{Q}_\ell \otimes_{\mathbb{Z}_\ell} \varprojlim \mu_{\ell^n}$: indeed, we are regarding $\mu_{\ell^n} \subset \overline{\mathbb{F}}_q$ as a $\mathbb{Z}/\ell^n\mathbb{Z}$-module via the multiplication $m \cdot \zeta := \zeta^m$.

Proposition 2.10. *Let X be a smooth proper scheme over a finite field $\mathbb{F}_q$ of cardinality $q = p^r$ with p prime. Write $\sigma \in \operatorname{Gal}(\overline{\mathbb{F}}_q/\mathbb{F}_q)$ for the Frobenius automorphism $x \mapsto x^q$. Let $\ell \neq p$ be a prime and let $\sigma^*(0)$ denote the automorphism of $\mathrm{H}^2_{\acute{e}t}(\overline{X}, \mathbb{Q}_\ell)$ induced by σ. Then $\rho(\overline{X})$ is bounded above by the number of eigenvalues of $\sigma^*(0)$, counted with multiplicity, of the form ζ/q, where ζ is a root of unity.*

Proof. Write σ^* for the automorphisms of $\operatorname{NS} \overline{X}$ induced by σ. The divisor classes generating $\operatorname{NS} \overline{X}$ are defined over a *finite* extension of k, so some power of σ^* acts as the identity on $\operatorname{NS} \overline{X}$. Hence, all eigenvalues of σ^* are roots of unity. Using the injection (7), we deduce that $\rho(\overline{X})$ is bounded above by the number of eigenvalues of $\sigma^*(1)$ operating on $\mathrm{H}^2_{\text{ét}}(X_{\bar{k}}, \mathbb{Q}_\ell(1))$ that are roots of unity. The isomorphism (8) shows that this number is in turn equal to the number of eigenvalues of $\sigma^*(0)$ operating on $\mathrm{H}^2_{\text{ét}}(X_{\bar{k}}, \mathbb{Q}_\ell)$ of the form ζ/q, where ζ is a root of unity. ∎

Remark 2.11. Let $\mathbb{F} \subseteq \overline{\mathbb{F}}_q$ be a finite extension of $\mathbb{F}_q$. The Tate conjecture [Tat65, p. 98] implies that

$$c(\operatorname{NS} X_{\mathbb{F}} \otimes \mathbb{Q}_\ell) = \mathrm{H}^2_{\text{ét}}(\overline{X}, \mathbb{Q}_\ell(1))^{\operatorname{Gal}(\overline{\mathbb{F}}_q/\mathbb{F})}.$$

One can deduce that the upper bound in Proposition 2.10 is sharp (exercise!). This conjecture has now been established for K3 surfaces X when q is odd [Nyg83, NO85, Cha13, Mau14, MP15], and also for q even if the geometric Picard rank of the surface is ≥ 2 [Cha16].

Proposition 2.10 implies that knowledge of the characteristic polynomial of σ^* acting on $\mathrm{H}^2_{\text{ét}}(\overline{X}, \mathbb{Q}_\ell)$ gives an upper bound for $\rho(\overline{X})$. It turns out that it is easier to calculate the characteristic polynomial of $(\sigma^*)^{-1}$, because we can relate this problem to point counts for X over a finite number of finite extensions of $\mathbb{F}_q$. To this end, we take a moment to understand what $(\sigma^*)^{-1}$ looks like.

2.3.1. *Absolute Frobenius.* For a scheme Z over a finite field $\mathbb{F}_q$ (with $q = p^r$), we let $F_Z \colon Z \to Z$ be the absolute Frobenius map: this map is the identity on points, and $x \mapsto x^p$ on the structure sheaf; it is *not* a morphism of $\mathbb{F}_q$-schemes. Set $\Phi_Z = F_Z^r$; the map $\Phi_Z \times 1 \colon Z \times \overline{\mathbb{F}}_q \to Z \times \overline{\mathbb{F}}_q$ induces a linear transformation $\Phi_Z^* \colon \mathrm{H}^2_{\text{ét}}(\overline{Z}, \mathbb{Q}_\ell) \to \mathrm{H}^2_{\text{ét}}(\overline{Z}, \mathbb{Q}_\ell)$. The action of F_Z on $Z_{\text{ét}}$ is (naturally equivalent to) the identity [Mil80, VI Lemma 13.2], and since $F_{\overline{Z}}^r = F_Z^r \times F_{\bar{k}}^r = \Phi_Z \times \sigma$, the maps Φ_Z^* and $\sigma^*(0)$ operate as each other's inverses on $\mathrm{H}^2_{\text{ét}}(\overline{Z}, \mathbb{Q}_\ell)$. Using the notation of Proposition 2.10, we conclude that the number of eigenvalues of $\sigma^*(0)$ operating on $\mathrm{H}^2_{\text{ét}}(\overline{X}, \mathbb{Q}_\ell)$ of the form ζ/q is equal to the number of eigenvalues of Φ_X^* operating on $\mathrm{H}^2_{\text{ét}}(\overline{X}, \mathbb{Q}_\ell)$ of the form $q\zeta$, where ζ is a root of unity.

2.4. **Upper bounds I: Putting everything together.**

Theorem 2.12. *Let R be a discrete valuation ring of a number field K, with residue field $k \simeq \mathbb{F}_q$. Fix an algebraic closure $\overline{K}$ of K, and let $\overline{R}$ be the integral closure of R in $\overline{K}$. Choose a nonzero prime $\mathfrak{p} \in \overline{R}$ so that $\bar{k} = \overline{R}/\mathfrak{p}$ is an algebraic closure of k. Let $\ell \neq \operatorname{char} k$ be a prime number.*

Let $\mathcal{X} \to R$ be a smooth proper morphism of relative dimension 2, and assume that the surfaces $\mathcal{X}_{\overline{K}}$ and $\mathcal{X}_{\bar{k}}$ are geometrically integral. There are natural injective homomorphisms of $\mathbb{Q}_\ell$-inner product spaces

$$\operatorname{NS} \mathcal{X}_{\overline{K}} \otimes \mathbb{Q}_\ell \hookrightarrow \operatorname{NS} \mathcal{X}_{\bar{k}} \otimes \mathbb{Q}_\ell \hookrightarrow \mathrm{H}^2_{\acute{e}t}(\mathcal{X}_{\bar{k}}, \mathbb{Q}_\ell(1))$$

and the second map is compatible with $\operatorname{Gal}(\bar{k}/k)$-actions. Consequently, $\rho(\mathcal{X}_{\overline{K}})$ is bounded above by the number of eigenvalues of $\Phi_{\mathcal{X}_k}^*$ operating on $\mathrm{H}^2_{\acute{e}t}(\mathcal{X}_{\bar{k}}, \mathbb{Q}_\ell)$, counted with multiplicity, of the form $q\zeta$, where ζ is a root of unity. □

Convention 2.13. We will apply Theorem 2.12 to K3 surfaces X over a number field K. In such cases, we will speak of a finite prime $\mathfrak{p} \subseteq \mathcal{O}_K$ of good reduction for X. The model $\mathcal{X} \to \operatorname{Spec} R$ with $R = (\mathcal{O}_K)_{\mathfrak{p}}$ satisfying the hypotheses of Theorem 2.12 will be implicit, and we will write $\overline{X}$ for the ($\overline{K}$-isomorphic) scheme $\mathcal{X}_{\overline{K}}$, and $\overline{X}_{\mathfrak{p}}$ for $\mathcal{X}_{\bar{k}}$.

Keep the notation of Theorem 2.12. The number of eigenvalues of $\Phi^*_{\mathcal{X}_k}$ of the form $q\zeta$ can be read off from the characteristic polynomial $\psi_q(x)$ of this linear operator. To compute this characteristic polynomial, we use two ideas. First, the characteristic polynomial of a linear operator on a finite dimensional vector space can be recovered from knowing traces of sufficiently many powers of the linear operator, as follows.

Theorem 2.14 (Newton's identities)**.** *Let T be a linear operator on a vector space V of finite dimension n. Write t_i for the trace of the i-fold composition T^i of T, and define*

$$a_1 := -t_1 \qquad \text{and} \qquad a_k := -\frac{1}{k}\left(t_k + \sum_{j=1}^{k-1} a_j t_{k-j}\right) \quad \text{for } k = 2, \ldots, n.$$

Then the characteristic polynomial of T is equal to

$$\det(x \cdot Id - T) = x^n + a_1 x^{n-1} + \cdots + a_{n-1}x + a_n.$$

Second, the traces of powers of $\Phi^*_{\mathcal{X}_k}$ operating on $\mathrm{H}^2_{\text{ét}}(\mathcal{X}_{\bar{k}}, \mathbb{Q}_\ell)$ can be recovered from the Lefschetz trace formula

$$\mathrm{Tr}\left((\Phi^*_{\mathcal{X}_k})^i\right) = \#\mathcal{X}_k(\mathbb{F}_{q^i}) - 1 - q^{2i};$$

see [Man86, §27] for a proof of this formula in the surface case. When $\mathcal{X}_k$ is a K3 surface, we have $n = 22$, so at first glance we have to count points over $\mathbb{F}_{q^i}$ for $i = 1, \ldots, 22$. However, the characteristic polynomial of $\Phi^*_{\mathcal{X}_k}$ happens to satisfy a functional equation, coming from the Weil conjectures (which have all been proved):

$$q^{22}\psi_q(x) = \pm x^{22}\psi_q(q^2/x).$$

If we are lucky, counting points over $\mathbb{F}_{q^i}$ for $i = 1, \ldots, 11$ will be enough to determine the sign of the functional equation, and thus allow us to compute $\psi_q(x)$. If we are unlucky, one can always compute two possible characteristic polynomials, one for each possible sign in the functional equation, and discard the polynomial whose roots provably have absolute value different from q (i.e., absolute value distinct from that predicted by the Weil conjectures). In practice, if we already know a few explicit divisor classes on $\mathcal{X}_{\bar{k}}$, we can cut down the amount of point counting required to determine $\psi_q(x)$. For example, knowing that the hyperplane class is fixed by Galois tells us that $(x-q)$ divides $\psi_q(x)$; this information can be used to get away with point count counts for $i = 1, \ldots, 10$ only. More generally, if one already knows an explicit submodule $M \subseteq \mathrm{NS}\,\mathcal{X}_{\bar{k}}$ as a Galois module, then the characteristic polynomial $\psi_M(x)$ of Frobenius acting on M can be computed, and since $\psi_M(x) \mid \psi_q(x)$, one can compute $\psi_q(x)$ with only a few point counts, depending on the rank of M.

Exercise 2.15. Show that if M has rank r then counting points on $\mathcal{X}_k(\mathbb{F}_{q^i})$ for $i = 1, \ldots, \lceil (22-r)/2 \rceil$ suffices to determine the two possible polynomials $\psi_q(x)$ (one for each possible sign in the functional equation).

Example 2.16 ([HVA13, §5.3])**.** In the polynomial ring $\mathbb{F}_3[x, y, z, w]$, give weights 1, 1, 1 and 3, respectively, to the variables x, y, z and w, and let $\mathbb{P}_{\mathbb{F}_3}(1, 1, 1, 3) = \mathrm{Proj}\,\mathbb{F}_3[x, y, z, w]$ be the corresponding weighted projective plane. We choose a polynomial $p_5(x,y,z) \in \mathbb{F}_3[x,y,z]_5$ so that the hypersurface X given by

$$w^2 = 2y^2(x^2+2xy+2y^2)^2 + (2x+z)p_5(x,y,z) \tag{9}$$

is smooth, hence a K3 surface (of degree 2). For example, take

$$\begin{aligned} p_5(x,y,z) = x^5 &+ x^4y + x^3yz + x^2y^3 + x^2y^2z + 2x^2z^3 \\ &+ xy^4 + 2xy^3z + xy^2z^2 + y^5 + 2y^4z + 2y^3z^2 + 2z^5. \end{aligned}$$

The projection $\pi\colon \mathbb{P}(1,1,1,3) \dashrightarrow \operatorname{Proj}\mathbb{F}_3[x,y,z]$ restricts to a double cover morphism $\pi\colon X \to \mathbb{P}^2_{\mathbb{F}_3}$, branched along the vanishing of the right hand side of (9).

Let $N_i := \#X(\mathbb{F}_{3^i})$; counting points we find

N_1	N_2	N_3	N_4	N_5	N_6	N_7	N_8	N_9	N_{10}
7	79	703	6607	60427	532711	4792690	43068511	387466417	3486842479

.

Applying the procedure described above, this is enough information to determine the characteristic polynomial $\psi_3(x)$. The sign of the functional equation for $\psi_3(x)$ is negative—a positive sign gives rise to roots of absolute value $\neq 3$. Setting $\widetilde{\psi}(x) = 3^{-22}\psi_3(3x)$, we obtain a factorization into irreducible factors as follows:

$$\begin{aligned} \widetilde{\psi}(x) = \frac{1}{3}(x-1)(x+1)(3x^{20} &+ 3x^{19} + 5x^{18} + 5x^{17} + 6x^{16} + 2x^{15} + 2x^{14} \\ &- 3x^{13} - 4x^{12} - 8x^{11} - 6x^{10} - 8x^9 - 4x^8 \\ &- 3x^7 + 2x^6 + 2x^5 + 6x^4 + 5x^3 + 5x^2 + 3x + 3). \end{aligned}$$

The roots of the degree 20 factor of $\psi(x)$ are not integral, so they are not roots of unity. We conclude that $\rho(\overline{X}) \leq 2$.

