PRIMES, ELLIPTIC CURVES AND CYCLIC GROUPS: RESEARCH PROJECTS

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(PROJECT 1) A reformulation of the Prime Number Theorem for Primes in Arithmetic Progressions is that GRH for Dirichlet L-functions implies that

$$\psi(x,m,a) = \frac{\psi(x)}{\phi(m)} + \mathrm{O}\left(\sqrt{x}\,(\log x)^2\right).$$

In *Primes in arithmetic progressions*, Michigan Math. J. 17, 1970, pp. 33–39, H.L. Montgomery conjectured that the error term above should decrease when m increases; more precisely, he conjectured that

$$\psi(x,m,a) = \frac{\psi(x)}{\phi(m)} + \mathrm{O}_\varepsilon\left(\frac{x^{\frac{1}{2}+\varepsilon}}{\sqrt{m}}\right).$$

This would imply the Elliot-Halberstam Conjecture, which has many important arithmetic consequences.

It is natural to ask whether Montgomery's Conjecture has a generalization in the Chebotarev context, as done by M.R. Murty, V. K. Murty and N. Saradha in Subsection 3.13 of their paper *Modular forms and the Chebotarev Density Theorem,* American Journal of Mathematics 110, 1998, pp. 253–281.

In *Remarks on the error term in Chebotarev Density Theorem,* arxiv: 1305.5498v1 [math.NT] (23 May 2013), Joël Bellaïche addressed the question posed by Murty, Murty, and Saradha and answered it negatively in a certain range.

The project consists in predicting the correct range and error term that would answer affirmatively Montgomery's Conjecture in the context of primes splitting completely in the division fields of an elliptic curve $E/\mathbb{Q}$. In particular, the project will involve a considerate amount of numerical experiments.

PARTICIPATING STUDENTS: Travis Morrison (Pennsylvania State University) and Jesse Thorne (Emory University).

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(PROJECT 2) Following the ideas of A.C. Cojocaru and M.R. Murty in *Cyclicity of elliptic curves modulo p and elliptic curve analogues of Linnik's problem,* Mathematische Annalen 330, 2004, pp. 601–625, in Theorem 45 of *Primes, elliptic curves and cyclic groups: a synopsis* by A.C. Cojocaru

`(http://homepages.math.uic.edu/ cojocaru/primes-ec-cyclic.pdf),`

for an elliptic curve $E/\mathbb{Q}$ we highlight the constant

$$C_{\text{mean}}(E) := \sum_{m \geq 1} \frac{1}{[\mathbb{Q}(E[m]) : \mathbb{Q}]}.$$

The project consists in understanding product formulae for this constant (and related such constants) and in proving average results in the style of those of N. Jones in *Averages of elliptic curve constants,* Math. Ann. 345, 2009, pp. 685–710.

PARTICIPATING STUDENTS: Renee Bell (MIT), Clifford Blakestad (University of Colorado), Alexander Cowan (Columbia University), Vlad Matei (University of Wiscons at Madison), Geoffrey Smith (Harvard University), and Isabel Vogt (MIT).

(PROJECT 3) Given an elliptic curve $E/\mathbb{Q}$ and an integer t, a well-known conjecture of S. Lang and H. Trotter from 1976 predicts the asymptotic behaviour of

$$\#\{p \leq x : a_p = t\}.$$

In *Arithmetic properties of the Frobenius traces defined by a rational abelian variety,* arxiv:1504.00902v2 [math.NT] (10 Aug. 2015) by A.C. Cojocaru, R. Davis, A. Silverberg and K.E. Stange, the authors formulate and investigate a generalization of the Lang-Trotter Conjecture in the case of an abelian variety $A/\mathbb{Q}$ with a trivial endomorphism ring.

In this project, we will focus on a variation of the above work in the case of an abelian surface $A/\mathbb{Q}$ with a non-trivial endomorphism ring.

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