On the other hand, inspecting the right hand side of (9), we see that the line $2x+z=0$ on $\mathbb{P}^2$ is a tritangent line to the branch curve of the double cover morphism π. The components of the pullback of this line intersect according to the following Gram matrix

$$\begin{pmatrix} -2 & 3 \\ 3 & -2 \end{pmatrix}$$

which has determinant $-5 \neq 0$, and thus they generate a rank 2 sublattice L of $\operatorname{NS}\overline{X}$. We conclude that $\rho(\overline{X}) = 2$. Since the determinant of the lattice L is not divisible by a square, the lattice L must be saturated in $\operatorname{NS}\overline{X}$, so $\operatorname{NS}\overline{X} = L$.

By Theorem 2.12, any K3 surface over $\mathbb{Q}$ whose reduction at $p=3$ is isomorphic to X has geometric Picard rank at most 2.

2.5. **Upper bounds II.** Keep the notation of Theorem 2.12. It is natural to wonder how good the upper bound furnished by Theorem 2.12 really is, at least for K3 surfaces, which are the varieties that concern us. The Weil conjectures tell us that the eigenvalues of $\Phi^*_{\mathcal{X}_{\bar{k}}}$

operating on $\mathrm{H}^2_{\text{ét}}(\mathcal{X}_{\bar{k}}, \mathbb{Q}_\ell)$ have absolute value[4] q. Since the characteristic polynomial of $\Phi^*_{\mathcal{X}_{\bar{k}}}$ lies in $\mathbb{Q}[x]$, the eigenvalues not of form $q\zeta$ must come in complex conjugate pairs. In particular, the total number of eigenvalues that *are* of the form $q\zeta$ must have the same parity as the ℓ-adic Betti number $b_2 = \dim_{\mathbb{Q}_\ell} \mathrm{H}^2_{\text{ét}}(\mathcal{X}_{\bar{k}}, \mathbb{Q}_\ell)$. For a K3 surface, $b_2 = 22$ because the l-adic Betti numbers coincide with the usual Betti numbers (use [Mil80, Theorem 3.12]).

We conclude, for example, that Theorem 2.12 by itself cannot be used to construct a projective K3 surface over a number field of geometric Picard rank 1. This was a distressing state of affairs, since it is a classical fact that outside a countable union of divisors, the points in the coarse moduli space $\mathcal{K}_{2d}$ of complex K3 surfaces of degree $2d$ represent K3 surfaces of geometric Picard rank 1. The complement of these divisors is not empty (by the Baire category theorem!), but since number fields are countable, it was conceivable that there did not exist K3 surfaces over number fields of geometric Picard rank 1. Terasoma and Ellenberg showed that such surfaces do exist [Ter85, Ell04], and van Luijk constructed the first explicit examples [vL07].

2.5.1. *van Luijk's method.* The idea behind van Luijk's method [vL07] is beautiful in its simplicity: use information at *two* primes of good reduction. See Convention 2.13 to understand the notation below.

Proposition 2.17. *Let X be a K3 surface over a number field K, and let $\mathfrak{p}$ and $\mathfrak{p}'$ be two finite places of good reduction. Suppose that $\operatorname{NS} \overline{X}_{\mathfrak{p}} \simeq \mathbb{Z}^n$ and $\operatorname{NS} \overline{X}_{\mathfrak{p}'} \simeq \mathbb{Z}^n$, and that the discriminants $\operatorname{Disc}\left(\operatorname{NS} \overline{X}_{\mathfrak{p}}\right)$ and $\operatorname{Disc}\left(\operatorname{NS} \overline{X}_{\mathfrak{p}'}\right)$ are different in $\mathbb{Q}^\times/\mathbb{Q}^{\times 2}$. Then $\rho(\overline{X}) \leq n-1$.*

Proof. By Theorem 2.12, we know that $\rho(\overline{X}) \leq n$. If $\rho(\overline{X}) = n$, then $\operatorname{NS} \overline{X}$ is a full rank sublattice of both $\operatorname{NS} \overline{X}_{\mathfrak{p}}$ and $\operatorname{NS} \overline{X}_{\mathfrak{p}'}$. This implies that $\operatorname{Disc} \operatorname{NS}(\overline{X})$ is equal to *both* $\operatorname{Disc}\left(\operatorname{NS} \overline{X}_{\mathfrak{p}}\right)$ and $\operatorname{Disc}\left(\operatorname{NS} \overline{X}_{\mathfrak{p}'}\right)$ as elements of $\mathbb{Q}^\times/\mathbb{Q}^{\times 2}$, so the discriminants of the reductions are equal in $\mathbb{Q}^\times/\mathbb{Q}^{\times 2}$. This is a contradiction. ☐

Example 2.18 ([vL07, §3])**.** The following is van Luijk's original example. Set

$$\begin{aligned} f = {} & x^3 - x^2y - x^2z + x^2w - xy^2 - xyz + 2xyw + xz^2 + 2xzw \\ & + y^3 + y^2z - y^2w + yz^2 + yzw - yw^2 + z^2w + zw^2 + 2w^3, \end{aligned}$$

and let X be the quartic surface in $\mathbb{P}^3_{\mathbb{Q}} = \operatorname{Proj} \mathbb{Q}[x, y, z, w]$ given by

$$wf + 2z(xy^2 + xyz - xz^2 - yz^2 + z^3) - 3(z^2 + xy + yz)(z^2 + xy) = 0.$$

One can check (using the Jacobian criterion), that X is smooth, and that X has good reduction at $p = 2$ and 3. Let $\psi_p(x)$ denote the characteristic polynomial of Frobenius acting on $\mathrm{H}^2_{\text{ét}}(\overline{X}_p, \mathbb{Q}_\ell)$, and let $\widetilde{\psi}_p(x) = p^{-22}\psi_p(px)$. Proceeding as in Example 2.16, we use

[4]When we say absolute value here we mean any archimedean absolute value of the field obtained by adjoining to K the eigenvalues of $\Phi^*_{\mathcal{X}_{\bar{k}}}$.

point counts to compute

$$\widetilde{\psi}_2(x) = \frac{1}{2}(x-1)^2(2x^{20}+x^{19}-x^{18}+x^{16}+x^{14}+x^{11}+2x^{10}+x^9+x^6+x^4-x^2+x+2)$$

$$\widetilde{\psi}_3(x) = \frac{1}{3}(x-1)^2(3x^{20}+x^{19}-3x^{18}+x^{17}+6x^{16}-6x^{14}+x^{13}+6x^{12}-x^{11}-7x^{10}-x^9 \\ +6x^8+x^7-6x^6+6x^4+x^3-3x^2+x+3)$$

The roots of the degree 20 factors of $\widetilde{\psi}_p(x)$ are not integral for $p=2$ and 3, so they are not roots of unity. We conclude that $\rho(\overline{X}_2)$ and $\rho(\overline{X}_3)$ are both less than or equal to 2.

Next, we compute $\operatorname{Disc}(\operatorname{NS}\overline{X}_p)$ for $p=2$ and 3 by finding explicit generators for $\operatorname{NS}\overline{X}_p$. For $p=2$ note that, besides the hyperplane section H (i.e., the pullback of $\mathscr{O}_{\mathbb{P}^3}(1)$ to $\overline{X}_2$), the surface $\overline{X}_2$ contains the conic

$$C\colon w = z^2+xy = 0.$$

We have $H^2 = 4$ (it's the degree of $\overline{X}_2$ in $\mathbb{P}^3$), and $C\cdot H = \deg C = 2$. Finally, by the adjunction formula $C^2=-2$ because C has genus 0 and the canonical class on $\overline{X}_2$ is trivial. All told, we have produced a rank two sublattice of $\operatorname{NS}\overline{X}_2$ of discriminant

$$\det\begin{pmatrix}4 & 2\\ 2 & -2\end{pmatrix} = -12.$$

We conclude that $\operatorname{Disc}(\operatorname{NS}\overline{X}_2) = -3 \in \mathbb{Q}^\times/\mathbb{Q}^{\times 2}$.

For $p=3$, the surface $\overline{X}_3$ contains the hyperplane class H and the line $L\colon w=z=0$, giving a rank two sublattice of $\operatorname{NS}\overline{X}_3$ of discriminant

$$\det\begin{pmatrix}4 & 1\\ 1 & -2\end{pmatrix} = -9.$$

Thus $\operatorname{Disc}(\operatorname{NS}\overline{X}_3) = -1 \in \mathbb{Q}^\times/\mathbb{Q}^{\times 2}$. Proposition 2.17 implies that $\rho(\overline{X}) \le 1$, and since $\operatorname{NS}\overline{X}$ contains the hyperplane class, we conclude that $\rho(\overline{X}) = 1$.

2.6. **Further techniques.** In Examples 2.16 and 2.18 above, we computed the discriminant of the Néron-Severi lattice for some K3 surfaces by exhibiting explicit generators. What if we don't have explicit generators? In [Klo07] Kloosterman gets around this problem by using that Artin-Tate conjecture, which states that for a K3 surface X over a finite field $\mathbb{F}_q$ the Brauer group $\operatorname{Br} X := \mathrm{H}^2_{\text{ét}}(X,\mathbb{G}_m)_{\text{tors}}$ of X is finite and

$$\lim_{x\to q}\frac{\psi_q(x)}{(x-q)^{\rho(X)}} = q^{21-\rho(X)}\#\operatorname{Br} X|\operatorname{Disc}(\operatorname{NS} X)|, \tag{10}$$

where $\rho(X) = \operatorname{rk}(\operatorname{NS} X)$. The Artin-Tate conjecture follows from the Tate conjecture when $2\nmid q$ [Mil75], and the Tate conjecture is now known to hold in odd characteristic; see Remark 2.11. Assume then that q is odd. Pass to the finite extension of the ground field so that $\operatorname{NS} X = \operatorname{NS}\overline{X}$. Since the Artin-Tate conjecture holds, so in particular $\operatorname{Br} X$ is finite, a theorem of Lorenzini, Liu and Raynaud states that the quantity $\#\operatorname{Br} X$ is a square [LLR05].

Hence (10) can be used to compute $|\operatorname{Disc}(\operatorname{NS}\overline{X})|$ as an element of $\mathbb{Q}^\times/\mathbb{Q}^{\times 2}$.

Elsenhans and Jahnel have made several contributions to the computation of Néron-Severi groups of K3 surfaces. For example, in [EJ11a], they explain that one can use the Galois module structures of Néron-Severi groups to refine Proposition 2.17. Let X be a K3 surface over a number field K, and let $\mathfrak{p}$ be a finite place of good reduction for X, with residue field k (see Convention 2.13). The specialization map

$$\operatorname{sp}_{\overline{K},\bar{k}} \otimes \operatorname{id}\colon \operatorname{NS}\overline{X} \otimes_{\mathbb{Z}} \mathbb{Q} \to \operatorname{NS}\overline{X}_{\mathfrak{p}} \otimes_{\mathbb{Z}} \mathbb{Q}$$

is an injective homomorphism. The $\mathbb{Q}$-vector space $\operatorname{NS}\overline{X}_{\mathfrak{p}} \otimes_{\mathbb{Z}} \mathbb{Q}$ is a $\operatorname{Gal}(\bar{k}/k)$-representation, while the $\mathbb{Q}$-vector space $\operatorname{NS}\overline{X} \otimes_{\mathbb{Z}} \mathbb{Q}$ is a $\operatorname{Gal}(\overline{K}/K)$-representation. Let L denote the kernel of the latter representation.

Exercise 2.19. Show that the field extension L/K is finite and unramified at $\mathfrak{p}$.

Exercise 2.19 shows that, after choosing a prime $\mathfrak{q}$ in L lying above $\mathfrak{p}$, there is a unique lift of Frobenius to L, which together with the specialization map, makes $\operatorname{NS}\overline{X} \otimes_{\mathbb{Z}} \mathbb{Q}$ a $\operatorname{Gal}(\bar{k}/k)$-submodule of $\operatorname{NS} X_{\bar{k}} \otimes_{\mathbb{Z}} \mathbb{Q}$. By understanding the $\operatorname{Gal}(\bar{k}/k)$-submodules of $\operatorname{NS} X_{\bar{k}} \otimes_{\mathbb{Z}} \mathbb{Q}$ as we vary over several primes of good reduction, we can find restrictions on the structure of $\operatorname{NS}\overline{X} \otimes_{\mathbb{Z}} \mathbb{Q}$, and often compute $\rho(\overline{X})$.

The main tool is the characteristic polynomial χ_{Frob} of Frobenius as an endomorphism of $\operatorname{NS}\overline{X}_{\mathfrak{p}} \otimes_{\mathbb{Z}} \mathbb{Q}$. If χ_{Frob} has simple roots, then $\operatorname{Gal}(\bar{k}/k)$-submodules of $\operatorname{NS}\overline{X}_{\mathfrak{p}} \otimes_{\mathbb{Z}} \mathbb{Q}$ are in bijection with the monic polynomials dividing χ_{Frob}.

Recall that $\operatorname{NS}\overline{X}_{\mathfrak{p}} \otimes_{\mathbb{Z}} \mathbb{Q}_\ell$ is a $\operatorname{Gal}(\bar{k}/k)$-submodule of $\mathrm{H}^2_{\text{ét}}(\overline{X}_{\mathfrak{p}}, \mathbb{Q}_\ell(1))$ via the cycle class map, so χ_{Frob} divides the characteristic polynomial $\widetilde{\psi}_{\mathfrak{p}}$ of Frobenius acting on $\mathrm{H}^2_{\text{ét}}(X_{\bar{k}}, \mathbb{Q}_\ell(1))$, and we have seen that the roots of χ_{Frob} are roots of unity (because some power of Frobenius acts as the identity). Therefore, χ_{Frob} divides the product of the cyclotomic polynomials that divide $\widetilde{\psi}_{\mathfrak{p}}$. The Tate conjecture implies that χ_{Frob} is in fact equal to this product. So let V_{Tate} denote the highest dimensional $\mathbb{Q}_\ell$-subspace of $\mathrm{H}^2_{\text{ét}}(\overline{X}_{\mathfrak{p}}, \mathbb{Q}_\ell(1))$ on which all the eigenvalues of Frobenius are roots of unity. Let $L \subset \operatorname{NS}\overline{X}_{\mathfrak{p}}$ be a sublattice; typically, L will be generated by the classes of explicit divisors we are aware of on $\overline{X}_{\mathfrak{p}}$. If we are lucky, there are very few possibilities for $\operatorname{Gal}(\bar{k}/k)$-submodules of the quotient $V_{\mathrm{Tate}}/(L \otimes_{\mathbb{Z}} \mathbb{Q}_\ell)$, which we compare as we vary over finite places of good reduction. This is best explained through an example.

Example 2.20 ([EJ11a, §5])**.** The following is an example of a K3 surface X over $\mathbb{Q}$ with good reduction at $p = 3$ and 5, such that $\rho(\overline{X}_3) = 4$ and $\rho(\overline{X}_5) = 14$, for which we can show that $\rho(\overline{X}) = 1$ using only information at these two primes. Let $\mathcal{X}$ be the subscheme of

$\mathbb{P}(1,1,1,3) = \operatorname{Proj} \mathbb{Z}_{(15)}[x,y,z,w]$ given by $w^2 = f_6(x,y,z)$, where

$$f_6(x,y,z) \equiv 2x^6 + x^4y^2 + 2x^3y^2z + x^2y^2z^2 + x^2yz^3 + 2x^2z^4 + xy^4z + xy^3z^2 + xy^2z^3 + 2xz^5 + 2y^6 + y^4z^2 + y^3z^3 \bmod 3,$$

$$f_6(x,y,z) \equiv y^6 + x^4y^2 + 3x^2y^4 + 2x^5z + 3xz^5 + z^6 \bmod 5.$$

Set $X = \mathcal{X}_{\mathbb{Q}}$. Counting the elements of $\mathcal{X}_{\mathbb{F}_3}(\mathbb{F}_{3^n})$ for $n = 1, \ldots, 10$, we compute the characteristic polynomial of Frobenius on $\mathrm{H}^2_{\text{ét}}(\mathcal{X}_{\overline{\mathbb{F}}_3}, \mathbb{Q}_\ell(1))$ (here $\ell \neq 3$ is a prime) and we get

$$\widetilde{\phi}_3(x) = \frac{1}{3}(x-1)^2(x^2+x+1) (3x^{18} + 5x^{17} + 7x^{16} + 10x^{15} + 11x^{14} + 11x^{13} + 11x^{12} + 10x^{11} + 9x^{10} + 9x^9 + 9x^8 + 10x^7 + 11x^6 + 11x^5 + 11x^4 + 10x^3 + 7x^2 + 5x + 1)$$

Let $L \subset \operatorname{NS} \mathcal{X}_{\overline{\mathbb{F}}_3}$ be the rank 1 sublattice generated by the pullback of the class of a line for the projection $\mathcal{X}_{\mathbb{F}_3} \to \mathbb{P}^2_{\mathbb{F}_3}$ (i.e., the "hyperplane class"). The characteristic polynomial of Frobenius acting on $V_{\mathrm{Tate}}/(L \otimes_{\mathbb{Z}} \mathbb{Q}_\ell)$ is $(x-1)(x^2+x+1)$, which has simple roots. We conclude that, for each dimension 1, 2, 3, and 4, there is at most one $\operatorname{Gal}(\overline{\mathbb{F}}_3/\mathbb{F}_3)$-invariant vector subspace of $\operatorname{NS} \mathcal{X}_{\overline{\mathbb{F}}_3}$ that contains L.

Repeating this procedure[5] at $p = 5$, we find that the characteristic polynomial of Frobenius acting on $\mathrm{H}^2_{\text{ét}}(\mathcal{X}_{\overline{\mathbb{F}}_5}, \mathbb{Q}_\ell(1))$ is

$$\widetilde{\phi}_5(x) = \frac{1}{5}(x-1)^2(x^4+x^3+x^2+x+1)(x^8-x^7+x^5-x^4+x^3-x+1) (5x^8 - 5x^7 - 2x^6 + 3x^5 - x^4 + 3x^3 - 2x^2 - 5x + 5)$$

Again, let $L \subset \operatorname{NS} \mathcal{X}_{\overline{\mathbb{F}}_5}$ be the rank 1 sublattice generated by the pullback of the class of a line for the projection $\mathcal{X}_{\mathbb{F}_5} \to \mathbb{P}^2_{\mathbb{F}_5}$. The characteristic polynomial of Frobenius acting on $V_{\mathrm{Tate}}/(L \otimes_{\mathbb{Z}} \mathbb{Q}_\ell)$ is

$$(x-1)(x^4+x^3+x^2+x+1)(x^8-x^7+x^5-x^4+x^3-x+1)$$

which has simple roots. Thus, for each dimension 1, 2, 5, 6, 9, 10, 13, and 14 there is at most one $\operatorname{Gal}(\overline{\mathbb{F}}_5/\mathbb{F}_5)$-invariant vector subspace of $\operatorname{NS} \mathcal{X}_{\overline{\mathbb{F}}_5}$ that contains L.

Since $\operatorname{NS} \overline{X} \otimes_{\mathbb{Z}} \mathbb{Q}$ is a $\operatorname{Gal}(\overline{\mathbb{F}}_p/\mathbb{F}_p)$-invariant subspace of $\operatorname{NS} \mathcal{X}_{\overline{\mathbb{F}}_p}$ for $p = 3$ and 5, we already see that $\rho(\overline{X}) = 1$ or 2. If $\rho(\overline{X}) = 2$, then the discriminants of the $\operatorname{Gal}(\overline{\mathbb{F}}_p/\mathbb{F}_p)$-invariant subspaces of $\operatorname{NS} \mathcal{X}_{\overline{\mathbb{F}}_p}$ of rank 2 for $p = 3$ and 5 must be equal in $\mathbb{Q}^\times/\mathbb{Q}^{\times 2}$. These classes *modulo squares* of these discriminants can be calculated using the Artin-Tate formula (10), and they are, respectively -489 and -5. Hence $\rho(\overline{X}) = 1$.

---

[5]In the interest of transparency, one should add that brute-force point counting of $\mathbb{F}_{5^n}$-points of $\mathcal{X}_{\mathbb{F}_5}$ is usually not feasible for $n \geq 8$. However, the defining equation for $\mathcal{X}_{\mathbb{F}_5}$ contains no monomials involving both y and z. This "decoupling" allows for extra tricks that allow a refined brute-force approach to work. See [EJ08, Algorithm 17]. Alternatively, one can find several divisors on $\mathcal{X}_{\overline{\mathbb{F}}_5}$, given by irreducible components of the pullbacks of lines tritangent to the curve $f_6(x,y,z) = 0$ in $\mathbb{P}^2_{\overline{\mathbb{F}}_5}$, and thus compute a large degree divisor of $\widetilde{\phi}_5(x)$; see the discussion after Theorem 2.14.

Unless one uses p-adic cohomology methods to count points of a K3 surface over a finite field (e.g. [AKR10,CT14]), the slowest step in computing geometric Picard numbers using the above techniques is point counting. One is restricted to using small characteristics, typically 2, 3 and (sometimes) 5, and in practice, it can be difficult to write a model of a surface over a number field with good reduction at these small primes. Remarkably, Elsenhans and Jahnel proved a theorem that requires point counting in only *one* characteristic. Their result is quite general; we explain below how to use it in a concrete situation.

Theorem 2.21 ([EJ11b, Theorem 1.4])**.** *Let R be a discrete valuation ring with quotient field K of characteristic zero and perfect residue field k of characteristic $p > 0$. Write v for the valuation of R, and assume that $v(p) < p - 1$. Let $\pi\colon X \to \operatorname{Spec} R$ be a smooth proper morphism. Then the cokernel of the specialization homomorphism*

$$\mathrm{sp}\colon \operatorname{Pic} X_{\bar{K}} \to \operatorname{Pic} X_{\bar{k}}$$

is torsion-free. □

Recall that for a K3 surface the Picard group and the Néron-Severi group coincide (Proposition 1.8).

Example 2.22. Let $R = \mathbb{Z}_{(3)}$, so that $K = \mathbb{Q}$ and $k = \mathbb{F}_3$. Let X be the K3 surface in $\mathbb{P}(1,1,1,3) = \operatorname{Proj} \mathbb{Z}_{(3)}[x,y,z,w]$ given by

$$w^2 = 2y^2(x^2 + 2xy + 2y^2)^2 + (2x + z)p_5(x,y,z) + 3p_6(x,y,z),$$

where $p_5(x,y,z)$ is the same polynomial as in Example 2.16, and $p_6(x,y,z) \in \mathbb{Z}_{(3)}[x,y,z]_6$ is a polynomial of degree 6 such that X is smooth as a $\mathbb{Z}_{(3)}$-scheme.. We saw in Example 2.16 that $\operatorname{NS} X_{\overline{\mathbb{F}}_3} = \operatorname{Pic} X_{\overline{\mathbb{F}}_3}$ has rank 2 and is generated by the pullbacks C and C' for $X_{\overline{\mathbb{F}}_3} \to \mathbb{P}^2_{\overline{\mathbb{F}}_3}$ of the tritangent line $2x + z = 0$. Theorem 2.21 tell us that if $\operatorname{NS} X_{\overline{\mathbb{Q}}}$ has rank 2, then C and C' lift to classes $\widetilde{C}$ and $\widetilde{C}'$, respectively, in $\operatorname{NS} X_{\overline{\mathbb{Q}}}$. The Riemann-Roch theorem shows that $\widetilde{C}$ and $\widetilde{C}'$ are effective, and an intersection number computation shows that $\widetilde{C}$ and $\widetilde{C}'$ must be components of the pullback of a line tritangent to the branch curve of the projection $X_{\overline{\mathbb{Q}}} \to \mathbb{P}^2_{\overline{\mathbb{Q}}}$. But now the presence of $p_6(x,y,z)$ could wreck havoc here, and there may not be a line that is tritangent to the branch curve in characteristic zero!

For a particular $p_6(x,y,z)$, how does one look for a line tritangent to the curve

$$2y^2(x^2 + 2xy + 2y^2)^2 + (2x + z)p_5(x,y,z) + 3p_6(x,y,z) = 0$$

in $\mathbb{P}^2_{\overline{\mathbb{Q}}}$? One can use Gröbner bases and [EJ08, Algorithm 8] to carry out this task (on a computer!). Alternatively, one could use a different prime p of good reduction for $X_{\mathbb{Q}}$ and look for tritangent lines to the branch curve of the projection $X_{\overline{\mathbb{F}}_p} \to \mathbb{P}^2_{\overline{\mathbb{F}}_p}$, still using [EJ08, Algorithm 8], hoping of course that there is no such line. No point counting is needed in this second approach, but the Gröbner bases computations over finite fields that take place under the hood are much simpler than the corresponding computations over $\mathbb{Q}$.

Exercise 2.23. Fill in the details in the Example 2.22 to show that $\widetilde{C}$ and $\widetilde{C}'$ must be components of the pullback of a line tritangent to the branch curve of the projection $X_{\overline{\mathbb{Q}}} \to \mathbb{P}^2_{\overline{\mathbb{Q}}}$.

Exercise 2.24. Implement [EJ08, Algorithm 8] in your favorite platform, and use it to write down a specific homogeneous polynomial $p_6(x, y, z)$ of degree 6 for which you can prove that the surface $X_{\mathbb{Q}}$ of Example 2.22 has geometric Picard rank 1.

2.7. **More on the specialization map.** Let X be a K3 surface over a number field K, and let $\mathfrak{p}$ be a finite place of good reduction for X (see Convention 2.13). We have used the injectivity of the specialization map $\mathrm{sp}_{\overline{K},\bar{k}}\colon \mathrm{NS}\,\overline{X} \to \mathrm{NS}\,\overline{X}_{\mathfrak{p}}$ to glean information about the geometric Picard number $\rho(\overline{X})$ of X. On the other hand, we also know that $\rho(\overline{X}_{\mathfrak{p}})$ is even, whereas $\rho(\overline{X})$ can be odd, so the specialization map need not be surjective. In [EJ12], Elsenhans and Jahnel asked if there is always a finite place $\mathfrak{p}$ of good reduction such that $\rho(\overline{X}_{\mathfrak{p}}) - \rho(\overline{X}) \leq 1$.

Using Hodge theory, Charles answers this question in [Cha14]. Although the answer to the original question is "no", Charles' investigation yields sharp bounds for the difference $\rho(\overline{X}_{\mathfrak{p}}) - \rho(\overline{X})$. We introduce some notation to explain his results.

Let $T_{\mathbb{Q}}$ be the orthogonal complement of $\mathrm{NS}\,X_{\mathbb{C}}$ inside the singular cohomology group $\mathrm{H}^2(X_{\mathbb{C}}, \mathbb{Q})$ with respect to the cup product pairing; $T_{\mathbb{Q}}$ is a sub-Hodge structure of $\mathrm{H}^2(X_{\mathbb{C}}, \mathbb{Q})$. Write E for the endomorphism algebra of $T_{\mathbb{Q}}$. It is known that E is either a totally real field or a CM field[6]; see [Zar83].

Theorem 2.25 ([Cha14, Theorem 1])**.** *Let X, $T_{\mathbb{Q}}$ and E be as above.*

(1) If E is a CM field or if the dimension of $T_{\mathbb{Q}}$ as an E-vector space is even, then there exist infinitely many places $\mathfrak{p}$ of good reduction for X such that $\rho(\overline{X}_{\mathfrak{p}}) = \rho(\overline{X})$.

(2) If E is a totally real field and the dimension of $T_{\mathbb{Q}}$ as an E-vector space is odd, and if $\mathfrak{p}$ is a finite place of good reduction for X of residue characteristic ≥ 5, then

$$\rho(\overline{X}_{\mathfrak{p}}) \geq \rho(\overline{X}) + [E : \mathbb{Q}].$$

Equality holds for infinitely many places of good reduction.

Theorem 2.25 gives a theoretical algorithm for computing the geometric Picard number of a K3 surface X defined over a number field, provided the Hodge conjecture for codimension 2 cycles holds for $X \times X$. The idea is to run three processes in parallel; see [Cha14, §5] for details.

(1) Find divisors on $\overline{X}$ however you can (worst case scenario: start ploughing through Hilbert schemes of curves in the projective space where X is embedded and check whether the curves you see lie on $\overline{X}$). Use the intersection pairing to compute the rank of the span of the divisors you find. This will give a lower bound $\rho'(\overline{X})$ for $\rho(\overline{X})$.

[6]Recall a CM field K is a totally imaginary quadratic extension of a totally real number field.

(2) If the Hodge conjecture holds for $X \times X$, then elements of E are induced by codimension 2 cycles. Find codimension 2 cycles on $X \times X$ (again, worst case scenario one can use Hilbert schemes of surfaces on a projective space where $X \times X$ is embedded to look for surfaces that lie on $X \times X$). Use these cycles to compute the degree $[E : \mathbb{Q}]$.
(3) Systematically compute $\rho(\overline{X}_{\mathfrak{p}})$ at places of good reduction.

After a finite amount of computation, Theorem 2.25 guarantees we will have computed $\rho(\overline{X})$: Suppose that after a finite number of steps in the first process we have computed a lower bound $\rho'(\overline{X})$ that is sharp, i.e., $\rho'(\overline{X}) = \rho(\overline{X})$, but say we can't yet justify this equality. If E is a CM field or if the dimension of $T_{\mathbb{Q}}$ as an E-vector space is even, then Theorem 2.25 (1) guarantees that eventually $\rho'(\overline{X}) = \rho(\overline{X}_{\mathfrak{p}})$ for some prime $\mathfrak{p}$ of good reduction. The third process will allow us to conclude $\rho(\overline{X}) = \rho'(\overline{X})$ in this case. If E is a totally real field and the dimension of $T_{\mathbb{Q}}$ as an E-vector space is odd, then the second process allows us to compute $[E : \mathbb{Q}]$, and the third process will eventually give a prime $\mathfrak{p}$ of good reduction such that $\rho(\overline{X}_{\mathfrak{p}}) = \rho'(\overline{X}) + [E : \mathbb{Q}]$, proving that $\rho(\overline{X}) = \rho'(\overline{X})$ in this case as well, using Theorem 2.25 (2). Of course, we should keep running the first process in the meantime in case the lower bound $\rho'(\overline{X})$ is not yet sharp! But eventually it will be, and we will have computed $\rho(\overline{X})$.

This algorithm is not really practical, but it shows that the problem can be solved, in principle. Recent work of Poonen, Testa, and van Luijk shows that there is an *unconditional* algorithm to compute $\operatorname{NS} \overline{X}$, as a Galois module, for a K3 surface X defined over a finitely generated field of characteristic $\neq 2$ [PTvL15, §8]. For K3 surfaces of degree 2 over a number field, there is also work by Hassett, Kresch and Tschinkel on this problem [HKT13].

3. Brauer groups of K3 surfaces

3.1. Generalities.

References: [CT92, CTS87, Sko01, CT03, VA13]

Through this section, k denotes a number field. Call a smooth, projective geometrically integral variety over k a nice k-variety. Let X be a nice k-variety; is $X(k) \neq \emptyset$? There appears to be no algorithm that could answer this question in this level of generality[7]. On the other hand, the Lang-Nishimura Lemma[8] assures us that if X and Y are nice k-varieties, k-birational to each other, then

$$X(k) \neq \emptyset \iff Y(k) \neq \emptyset.$$

This suggests we narrow down the scope of the original question by fixing some k-birational invariants of X (like dimension). It also suggests we look at birational invariants of X that

[7]Hilbert's tenth problem over k asks for such an algorithm. The problem is open even for $k = \mathbb{Q}$, but it is known that no such algorithm exists for large subrings of $\mathbb{Q}$ [Poo03].

[8]See [RY00, Proposition A.6] for a short proof of this result due to Kollár and Szabó.

have some hope of capturing arithmetic. The Brauer group $\operatorname{Br} X := \mathrm{H}^2_{\text{ét}}(X, \mathbb{G}_m)$ is precisely such an invariant [Gro68, Corollaire 7.3].

Let k_v denote the completion of k at a place v of k. Since $k \hookrightarrow k_v$, an obvious necessary condition for $X(k) \neq \emptyset$ is $X(k_v) \neq \emptyset$ for all places v. Detecting if $X(k_v) \neq \emptyset$ is a relatively easy task, thanks to the Weil conjectures and Hensel's lemma (at least for finite places of good reduction and large enough residue field—see §5 of Viray's Arizona Winter School notes, for example [Vir15]). That these weak necessary conditions are not sufficient has been known for decades [Lin40, Rei42]; see [CT92] for a beautiful, historical introduction to this topic.

Let $\mathbb{A}_k$ denote the ring of adeles of k. A nice k-variety such that $X(\mathbb{A}_k) = \prod_v X(k_v) \neq \emptyset$ and $X(k) = \emptyset$ is called a counterexample to the Hasse principle[9]. In 1970 Manin observed that the Brauer group of a variety could be used to explain several of the known counterexamples to the Hasse principle. More precisely, for any subset $\mathcal{S} \subseteq \operatorname{Br} X$, Manin constructed an obstruction set $X(\mathbb{A}_k)^{\mathcal{S}}$ satisfying

$$X(k) \subseteq X(\mathbb{A}_k)^{\mathcal{S}} \subseteq X(\mathbb{A}_k),$$

and he observed that it was possible to have $X(\mathbb{A}_k) \neq \emptyset$, yet $X(\mathbb{A}_k)^{\mathcal{S}} = \emptyset$, and thus $X(k) = \emptyset$. Whenever this happens, we say there is a Brauer-Manin obstruction to the Hasse principle. We will not define the sets $X(\mathbb{A}_k)^{\mathcal{S}}$ here; the focus of these notes is on trying to write down, in a convenient way, the input necessary to compute the sets $X(\mathbb{A}_k)^{\mathcal{S}}$, namely elements of $\operatorname{Br} X$ expressed, for example, as central simple algebras over the function field $\mathbf{k}(X)$. For details on how to define $X(\mathbb{A}_k)^{\mathcal{S}}$, see [Sko01, §5.2], [VA13, §3] and [CT15, Vir15].

3.2. **Flavors of Brauer elements.** For a map of schemes $X \to Y$, étale cohomology furnishes a map of Brauer groups $\operatorname{Br} Y \to \operatorname{Br} X$; it also recovers Galois cohomology when $X = \operatorname{Spec} K$ for a field K. In fact,

$$\operatorname{Br} \operatorname{Spec}(K) = \mathrm{H}^2_{\text{ét}}(\operatorname{Spec} K, \mathbb{G}_m) \simeq \mathrm{H}^2\left(\operatorname{Gal}(\overline{K}/K), \overline{K}^{\times}\right) = \operatorname{Br} K,$$

where $\overline{K}$ is a separable closure of K, and $\operatorname{Br} K$ is the (cohomological) Brauer group of K.

For a nice k-variety X, write $\overline{X}$ for $X \times_{\operatorname{Spec} k} \operatorname{Spec} \bar{k}$, where $\bar{k}$ is a separable closure of k. There is a filtration of the Brauer group

$$\operatorname{Br}_0 X \subseteq \operatorname{Br}_1 X \subseteq \operatorname{Br} X,$$

where

$$\begin{aligned} \operatorname{Br}_0 X &:= \operatorname{im}(\operatorname{Br} k \to \operatorname{Br} X), && \text{arising from the structure morphism } X \to \operatorname{Spec} k \text{, and} \\ \operatorname{Br}_1 X &:= \ker\left(\operatorname{Br} X \to \operatorname{Br} \overline{X}\right), && \text{arising from extension of scalars } \overline{X} \to X. \end{aligned}$$

[9]The equality $X(\mathbb{A}_k) = \prod_v X(k_v)$ follows from projectivity of X, because $X(\mathcal{O}_k) = X(k)$ in this case; here $\mathcal{O}_k$ denotes the ring of integers of k.

Elements in $\operatorname{Br}_0 X$ are called constant; class field theory shows that if $\mathcal{S} \subseteq \operatorname{Br}_0 X$, then $X(\mathbf{A})^{\mathcal{S}} = X(\mathbf{A})$, so these elements cannot obstruct the Hasse principle. Elements in $\operatorname{Br}_1 X$ are called algebraic; the remaining elements of the Brauer group are transcendental.

The Leray spectral sequence for $X \to \operatorname{Spec} k$ and $\mathbb{G}_m$

$$E_2^{p,q} := \mathrm{H}^p\left(\operatorname{Gal}(\bar{k}/k), \mathrm{H}^q_{\text{ét}}(\overline{X}, \mathbb{G}_m)\right) \Rightarrow \mathrm{H}^{p+q}_{\text{ét}}(X, \mathbb{G}_m)$$

gives rise to an exact sequence of low-degree terms, which yields an isomorphism

$$\operatorname{Br}_1 X / \operatorname{Br}_0 X \xrightarrow{\sim} \mathrm{H}^1(\operatorname{Gal}(\bar{k}/k), \operatorname{Pic} \overline{X}). \tag{11}$$

Exercise 3.1. Fill in the necessary details to prove the map in (11) is indeed an isomorphism. You will need the vanishing of $\mathrm{H}^3(\operatorname{Gal}(\bar{k}/k), (\bar{k})^\times)$ for a number field k, due to Tate; see [NSW08, 8.3.11(iv)].

Roughly speaking, the isomorphism (11) tells us that the Galois action on $\operatorname{Pic} \overline{X}$ determines the algebraic part of the Brauer group. There are whole classes of varieties for which $\operatorname{Br} X = \operatorname{Br}_1 X$, e.g., curves [Gro68, Corollaire 5.8] or rational varieties, by the birational invariance of the Brauer group and the following exercise.

Exercise 3.2. Show that $\operatorname{Br} \mathbb{P}^n_{\bar{k}} = 0$. Hint: use the Kummer sequence in étale cohomology to show that $\operatorname{Br} \mathbb{P}^n_{\bar{k}}[\ell] = 0$ for every prime ℓ, and the inclusion $\operatorname{Br} \mathbb{P}^n_{\bar{k}} \hookrightarrow \operatorname{Br} \mathbf{k}(\mathbb{P}^n_{\bar{k}})$ coming from the generic point of $\mathbb{P}^n_{\bar{k}}$ to see that $\operatorname{Br} \mathbb{P}^n_{\bar{k}}$ is torsion (see §3.3 below).

Exercise 3.3. Let X be a nice k-variety of dimension 2. Show that if the Kodaira dimension of X is negative then $\operatorname{Br} X = \operatorname{Br}_1 X$.

3.3. **Computing algebraic Brauer-Manin obstructions.** On a nice k-variety X with function field $\mathbf{k}(X)$, the inclusion $\operatorname{Spec} \mathbf{k}(X) \to X$ gives rise to a map $\operatorname{Br} X \to \operatorname{Br} \mathbf{k}(X)$ via functoriality of étale cohomology. This map is injective; see [Mil80, Example III.2.22]. When trying to compute the obstruction sets $X(\mathbb{A}_k)^{\mathcal{S}}$, at least when $\mathcal{S} \subseteq \operatorname{Br}_1 X$, one often tries to compute the right hand side of (11); one then tries to invert the map (11) and embed $\operatorname{Br}_1(X)$ into $\operatorname{Br} \mathbf{k}(X)$, thus representing elements of $\operatorname{Br}_1 X$ as central simple algebras over $\mathbf{k}(X)$. This kind of representation is convenient for the computation of the obstruction sets $X(\mathbb{A}_k)^{\mathcal{S}}$. See, for example, [Sko01, p. 145] and [KT04, KT08, CT15, Vir15] for some explicit calculations along these lines, and [KT04], [VA08, §3] and [VA13, §3.5] for ideas on how to invert the isomorphism (11).

3.4. **Colliot-Thélène's conjecture.** Before moving on to K3 surfaces, we mention a conjecture of Colliot-Thélène [CT03], whose origins date back to work of Colliot-Thélène and Sansuc in the case of surfaces [CTS80, Question k_1]. Recall a rationally connected variety Y over an algebraically closed field K is a smooth projective integral variety such that any two closed points lie in the image of some morphism $\mathbb{P}^1_K \to Y$. For surfaces, rational connectedness is equivalent to rationality.

Conjecture 3.4 (Colliot-Thélène)**.** Let X be a nice variety over a number field k. Suppose that X is geometrically rationally connected. Then $X(\mathbb{A}_k)^{\operatorname{Br} X} \neq \emptyset \implies X(k) \neq \emptyset$.

Conjecture 3.4 remains wide open even for geometrically rational surfaces, including, for example, cubic surfaces. See Colliot-Thélène's Arizona Winter School notes [CT15] for more on this conjecture, including evidence for it and progress towards it.

3.5. **Skorobogatov's conjecture.** Based on growing evidence [CTSSD98, SSD05, IS15a, HS16], Skorobogatov has put forth [Sko09] the following conjecture.

Conjecture 3.5 (Skorobogatov)**.** Let X be a projective K3 surface over a number field k. Then $X(\mathbb{A}_k)^{\operatorname{Br} X} \neq \emptyset \implies X(k) \neq \emptyset$.

Remark 3.6. The analogous conjecture for other surfaces of Kodaira dimension 0 is false: Skorobogatov has constructed counter examples of bi-elliptic surfaces for which $X(\mathbb{Q}) = \emptyset$ while $X(\mathbb{A}_{\mathbb{Q}})^{\operatorname{Br} X} \neq \emptyset$. Using [VAV11] as a starting point, Balestrieri, Berg, Manes, Park and Viray constructed an Enriques surface over $\mathbb{Q}$ satisfying the analogous conclusion [BBM+16].

3.6. **Transcendental Brauer elements on K3 surfaces: An introduction.**

References: [SZ08, SZ12, Wit04, Ier10, ISZ11, Pre13, IS15a, New16]

We have seen that there are no transcendental elements of the Brauer group for curves and surfaces of negative Kodaira dimension. The first place we might see such elements is on surfaces of Kodaira dimension zero. K3 surfaces fit this profile. In fact, if X is an algebraic K3 surface over a number field, the group $\operatorname{Br} \overline{X}$ is quite large: there is an exact sequence

$$0 \to (\mathbb{Q}/\mathbb{Z})^{22-\rho} \to \operatorname{Br} \overline{X} \to \bigoplus_{\ell \text{ prime}} \mathrm{H}^3_{\text{ét}}(\overline{X}, \mathbb{Z}_\ell(1))_{\text{tors}} \to 0,$$

where $\rho = \rho(\overline{X})$ is the geometric Picard number of X; see [Gro68, (8.7) and (8.9)]. Moreover, since X is a surface, [Gro68, (8.10) and (8.11)] gives, for each prime ℓ, a perfect pairing of finite abelian groups

$$\left(\operatorname{Br} \overline{X}/(\mathbb{Q}/\mathbb{Z})^{22-\rho}\right)\{\ell\} \times \operatorname{NS} \overline{X}\{\ell\} \to \mathbb{Q}_\ell/\mathbb{Z}_\ell,$$

where $A\{\ell\}$ denotes the ℓ-primary torsion of A. Since $\operatorname{NS} \overline{X}$ is torsion-free (by Proposition 1.8 and the fact that $\operatorname{Num} \overline{X}$ is torsion free, essentially by definition), we conclude that $\operatorname{Br} \overline{X} \simeq (\mathbb{Q}/\mathbb{Z})^{22-\rho}$. (Alternatively, one can embed $k \hookrightarrow \mathbb{C}$, and use the vanishing of the singular cohomology group $\mathrm{H}^3(X_{\mathbb{C}}, \mathbb{Z})$ and comparison theorems [Mil80, III.3.12].)

This result doesn't necessarily imply that $\operatorname{Br} X$ has infinitely many transcendental elements, because it's possible that most elements of $\operatorname{Br} \overline{X}$ might not descend to the ground field. This is indeed the case, as shown by the following remarkable theorem of Skorobogatov and Zarhin.

Theorem 3.7 ([SZ08, Theorem 1.2])**.** *If X is an algebraic K3 surface over a number field k, then the group $\operatorname{Br} X/\operatorname{Br}_0 X$ is finite.* □

It is natural to ask what the possible isomorphism types of $\operatorname{Br} X/\operatorname{Br}_0 X$ are (or for that matter $\operatorname{Br} X/\operatorname{Br}_1 X$), at least at first as abstract abelian groups. A related question is: what prime numbers can divide the order of elements of $\operatorname{Br} X/\operatorname{Br}_0 X$? These kinds of questions have prompted much recent work on Brauer groups of K3 surfaces (e.g., [SZ12, ISZ11, IS15a, New16]), particularly on surfaces with high geometric Picard rank. Two recent striking results [IS15a, New16] on the transcendental odd-torsion of the Brauer group are the following (for a finite abelian group A, write A_{odd} for its subgroup of odd order elements).

Theorem 3.8 ([IS15a, IS15b]). *Let $X_{[a,b,c,d]}$ be a smooth quartic in $\mathbb{P}^3_{\mathbb{Q}}$ given by*

$$ax^4 + by^4 = cz^4 + dw^4.$$

Then

$$\left(\operatorname{Br} X_{[a,b,c,d]}/\operatorname{Br}_0 X_{[a,b,c,d]}\right)_{\text{odd}} = (\operatorname{Br} \overline{X}_{[a,b,c,d]})^{\operatorname{Gal}(\overline{\mathbb{Q}}/\mathbb{Q})}_{\text{odd}} \simeq \begin{cases} \mathbb{Z}/3\mathbb{Z} & \text{if } -3abcd \in \langle -4\rangle \mathbb{Q}^{\times 4}, \\ \mathbb{Z}/5\mathbb{Z} & \text{if } 5^3abcd \in \langle -4\rangle \mathbb{Q}^{\times 4}, \\ 0 & \text{otherwise.} \end{cases}$$

Furthermore, transcendental elements of odd order on $X_{[a,b,c,d]}$ never obstruct the Hasse principle, but they can obstruct weak approximation.

This work builds on earlier work by Bright, Ieronymou, Skorobogatov, and Zarhin [Bri11, SZ12, ISZ11]. Curiously, transcendental elements of order 5 on surfaces of the form $X_{[a,b,c,d]}$ *always* obstruct weak approximation (density of $X(k)$ in $X(\mathbb{A}_k)$ for the product topology of the v-adic topologies); it is also possible for transcendental elements of order 3 to obstruct weak approximation. The first example of such an obstruction was found by Preu [Pre13] on the surface $X_{[1,3,4,9]}$. See [IS15b, Theorem 2.3] for precise conditions detailing when such obstructions arise.

Newton [New16] has found a similar statement for K3 surfaces that are Kummer for the abelian surface $E \times E$, where E is an elliptic curve with complex multiplication.

Theorem 3.9 ([New16]). *Let $E/\mathbb{Q}$ be an elliptic curve with complex multiplication by the full ring of integers of an imaginary quadratic field. Let X be the Kummer K3 surface associated to the abelian surface $E \times E$. Suppose that $(\operatorname{Br} X/\operatorname{Br}_1 X)_{\text{odd}} \neq 0$. Then $\operatorname{Br}_1 X = \operatorname{Br} \mathbb{Q}$ and*

$$\operatorname{Br} X/\operatorname{Br} \mathbb{Q} \simeq \mathbb{Z}/3\mathbb{Z}.$$

Moreover $X(\mathbb{A}_{\mathbb{Q}})^{\operatorname{Br} X} \subsetneq X(\mathbb{A}_{\mathbb{Q}})$; consequently, there is always a Brauer-Manin obstruction to weak approximation on X.

The surfaces of Theorem 3.9 always have rational points by their construction, but it would be interesting to understand the situation for the Hasse principle on torsors for these surfaces; it seems likely that Newton's method will also show that the Hasse principle cannot be obstructed by odd order transcendental Brauer elements for such torsors.

So far, no collection of odd order elements of the Brauer group has been shown to obstruct the Hasse principle on a K3 surface.

Question 3.10 ([IS15a]). Does there exist a K3 surface X over a number field k with $X(\mathbb{A}_k) \neq \emptyset$ such that $X(\mathbb{A}_k)^{(\operatorname{Br} X)_{\mathrm{odd}}} = \emptyset$?

As for transcendental Brauer elements of even order, Hassett and the author showed that they can indeed obstruct the Hasse principle on a K3 surface. We looked at the other end of the Néron-Severi spectrum, i.e., at K3 surfaces of geometric Picard rank one (in fact, we used the technology developed in §2 to compute Picard numbers!).

Theorem 3.11 ([HVA13]). *Let X be a K3 surface of degree 2 over a number field k, with function field $\mathbf{k}(X)$, given as a sextic in the weighted projective space $\mathbb{P}(1,1,1,3) = \operatorname{Proj} k[x,y,z,w]$ of the form*

$$w^2 = -\frac{1}{2} \cdot \det \begin{pmatrix} 2A & B & C \\ B & 2D & E \\ C & E & 2F \end{pmatrix}, \tag{12}$$

where $A, \dots, F \in k[x,y,z]$ are homogeneous quadratic polynomials. Then the class $\mathscr{A}$ of the quaternion algebra $(B^2 - 4AD, A)$ in $\operatorname{Br}(\mathbf{k}(X))$ extends to an element of $\operatorname{Br}(X)$.

When $k = \mathbb{Q}$, there exist polynomials $A, \dots, F \in \mathbb{Z}[x,y,z]$ such that X has geometric Picard rank 1 and $\mathscr{A}$ gives rise to a transcendental Brauer-Manin obstruction to the Hasse principle on X.

For the second part of Theorem 3.11, one can take

$$\begin{aligned}
A &= -7x^2 - 16xy + 16xz - 24y^2 + 8yz - 16z^2, \\
B &= 3x^2 + 2xz + 2y^2 - 4yz + 4z^2, \\
C &= 10x^2 + 4xy + 4xz + 4y^2 - 2yz + z^2, \\
D &= -16x^2 + 8xy - 23y^2 + 8yz - 40z^2, \\
E &= 4x^2 - 4xz + 11y^2 - 4yz + 6z^2, \\
F &= -40x^2 + 32xy - 40y^2 - 8yz - 23z^2.
\end{aligned} \tag{13}$$

The reason to look at K3 surfaces with very low Picard rank is that these surfaces have little structure, e.g., they don't have elliptic fibrations or Kummer structures that one can use to construct or control transcendental Brauer elements [Wit04, SSD05, HS05, Ier10, Pre13, EJ13, IS15a, New16]. Our hope was to give a way to construct Brauer classes that did not depend on extra structure, that could be systematized for large classes of K3 surfaces. So far, we have been able to construct all the possible kinds of 2-torsion elements on K3 surfaces of degree 2 [HVAV11, HVA13, MSTVA16]; see §3.9 below.

Exercise 3.12. Let X be an algebraic K3 surface over $\mathbb{C}$. Prove that if $\rho(X) \geq 5$ then there is a map $\phi\colon X \to \mathbb{P}^1_{\mathbb{C}}$ whose general fiber is a smooth curve of genus 1. Hint: use the Hasse-Minkowski theorem to show there is class $C \in \operatorname{Pic} X$ with $C^2 = 0$. Use the linear system of this class (or a similar class of square zero) to produce the desired fibration.

3.7. **Transcendental Brauer elements on K3 surfaces: Hodge Theory.**

References: [vG05, Muk84, Căl02, HVAV11, HVA13, MSTVA16, IOOV16, Sko16]

The idea behind the construction of transcendental Brauer elements in [HVAV11, HVA13, MSTVA16] goes back to work of van Geemen [vG05], and is most easily explained using sheaf cohomology on complex K3 surfaces; most of this section can be properly rewritten using Kummer sequences for étale cohomology and comparison theorems, e.g., see [Sch05, Proposition 1.3]. The analytic point of view is a little easier to digest.

Let X be a complex K3 surface. Let $\operatorname{Br}' X = \mathrm{H}^2(X, \mathscr{O}_X^\times)_{\mathrm{tors}}$. Since $\mathrm{H}^3(X, \mathbb{Z}) = 0$, the long exact sequence in sheaf cohomology associated to the exponential sequence gives

$$0 \to \mathrm{H}^2(X, \mathbb{Z})/c_1(\operatorname{NS} X) \to \mathrm{H}^2(X, \mathscr{O}_X) \to \mathrm{H}^2(X, \mathscr{O}_X^\times) \to 0$$

We apply the functor $\operatorname{Tor}_\bullet^{\mathbb{Z}}(\,\cdot\,, \mathbb{Q}/\mathbb{Z})$ to this short exact sequence of abelian groups. Note that $\operatorname{Tor}_1^{\mathbb{Z}}(\mathrm{H}^2(X, \mathscr{O}_X), \mathbb{Q}/\mathbb{Z}) = \mathrm{H}^2(X, \mathscr{O}_X)_{\mathrm{tors}} = 0$ and that $\mathrm{H}^2(X, \mathscr{O}_X) \otimes \mathbb{Q}/\mathbb{Z} = 0$ since $\mathbb{Q}/\mathbb{Z}$ is torsion and $\mathrm{H}^2(X, \mathscr{O}_X)$ is divisible. Hence

$$\operatorname{Br}' X \simeq \big(\mathrm{H}^2(X, \mathbb{Z})/c_1(\operatorname{NS} X)\big) \otimes \mathbb{Q}/\mathbb{Z}. \tag{14}$$

Let T_X be the orthogonal complement in $\mathrm{H}^2(X, \mathbb{Z})$ of $\operatorname{NS} X$ with respect to cup product. We call T_X the transcendental lattice of X. Write $T_X^\vee = \operatorname{Hom}(T_X, \mathbb{Z})$ for the dual of T_X.

Lemma 3.13. *The map*

$$\begin{aligned} \phi \colon \mathrm{H}^2(X, \mathbb{Z})/c_1(\operatorname{NS} X) &\to T_X^\vee \\ v + \operatorname{NS} X &\mapsto [t \mapsto \langle v, t\rangle] \end{aligned}$$

is an isomorphism of lattices.

Proof. First, observe that both $\operatorname{NS} X$ and T_X are primitive sublattices of $\mathrm{H}^2(X, \mathbb{Z})$: for the former lattice, note that $\mathrm{H}^2(X, \mathbb{Z})/c_1(\operatorname{NS} X)$ injects into $\mathrm{H}^2(X, \mathscr{O}_X)$, which is torsion-free, and that c_1 is an injective map, because $\mathrm{H}^1(X, \mathscr{O}_X) = 0$, by definition of a K3 surface. For the latter, use Exercise 1.26(1).

Since $\operatorname{NS} X$ is a primitive sublattice of $\mathrm{H}^2(X, \mathbb{Z})$, we have $T_X^\perp = \operatorname{NS} X$, by Exercise 1.26(2). Injectivity of the map ϕ follows: if $\phi(v + \operatorname{NS} X) = 0$, then $v \in T_X^\perp = \operatorname{NS} X$, so $v + \operatorname{NS} X$ is the trivial class in $\mathrm{H}^2(X, \mathbb{Z})/c_1(\operatorname{NS} X)$.

Consider the short exact sequence of abelian groups

$$0 \to T_X \to \mathrm{H}^2(X, \mathbb{Z}) \to \mathrm{H}^2(X, \mathbb{Z})/T_X \to 0.$$

Apply the functor $\operatorname{Ext}_{\mathbb{Z}}^\bullet(\,\cdot\,, \mathbb{Z})$. Since $\mathrm{H}^2(X, \mathbb{Z})/T_X$ is torsion free, we have

$$\operatorname{Ext}_{\mathbb{Z}}^1\big(\mathrm{H}^2(X, \mathbb{Z})/T_X, \mathbb{Z}\big) = 0$$

so the natural map

$$\operatorname{Hom}_{\mathbb{Z}}(\mathrm{H}^2(X, \mathbb{Z}), \mathbb{Z}) \to T_X^\vee$$

is surjective. Since $\mathrm{H}^2(X, \mathbb{Z})$ is unimodular, and hence self dual, this means that every element of $T_X^\vee$ has the form $v \mapsto \langle \lambda, v \rangle$ for some $\lambda \in \mathrm{H}^2(X, \mathbb{Z})$. This gives surjectivity of ϕ. ☐

Proposition 3.14. *Let X be a complex K3 surface. There are isomorphisms of abelian groups*

$$\operatorname{Br} X \simeq T_X^\vee \otimes \mathbb{Q}/\mathbb{Z} \simeq \operatorname{Hom}_{\mathbb{Z}}(T_X, \mathbb{Q}/\mathbb{Z}).$$

Proof. This follows from (14) and Lemma 3.13. ☐

Informally, Proposition 3.14 tells us there are bijections

$$\begin{aligned} & \{\text{cyclic subgroups of } \operatorname{Br}' X \text{ of order } n\} \\ \stackrel{1-1}{\longleftrightarrow}\ & \{\text{surjections } T_X \twoheadrightarrow \mathbb{Z}/n\mathbb{Z}\} \\ \stackrel{1-1}{\longleftrightarrow}\ & \{\text{sublattices } \Gamma \subseteq T_X \text{ of index } n \text{ with cyclic quotient and generator}\} \end{aligned} \tag{15}$$

where the last bijection comes from

($\longrightarrow$) taking the kernel of the surjection $T_X \twoheadrightarrow \mathbb{Z}/n\mathbb{Z}$.

($\longleftarrow$) taking the cokernel of the inclusion $\Gamma \subseteq T_X$.

In what follows, we will focus on the case where $n = p$ is a prime number, in which case (15) tells us that subgroups of order p of $\operatorname{Br}' X$ are in one-to-one correspondence with sublattices of index p of T_X. Since we are working over a ground field that is already algebraically closed, this discussion asserts that sublattices of T_X contain information about the transcendental classes of K3 surfaces!

3.8. **First examples: work of van Geemen** [vG05, §9]**.** Let's implement the above idea in the simplest possible case. Consider an complex algebraic K3 surface X with $\operatorname{NS} X \simeq \mathbb{Z}h$, $h^2 = 2$. We will study sublattices of index 2 in T_X, up to isometry, corresponding by (15) to elements of order 2 in $\operatorname{Br}' X$.

First, a primitive embedding

$$\operatorname{NS} X = \langle h \rangle \hookrightarrow \Lambda_{\mathrm{K3}} = U^{\oplus 3} \oplus E_8(-1)^{\oplus 2}$$

exists by Theorem 1.27. Let $\{e, f\}$ be a basis for the first summand of Λ_{K3} equal to the hyperbolic plane U, with intersection matrix

$$\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix},$$

A primitive embedding $\langle h \rangle \hookrightarrow \Lambda_{\mathrm{K3}}$ is also unique up to isometry by [Nik79, Theorem 1.14.4], so we may assume that $h = e + f$. Let $v = e - f$; we have $v^2 = -2$, $\langle h, v \rangle = 0$, and

$$T_X \simeq \langle v \rangle \oplus \Lambda', \qquad \text{where } \Lambda' = U^{\oplus 2} \oplus E_8(-1)^{\oplus 2}.$$

The lattice Λ' is unimodular (hence equal to its dual lattice), so every $\phi \in \operatorname{Hom}(\Lambda', \mathbb{Z})$ is of the form

$$\phi_\lambda \colon \Lambda' \to \mathbb{Z}, \qquad v \mapsto \langle v, \lambda \rangle.$$

for some $\lambda \in \Lambda'$. In other words, the map

$$\Lambda' \to \operatorname{Hom}(\Lambda', \mathbb{Z}), \qquad \lambda \mapsto \phi_\lambda$$

is an isomorphism. Tensoring with $\mathbb{Z}/2\mathbb{Z}$ we get an isomorphism

$$\Lambda'/2\Lambda' \to \operatorname{Hom}(\Lambda', \mathbb{Z}/2\mathbb{Z}), \qquad \lambda + 2\Lambda' \mapsto \phi_\lambda \otimes \mathrm{id}_{\mathbb{Z}/2\mathbb{Z}}$$

Hence, a surjection $T_X \to \mathbb{Z}/2\mathbb{Z}$ has the form

$$\begin{aligned} &\alpha \colon T_X \to \mathbb{Z}/2\mathbb{Z} \\ &nv + \lambda' \mapsto a_\alpha n + \langle \lambda', \lambda_\alpha \rangle \bmod 2, \end{aligned} \tag{16}$$

for some $\lambda_\alpha \in \Lambda'$, determined only up to an element of $2\Lambda'$, and some $a_\alpha \in \{0, 1\}$. We classify these surjections by studying their kernels (see (15)). These kernels are lattices which, by Theorem 1.25, are determined up to isomorphism by their rank, signature, and discriminant quadratic forms. Recall that the discriminant quadratic form of a lattice $(L, \langle\ ,\ \rangle)$ is

$$q_L \colon L^\vee/L \to \mathbb{Q}/2\mathbb{Z} \qquad x + L \mapsto \langle x, x \rangle \bmod 2\mathbb{Z}.$$

Proposition 3.15 ([vG05, Proposition 9.2])**.** *Let X be a complex algebraic K3 surface with $\operatorname{NS} X \simeq \mathbb{Z}h$, $h^2 = 2$. Let $\alpha \colon T_X \to \mathbb{Z}/2\mathbb{Z}$ be a surjective map as above, and put $\Gamma_\alpha = \ker \alpha$. Then*

(1) If $a_\alpha = 0$ then $\Gamma_\alpha^\vee/\Gamma_\alpha \simeq (\mathbb{Z}/2\mathbb{Z})^3$. There are $2^{20} - 1$ such lattices Γ_α, all isomorphic to each other.

(2) If $a_\alpha = 1$ then $\Gamma_\alpha^\vee/\Gamma_\alpha \simeq \mathbb{Z}/8\mathbb{Z}$. There are 2^{20} such lattices Γ_α, sorted out into two isomorphism classes by their discriminant forms as follows:

(a) The even class*, where $\frac{1}{2}\langle \lambda_\alpha, \lambda_\alpha \rangle \equiv 0 \bmod 2$. There are $2^9(2^{10} + 1)$ such lattices.*

(b) The odd class*, where $\frac{1}{2}\langle \lambda_\alpha, \lambda_\alpha \rangle \equiv 1 \bmod 2$. There are $2^9(2^{10} - 1)$ such lattices.*

Proof. In all cases, the order of the discriminant group $\Gamma_\alpha^\vee/\Gamma_\alpha$ is $\operatorname{disc}(\Gamma_\alpha) = 2^2 \operatorname{disc}(T_X) = 8$, because Γ_α has index 2 in T_X. If $a_\alpha = 0$, then Γ_α has an orthogonal direct sum decomposition

$$\Gamma_\alpha = \langle v \rangle \oplus (\Gamma_\alpha \cap \Lambda'),$$

and we obtain a decomposition of the discriminant group

$$\Gamma_\alpha^\vee/\Gamma_\alpha = \langle v \rangle^\vee/\langle v \rangle \oplus (\Gamma_\alpha \cap \Lambda')^\vee/(\Gamma_\alpha \cap \Lambda') \simeq \mathbb{Z}/2\mathbb{Z} \oplus (\Gamma_\alpha \cap \Lambda')^\vee/(\Gamma_\alpha \cap \Lambda').$$

The discriminant group $(\Gamma_\alpha \cap \Lambda')^\vee/(\Gamma_\alpha \cap \Lambda')$ has order 4. Let $\mu \in \Lambda'$ satisfy $\langle \mu, \lambda_\alpha \rangle = 1$. One verifies that $\{\lambda/2, \mu\}$ generates a subgroup of order 4 in $(\Gamma_\alpha \cap \Lambda')^\vee/(\Gamma_\alpha \cap \Lambda')$, isomorphic to $(\mathbb{Z}/2\mathbb{Z})^2$ (do this!). The discriminant quadratic form is also determined up to isometry (check this!), so all the lattices Γ_α with $a_\alpha = 0$ are isometric. There are $2^{20}-1$ choices for λ_α, parametrized by elements in $\Lambda'/2\Lambda'$, except for the zero vector, which would give $\Gamma_\alpha = T_X$.

For the case $a_\alpha = 1$, we check that $w := \frac{1}{4}(-v + 2\lambda_\alpha)$ is in $\Gamma_\alpha^\vee$. The vector $4w$ is not in Γ_α (it is in T_X, but it is not in the kernel of the map α), but $8w \in \Gamma_\alpha$, so w has order 8 in the discriminant group, which is therefore isomorphic to $\mathbb{Z}/8\mathbb{Z}$. The discriminant form q_α of Γ_α is determined by its value on w, which is

$$q(w) = \langle w, w\rangle = \frac{-2 + 4\langle\lambda_\alpha, \lambda_\alpha\rangle}{16} = \frac{-1 + 2\langle\lambda_\alpha, \lambda_\alpha\rangle}{8} \bmod 2\mathbb{Z}$$

Two lattices Γ_α and $\Gamma_{\alpha'}$ of this form, with discriminant groups generated by w and w', respectively, are therefore equivalent if and only if there exists an integer x such that $q_\alpha(xw) = q_{\alpha'}(w')$. In other words, if and only if

$$x^2 \cdot \frac{-1 + 2\langle\lambda_\alpha, \lambda_\alpha\rangle}{8} \equiv \frac{-1 + 2\langle\lambda_{\alpha'}, \lambda_{\alpha'}\rangle}{8} \bmod 2\mathbb{Z}$$

On the other hand, a vector λ_α is determined only up to elements of $2\Lambda'$ and thus can always be modified (check!) to satisfy $\langle\lambda_\alpha, \lambda_\alpha\rangle = 0$ or 2; we assume a normalization like this. If $\langle\lambda_\alpha, \lambda_\alpha\rangle = \langle\lambda_{\alpha'}, \lambda_{\alpha'}\rangle$, then $x = 1$ will show two lattices are isomorphic. If $\langle\lambda_\alpha, \lambda_\alpha\rangle \neq \langle\lambda_{\alpha'}, \lambda_{\alpha'}\rangle$, then we are looking for an integer x such that

$$x^2 \cdot \frac{-1}{8} \equiv \frac{-1+4}{8} \bmod 2\mathbb{Z}$$

i.e., for an integer x such that $x^2 \equiv 13 \bmod 16$. No such integer exists. We conclude there are two isomorphism classes of lattices Γ_α with $a_\alpha = 1$, depending on the parity of $\frac{1}{2}\langle\lambda_\alpha, \lambda_\alpha\rangle$, as claimed. The count of the number of lattices of each type is left as an exercise. □

Exercise 3.16. Formulate and prove the analogue of Proposition 3.15 for complex algebraic K3 surfaces with $\operatorname{NS} X \simeq \mathbb{Z}h$, $h^2 = 2d$ (see [MSTVA16]). Can you do the case when $\operatorname{NS} X \simeq U$? Such K3 surfaces are endowed with elliptic fibrations (see Exercise 3.12). What about the case when $\rho(X) = 19$?

3.9. **From lattices to geometry.** Proposition 3.15 is nice, but how are we supposed to extract central simple algebras over the function field of a complex K3 surface from it? The hope here is that the lattices Γ_α of Proposition 3.15 are themselves isomorphic to a piece of the cohomology of a *different algebraic variety*, and that the isomorphism is really a shadow of some geometric correspondence that could shed light on the mysterious transcendental Brauer classes.

For example, in the notation of §3.8, an obvious sublattice of index 2 of $T_X = \langle v\rangle \oplus \Lambda'$ is $\Gamma := \langle 2v\rangle \oplus \Lambda'$. This lattice is in the even class of Propososition 3.15(2). Note that $\omega_X \in T_X \otimes \mathbb{C}$, so $\omega_X \in \Gamma \otimes \mathbb{C}$ as well. If we can re-embed Γ *primitively* in Λ_{K3}, say by a map $\iota\colon \Gamma \hookrightarrow \Lambda_{\mathrm{K3}}$, then $\iota_{\mathbb{C}}(\omega_X)$ will give a period point in the period domain Ω, and by the surjectivity of the period map (Theorem 1.24) there will exist a K3 surface Y with[10] $\omega_Y = \iota_{\mathbb{C}}(\omega_X)$ and $T_Y \simeq \iota(\Gamma)$. Discriminant and rank considerations imply that $\operatorname{NS} Y \simeq \mathbb{Z}h'$, $h'^2 = 8$, i.e., Y is a K3 surface of degree 8, with Picard rank 1.

Exercise 3.17. Show that there is indeed a primitive embedding $\iota\colon \Gamma \hookrightarrow \Lambda_{\mathrm{K3}}$. Hint: what would $\iota(\Gamma)^\perp$ have to look like as a lattice (including its discriminant form)? Could you apply Theorem 1.27 and [Nik79, Corollary 1.14.4] to this orthogonal complement instead?

Our discussion suggests there is a correspondence, up to isomorphism, between pairs (X, α) consisting of a K3 surface X of degree 2 and Picard rank 1 together with an even class $\alpha \in \operatorname{Br}' X$, and K3 surfaces of degree 8 and Picard rank 1. This is indeed the case; Mukai had already observed this in [Muk84, Example 0.9]. Starting with a K3 surface Y of degree 8 with $\operatorname{NS} Y \simeq \mathbb{Z}h'$, Mukai notes that the moduli space of stable sheaves E (with respect to h') of rank 2, determinant algebraically equivalent to h', and Euler characteristic 4, is birational to a K3 surface X of degree 2. The moduli space is in general not fine, and the obstruction to the existence of a universal sheaf is an element $\alpha \in \operatorname{Br}' X[2]$. See [Căl02, MSTVA16] for accounts of this phenomenon. Let $\pi_X\colon X \times Y \to X$ be the projection onto the first factor. In modern lingo, any $\pi_X^{-1}\alpha$-twisted universal sheaf on $X \times Y$ induces a Fourier-Mukai equivalence of bounded derived categories $\mathrm{D^b}(X, \alpha) \simeq \mathrm{D^b}(Y)$.

Before we explain a more geometric approach to the correspondence $(X, \alpha) \longleftrightarrow Y$, we pause to identify the varieties encoded by the remaining isomorphism classes of lattices from Proposition 3.15.

Proposition 3.18. *Let X be a complex algebraic K3 surface with $\operatorname{NS} X \simeq \mathbb{Z}h$, $h^2 = 2$. Let Γ_α be the kernel of a surjection $\alpha\colon T_X \to \mathbb{Z}/2\mathbb{Z}$. Let $\Gamma_\alpha(-1)$ denote the lattice Γ_α with its bilinear form scaled by -1.*

(1) If $\Gamma_\alpha^\vee/\Gamma \simeq (\mathbb{Z}/2\mathbb{Z})^3$, then there is an isometry

$$\Gamma_\alpha(-1) \simeq \langle h_1^2, h_1h_2, h_2^2\rangle^\perp \subseteq \mathrm{H}^4(Y, \mathbb{Z}),$$

where $Y \to \mathbb{P}^2 \times \mathbb{P}^2$ is a double cover branched along a smooth divisor of type $(2, 2)$ in $\mathbb{P}^2 \times \mathbb{P}^2$ and h_i is the pullback of $\mathscr{O}_{\mathbb{P}^2}(1)$ along the projection $\pi_i\colon Y \to \mathbb{P}^2$ for $i = 1, 2$.

(2) If $\Gamma_\alpha^\vee/\Gamma \simeq (\mathbb{Z}/8\mathbb{Z})$, then

(a) if Γ_α belongs to the even class, then there is an isometry

$$\Gamma_\alpha \simeq T_Y \subseteq \mathrm{H}^2(Y, \mathbb{Z}),$$

where T_Y is the transcendental lattice of a K3 surface of degree 8.

[10] Note the importance of primitivity of $\iota\colon \Gamma \hookrightarrow \Lambda_{\mathrm{K3}}$ here: T_Y must be a primitive sublattice of $\mathrm{H}^2(Y, \mathbb{Z})$; see the proof of Lemma 3.13.

(b) *if* Γ_α *belongs to the odd class, then there is an isometry*

$$\Gamma_\alpha(-1) \simeq \langle H^2, P\rangle^\perp \subseteq \mathrm{H}^4(Y,\mathbb{Z}),$$

where $Y \subseteq \mathbb{P}^5$ *is a cubic fourfold containing a plane* P*, with hyperplane section* H*.*

Proof. We have discussed the case (2)(a). However, all the statements can be deduced from Theorem 1.25 (see also [vG05, §§9.6–9.8]). For example, let $Y \subseteq \mathbb{P}^5$ be a cubic fourfold, and write H for a hyperplane section of Y. By the Hodge–Riemann relations, the lattice $\mathrm{H}^4(Y,\mathbb{Z})$ has signature $(21,2)$; it is unimodular by Poincaré duality, and it is odd (i.e. not even), because $\langle H^2, H^2\rangle = 3$. By the analogue of Theorem 1.13 for odd indefinite unimodular lattices [Ser73, §V.2.2], we have $\mathrm{H}^4(Y,\mathbb{Z}) \simeq \langle +1\rangle^{\oplus 21} \oplus \langle -1\rangle^{\oplus 2}$ If Y contains a plane P, then the Gram matrix for $\langle H^2, P\rangle$ is

$$\begin{pmatrix} 3 & 1 \\ 1 & 3 \end{pmatrix}$$

(see [Has00, §4.1] for the calculation of $\langle P, P\rangle$.). One checks that the rank, signature and discriminant form of $\langle H^2, P\rangle^\perp$ matches that of Γ_α. Applying Theorem 1.25 finishes the proof in this case. The other cases are left as exercises. □

Exercise 3.19. Let $Y \to \mathbb{P}^2 \times \mathbb{P}^2$ be a double cover branched along a smooth divisor of type $(2,2)$ in $\mathbb{P}^2 \times \mathbb{P}^2$.

(1) Compute the structure of the lattice $\mathrm{H}^4(Y,\mathbb{Z})$.

(2) For $i = 1,\ 2$, let h_i be the pullback of $\mathscr{O}_{\mathbb{P}^2}(1)$ along the projection $\pi_i \colon Y \to \mathbb{P}^2$. Compute the Gram matrix of the lattice $\langle h_1^2, h_1h_2, h_2^2\rangle$.

(3) Compute the rank, signature and discriminant quadratic form of $\langle h_1^2, h_1h_2, h_2^2\rangle^\perp$. Use this to establish Proposition 3.18(1).

Remark 3.20. The connection between cubic fourfolds containing a plane and K3 surfaces of degree 2 goes back at least to Voisin's proof of the Torelli theorem for cubic fourfolds [Voi86]. See also Hassett's work on this subject [Has00]. Fans of derived categories should consult [MS12].

The proof of Proposition 3.18 might make it seem like a numerical coincidence, but the discussion of the case (2)(a) before the Proposition suggests something deeper is going on. Let us describe the geometry that connects a pair (X,α) to the auxiliary variety Y.

Theorem 3.21. *Let* Y *be either*

(1) a K3 surface of degree 8 *with* $\mathrm{NS}\, Y \simeq \mathbb{Z}$*, or,*

(2) a smooth cubic fourfold containing a plane P *such that* $\mathrm{H}^4(Y,\mathbb{Z})_{\mathrm{alg}} \simeq \langle H^2, P\rangle$*, where* H *denotes a hyperplane section, or*

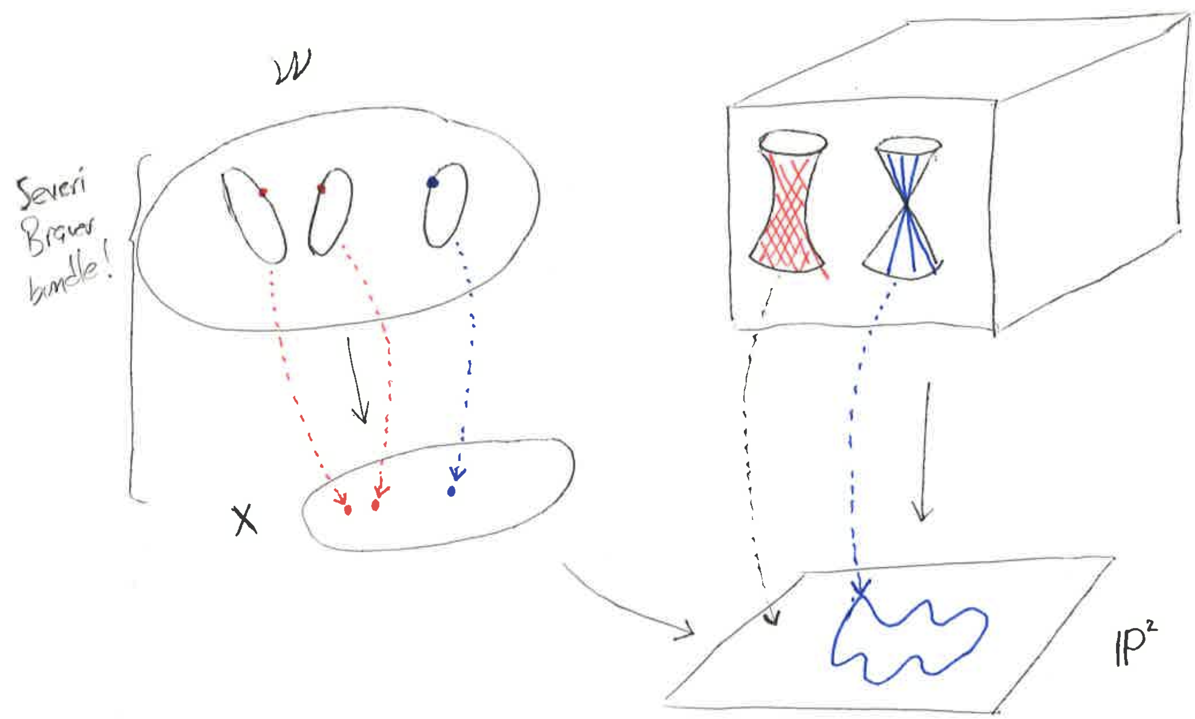

FIGURE 1. Pictorial representation of Theorem 3.21. Each point of $\mathcal{W}$ represents a linear subspace of maximal dimension in a fiber of the quadric bundle $Y' \to \mathbb{P}^2$.

(3) a smooth double cover of $\mathbb{P}^2 \times \mathbb{P}^2$ *branched over a smooth divisor of type* $(2,2)$ *such that* $\mathrm{H}^4(Y,\mathbb{Z})_{\mathrm{alg}} \simeq \langle h_1^2, h_1h_2, h_2^2\rangle$, *where* h_1, h_2 *are the respective pullbacks to* Y *of* $\mathscr{O}_{\mathbb{P}^2}(1)$ *along the two projections* $\pi_1, \pi_2 \colon Y \to \mathbb{P}^2$.

Then there is a quadric fibration $\pi\colon Y' \to \mathbb{P}^2$ *associated to* Y *such that, for general* Y, *the discriminant locus* $\Delta \subseteq \mathbb{P}^2$ *of* π *is a smooth curve of degree* 6, *and the Stein factorization for the relative variety of maximal isotropic subspaces* $\mathcal{W} \to \mathbb{P}^2$ *has the form*

$$\mathcal{W} \to X \to \mathbb{P}^2,$$

where X *is a double cover of* $\mathbb{P}^2$ *branched along* Δ, *and* $\mathcal{W} \to X$ *is a smooth* $\mathbb{P}^n$*-bundle for the analytic topology for some* $n \in \{1,3\}$.

So there it is! The surface X is a K3 surface of degree 2, and $\mathcal{W} \to X$ is a Severi-Brauer bundle representing a class $\alpha \in \mathrm{Br}' X[2]$. The bundle $\mathcal{W} \to X$ can be turned into a central simple algebra over the function field $\mathbf{k}(X)$ that is suitable for the computation of Brauer-Manin obstructions; see [HVAV11, HVA13, MSTVA16] for details. Figure 1 illustrates this idea.

Proof of Theorem 3.21. We explain how to construct the quadric bundles $Y' \to \mathbb{P}^2$. The rest of the theorem can be deduced from [HVAV11, Proposition 3.3]; see also [HVAV11, Theorem 5.1] in the case of cubic fourfolds, [HVA13, Theorem 3.2] for double covers of $\mathbb{P}^2 \times \mathbb{P}^2$, and [MSTVA16, Lemmas 13 and 14] for K3 surfaces of degree 8.

If Y is a K3 surface of degree 8 with $\operatorname{NS} Y \simeq \mathbb{Z}$, then it is a complete intersection of three quadrics $V(Q_0, Q_1, Q_2)$ in $\mathbb{P}^5 = \operatorname{Proj} \mathbb{C}[x_0, \ldots, x_5]$; see [Bea96, Chapter VIII, Exercise 11] or [IK13, Proposition 3.8]. There is a net of quadrics

$$Y' = \left\{([x, y, z], [x_0, \ldots, x_5]) \in \mathbb{P}^2 \times \mathbb{P}^5 : xQ_0 + yQ_1 + zQ_2 = 0\right\} \subseteq \mathbb{P}^2 \times \mathbb{P}^5,$$

and the projection to the first factor gives the desired bundle of quadrics $Y' \to \mathbb{P}^2$. For a general K3 surface Y, the singular fibers of $Y' \to \mathbb{P}^2$ will have rank 5, and thus the discriminant locus on $\mathbb{P}^2$ will be a smooth sextic curve.

If Y is a smooth cubic fourfold containing a plane P, then blowing up and projecting away from P gives a fibration into quadrics $Y' \to \mathbb{P}^2$. The discriminant locus on $\mathbb{P}^2$ where the fibers of the map drop rank is smooth already because Y does not contain another plane intersecting P along a line [Voi86, §Lemme 2], by hypothesis.

Finally, if $Y \to \mathbb{P}^2 \times \mathbb{P}^2$ is a double cover branched along a type $(2, 2)$-divisor, then the projections $\pi_i \colon Y \to \mathbb{P}^2$ give fibrations into quadrics. Smoothness of the discriminant loci is discussed in [HVA13, Lemma 3.1]. □

Remark 3.22. If Y is defined over a number field, then so is the output data $\mathcal{W} \to \mathbb{P}^2$ of the above construction. This gives a way of writing down transcendental Brauer classes on X defined over a number field(!), provided one uses Y as the starting data. The difficulty here is that one might like to use X as the starting data (over a number field), and compute all the possible Y over number fields that fit into the above recipe.

Remark 3.23. The results developed in [IOOV16, Sko16] contain as special cases extensions of Proposition 3.18 and Theorem 3.21 to K3 surfaces of degree 2 without restrictions on their Néron-Severi groups.

4. Uniform boundedness and K3 surfaces: some questions

Let X be a K3 surface over a number field k. In this section, we return to the question of possible orders of the finite quotient $|\operatorname{Br} X / \operatorname{Br}_0 X|$, and connect this question to the geometric correspondences we saw in Theorem 3.21. There is a strong analogy between torsion points on elliptic curves over number fields, and nonconstant Brauer classes of K3 surfaces over number fields. We start by exploring this idea: the analogy suggests it is conceivable that if one fixes just the right amount of data, e.g., a geometric lattice polarization, then there are only finitely many possibilities for $|\operatorname{Br} X / \operatorname{Br}_0 X|$.

4.1. **Torsion subgroups of elliptic curves.** Let E be an elliptic curve over a number field k. By the Mordell-Weil theorem, the group $E(k)$ is finitely generated and abelian. Hence

$$E(k) \cong E(k)_{\mathrm{tors}} \times \mathbb{Z}^r,$$

for some nonnegative integer r. In a 1966 survey paper, Cassels asserts it is a folklore conjecture that there are only finitely many possibilities for $E(k)_{\mathrm{tors}}$ [Cas66, §22]. Shortly thereafter, Manin showed that for each prime p there is a uniform bound on the p-primary torsion of elliptic curves over k:

Theorem 4.1 ([Man69])**.** *Let k be a number field; fix a prime p. There is a constant $c := c(k,p)$ such that $|E(k)_{\mathrm{tors}}| < c(k,p)$ for all elliptic curves E/k.*

Manin proved that the modular curve $X_1(p^r)$, which has high genus for all $r \gg 0$, has only finitely many k-points—before Faltings' theorem was known! Shortly thereafter, Ogg gave a precise conjecture for the possible orders of torsion points on elliptic curves over $\mathbb{Q}$ [Ogg75, Conjecture 1]. In a spectacular breakthrough, Mazur proved this conjecture, and classified all possibilities for $E(\mathbb{Q})_{\mathrm{tors}}$.

Theorem 4.2 ([Maz77, Theorem 8])**.** *Let $E/\mathbb{Q}$ be an elliptic curve. Then $E(\mathbb{Q})_{\mathrm{tors}}$ is isomorphic to one of the following 15 groups:*

$$\mathbb{Z}/n\mathbb{Z} \quad \text{for } 1 \le n \le 10 \text{ or } n = 12, \qquad \text{or} \qquad \mathbb{Z}/2\mathbb{Z} \times \mathbb{Z}/2n\mathbb{Z} \quad \text{for } 1 \le n \le 4.$$

In fact, Mazur showed that the only rational points of the modular curve $X_1(N)$ are the rational cusps if $N = 11$ or $N \ge 13$. After subsequent work establishing (strong) uniform boundedness of torsion over more classes of number fields [Kam92, KM95], Merel showed that in fact $\#E(k)_{\mathrm{tors}}$ could be bounded by a constant depending only on the *degree* of k:

Theorem 4.3 ([Mer96])**.** *Fix $d \ge 1$. There is a constant $c := c(d)$ such that $|E(k)_{\mathrm{tors}}| < c$ for all elliptic curves E over a number field k for which $[k : \mathbb{Q}] = d$.*

4.2. **From torsion on elliptic curves to Brauer groups of K3 surfaces.** Is there a Mazur/Merel Theorem for K3 surfaces? At first glance, this question makes no sense. K3 surfaces have no group structure: what would torsion subgroup even mean? Perhaps we can reinterpret the group $E(k)_{\mathrm{tors}}$ of an elliptic curve in such a way that it does not depend on the group structure of E, and then look for an analogue on K3 surfaces:

$$\begin{aligned}
E(k)_{\mathrm{tors}} &\simeq (\operatorname{Pic}^0 E)_{\mathrm{tors}} && \text{by [Sil09, III.3.4], taking Galois invariants,}\\
&\simeq (\operatorname{Pic} E)_{\mathrm{tors}} && \text{because only degree 0 line bundles are torsion,}\\
&\simeq \mathrm{H}^1(E, \mathscr{O}_E^\times)_{\mathrm{tors}} && \text{[Har77, Exercise III.4.5],}\\
&\simeq \mathrm{H}^1_{\text{ét}}(E, \mathbb{G}_m)_{\mathrm{tors}} && \text{[Mil80, III, Proposition 4.9],}\\
&\simeq \mathrm{H}^1_{\text{ét}}(E, \mathbb{G}_m)_{\mathrm{tors}}/\mathrm{H}^1_{\text{ét}}(\operatorname{Spec} k, \mathbb{G}_m) && \text{by Hilbert's Theorem 90.}
\end{aligned}$$

The quotient $\mathrm{H}^1_{\text{ét}}(E, \mathbb{G}_m)_{\mathrm{tors}}/\mathrm{H}^1_{\text{ét}}(\operatorname{Spec} k, \mathbb{G}_m)$ makes no reference to the group structure of E, and so it is defined for more general varieties. For a K3 surface X/k, we might thus consider the quotient

$$\mathrm{H}^2_{\text{ét}}(X, \mathbb{G}_m)_{\mathrm{tors}}/\mathrm{H}^2_{\text{ét}}(\operatorname{Spec} k, \mathbb{G}_m) = \operatorname{Br} X / \operatorname{Br}_0 X.$$

Theorem 3.7 guarantees that $\operatorname{Br} X/\operatorname{Br}_0 X$ is finite!

4.3. **Moduli spaces.** Understanding the arithmetic of the modular curves $X_0(N)$ and $X_1(N)$ is essential in proving Theorems 4.2 and 4.3. We should expect that defining and understanding moduli spaces for K3 surfaces with level structures coming from the Brauer group will be crucial in investigating uniform boundedness problems for Brauer groups on K3 surfaces. As with modular curves, one can start by studying the geometry of these spaces when defined as complex analytic varieties.

In this context, for example, Proposition 3.15 should have the following interpretation: let $\mathcal{K}_2^o$ denote the locus of the coarse moduli space of complex K3 surfaces of degree 2 whose points correspond to K3 surfaces of Picard rank 1; see [GHS13, §2.5] for a definition of this space. Then the locus of the (to be defined) moduli space $\mathcal{Y}_0(2,2)$ parametrizing pairs $(X, \langle\alpha\rangle)$, where X is a K3 surface of degree 2 and $0 \neq \alpha \in (\operatorname{Br} X)[2]$, such that $\rho(X) = 1$ has three components. Each component maps dominantly onto $\mathcal{K}_2^o$ via the forget map, with finite degree equal to the number of lattices in the corresponding isomorphism class of Proposition 3.15. Proposition 3.18 identifies each of these three components in turn as moduli spaces of other varieties, and Theorem 3.21 details geometric correspondences realizing the isomorphisms between the moduli spaces of objects in Proposition 3.18 and the components of $\mathcal{Y}_0(2,2)$. Compare this with the discussion in §3.9.

The lattice-theoretic calculations of [MSTVA16] show that if $p \nmid 2d$, then the analogous moduli space $\mathcal{Y}_0(2d,p)$ parametrizing pairs $(X, \langle\alpha\rangle)$, where X is a K3 surface of degree $2d$ and $0 \neq \alpha \in (\operatorname{Br} X)[p]$, has three components. One of these components can be identified, á la Mukai, with the moduli space $\mathcal{K}_{2dp^2}$ of K3 surfaces of degree $2dp^2$, and if $d = 1$ and $p \equiv 2 \bmod 3$, then another component is isomorphic to the moduli space $\mathcal{C}_{2p^2}$ of special cubic fourfolds of discriminant $2p^2$. Both $\mathcal{K}_{2dp^2}$ and $\mathcal{C}_{2p^2}$ are varieties of general type for $p \geq 11$ [GHS07, TVA16]. This leads us to propose the following challenge:

Challenge 4.4. Does there exist a K3 surface $X/\mathbb{Q}$ of degree 2 with $\rho(\overline{X}) = 1$, such that $(\operatorname{Br} X/\operatorname{Br}_0 X)[11] \neq 0$?

The above discussion is admittedly informal, but it should be possible to use ideas of Rizov [Riz06] to make it precise and arithmetic.

4.4. **Uniform boundedness.** We conclude by stating optimistic conjectures about Brauer groups of K3 surfaces over number fields suggested by the above discussion.

Conjecture 4.5 (Uniform boundedness)**.** Fix a number field k and a primitive lattice $L \hookrightarrow \Lambda_{K3} = U^{\oplus 3} \oplus E_8(-1)^{\oplus 2}$. Let X be a K3 surface over k such that $\operatorname{NS} \overline{X} \simeq L$. Then there is a constant $c(K, L)$, independent of X, such that

$$|\operatorname{Br} X/\operatorname{Br}_0 X| < c(k, L).$$

Conjecture 4.6 (Strong uniform boundedness)**.** Fix a positive integer n and a primitive lattice $L \hookrightarrow \Lambda_{K3} = U^{\oplus 3} \oplus E_8(-1)^{\oplus 2}$. Let X be a K3 surface over a number field k of degree

n such that $\mathrm{NS}\,\overline{X} \simeq L$. Then there is a constant $c(n, L)$, independent of X such that

$$|\operatorname{Br} X / \operatorname{Br}_0 X| < c(n, L).$$

If, for some lattice L, Conjecture 4.5 is verified with an effectively computable constant $c(k, L)$, then [KT11, Theorem 1] would imply that the obstruction set $X(\mathbb{A}_k)^{\operatorname{Br} X}$ is effectively computable for the corresponding surfaces. Skorobogatov's Conjecture 3.5 would then imply there is an effective way to determine if $X(k) \neq \emptyset$ for these K3 surfaces.

The relevant moduli spaces with level structures whose rational points would shed light on Conjectures 4.5 and 4.6, have dimension $20 - r$, where $r = \operatorname{rk} L$. These spaces tend to have trivial Albanese varieties (one can use the techniques of [Kon88] to see this); thus, determining the qualitative arithmetic of these spaces is a difficult problem for small values of r. However, special cases of these conjectures may be accessible, e.g., by taking specific L with $r = 19$ or 20, where the moduli spaces to be studied have dimension ≤ 1. This is the subject of upcoming joint work with Bianca Viray. More optimistically, recent work of the author with Dan Abramovich [AVA16a, AVA16b] gives "proofs-of-concept" for similar questions on abelian varieties, conditional on Lang's Conjecture and Vojta's Conjecture, respectively. These strong conjectures allow us to control the arithmetic of high-dimensional moduli spaces with level structures. It is our hope that once an arithmetic theory of moduli spaces of K3 surfaces with Brauer level structures is firmly in place, one may obtain similar conditional results strengthening the plausibility of Conjectures 4.5 and 4.6.

5. Epilogue: Results from the Arizona Winter School

We report on the work of three project groups that began at the Arizona Winter School.

5.1. **Picard groups of degree two K3 surfaces.** Using the techniques presented in §2 as a starting point, Bouyer, Costa, Festi, Nicholls, and West [BCF+16] have computed not only the geometric Picard rank, but the full Galois module structure for general members of the family of degree 2 K3 surfaces given by

$$X/\mathbb{Q}: \qquad w^2 = ax^6 + by^6 + cz^6 + dx^2y^2z^2.$$

Over $\overline{\mathbb{Q}}$, we can assume that $a = b = c = 1$; for general d, the authors showed that $\rho(\overline{X}) = 19$. Using explicit generators for $\mathrm{NS}(\overline{X})$, the authors are able to compute the Galois cohomology groups $\mathrm{H}^i(\mathrm{Gal}(\overline{\mathbb{Q}}/\mathbb{Q}), \mathrm{NS}(\overline{X}))$ for $0 \leq i \leq 2$, and hence compute the algebraic Brauer groups $\operatorname{Br}_1 X / \operatorname{Br}_0 X$ of this family; see §3.2. The case $d = 0$, where $\rho(\overline{X}) = 20$ is also studied in Nakahara's upcoming Ph. D. thesis.

5.2. **Rational points and derived equivalence.** Ascher, Dasaratha, Perry, and Zong constructed remarkable further examples of the kind appearing in Theorem 3.11 which showed that, over $\mathbb{Q}$, $\mathbb{Q}_2$ and $\mathbb{R}$, the existence of rational points on K3 surfaces need not be preserved by twisted derived equivalences ([ADPZ16]). This result stands in sharp contrast with the untwisted derived equivalence over finite fields and p-adic fields; see [Hon15, LO15] and [HT16, Corollary 35].

5.3. **Effective bounds for Brauer groups of Kummer surfaces.** Let A be a principally polarized abelian surface over a number field k, and let X be the associated Kummer surface. Building on ideas in [SZ08], Cantoral Farfán, Tang, Tanimoto, and Visse ([CFTTV16]) showed there is an effectively computable constant M, depending on the Faltings' height of A and $\mathrm{NS}(\overline{A})$, such that $|\operatorname{Br} X / \operatorname{Br}_1 X| < M$. By [KT11, Theorem 1], it follows that the Brauer-Manin set $X(\mathbf{A})^{\operatorname{Br} X}$ for these surfaces is effectively computable. Their work also yields *practical* algorithms for computing the quotient $\operatorname{Br}_1 X / \operatorname{Br}_0 X$ when $\rho(\overline{A}) = 1$ or 2.

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