SHEAVES, STACKS, AND SHTUKAS

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These are extended notes from a four-lecture series at the 2017 Arizona Winter School on the topic of perfectoid spaces. The appendix describes the proposed student projects (including contributions from David Hansen and Sean Howe). See the table of contents below for a list of topics covered.

These notes have been deliberately written to include *much more* material than could possibly be presented in four one-hour lectures. Certain material (including the definition of an analytic Huber ring and the extension of various basic results from Tate rings to analytic rings) is original to these notes.

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Convention 0.0.1. Throughout these lecture notes, the following conventions are in force unless specifically overridden.

- All rings are commutative and unital.
- A *complete* topological space is required to be Hausdorff (as usual).
- All Huber rings and pairs we consider are complete. (This is not the convention used in [163, Lecture 1].)

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1. Sheaves on analytic adic spaces

We begin by picking up where the first lecture of Weinstein [163, Lecture 1], on the adic spectrum associated to a Huber pair, leaves off. We collect the basic facts we need about the structure sheaf, vector bundles, and coherent sheaves on the adic spectrum. The approach is in some sense motivated by the analogy between the theories of "varieties" (here meaning schemes locally of finite type over a field) and of general schemes. In our version of this analogy, the building blocks of the finite-type case are affinoid algebras over a nonarchimedean field (with which we assume some familiarity, e.g., at the level of [21] or [64]), and we are trying to extend to more general Huber rings in order to capture examples that are very much not of finite type (notably perfectoid rings). However, this passage does not go quite as smoothly as in the theory of schemes, so some care is required to assemble a theory that is both expansive enough to include perfectoid rings, but robust enough to allow us to assert the general theorems we will need.

In order to streamline the exposition, we have opted to state most of the key theorems first without proof (see §1.2–1.4). We then follow with discussion of the overall strategy of proof of these theorems (see §1.6), and finally treat the technical details of the proofs (see §1.7–1.9). Along the way, we include some technical subsections that can be skimmed or skipped on first reading: one on the open mapping theorem (§1.1), one on Banach rings (§1.5), one on the étale topology (§1.10), and one on preadic spaces (§1.11).

Hypothesis 1.0.1. Throughout §1, let (A, A^+) be a fixed Huber pair (with A complete, as per our conventions) and put $X := \mathrm{Spa}(A, A^+)$. Unless otherwise specified, we assume also that A is *analytic* (see Definition 1.1.2); however, there is little harm done if the reader prefers to assume in addition that A is *Tate* (see Definition 1.1.2 and Remark 1.1.5).

1.1. **Analytic rings and the open mapping theorem.**

We begin with a brief technical discussion, which can mostly be skipped on first reading. This has to do with the fact that Huber's theory of adic spaces includes the theory of formal schemes as a subcase, but we are primarily interested in the complementary subcase.

Remark 1.1.1. In any Huber ring, the set of units is open: if x is a unit and y is sufficiently close to x, then $x^{-1}(x-y)$ is topologically nilpotent and its powers sum to an inverse of y. This implies that any maximal ideal is closed.

This observation is often used in conjunction with [85, Proposition 3.6(i)]: if $A \neq 0$, then $X \neq \emptyset$. For a derivation of this result, see Corollary 1.5.18.

Definition 1.1.2. Recall that the Huber ring A is said to be *Tate* (or sometimes *microbial*) if it contains a topologically nilpotent unit (occasionally called a *microbe* by analogy with terminology used in real algebraic geometry [47]; more commonly a *pseudouniformizer*).

More generally, we say that A is *analytic* if its topologically nilpotent elements generate the trivial ideal in A; Example 1.5.7 separates these two conditions. The term *analytic* is not standard (yet), but is motivated by Lemma 1.1.3 below. By convention, the zero ring is both Tate and analytic.

We say that a Huber pair (A, A^+) is *Tate* (resp. *analytic*) if A is Tate (resp. analytic).

Lemma 1.1.3. *The following conditions on a general Huber pair (A, A^+) are equivalent.*

(a) *The ring A is analytic.*
(b) *Any ideal of definition in any ring of definition generates the unit ideal in A.*

(c) *Every open ideal of A is trivial.*
(d) *For every nontrivial ideal I of A, the quotient topology on A/I is not discrete.*
(e) *The only discrete topological A-module is the zero module.*
(f) *The set X contains no point on whose residue field the induced valuation is trivial.*

Proof. We start with some easy implications:

- (b) implies (a) (any ideal of definition consists of topologically nilpotent elements);
- (b) and (c) are equivalent (any ideal of definition is open, and any open ideal contains an ideal of definition);
- (c) and (d) are equivalent (trivially);
- (e) implies (d) (trivially).

We next check that (a) implies (b). Suppose that A is analytic, A_0 is a ring of definition, and I is an ideal of definition. For any topologically nilpotent elements $x_1, \dots, x_n \in A$ which generate the unit ideal, for any sufficiently large m the elements $x_1^m, \dots, x_n^m$ belong to I and still generate the unit ideal in A.

At this point, we have the equivalence among (a)–(d). To add (e), we need only check that (c) implies (e), which we achieve by checking the contrapositive. Let M be a nonzero discrete topological A-module, and choose any nonzero $m \in M$. The map $A \to M$, $a \mapsto am$ is continuous; its kernel is a nontrivial open ideal of A.

We next check that (a) implies (f). If A is analytic, then for each $v \in X$, we can find a topologically nilpotent element $x \in A$ with $v(x) \neq 0$. We must then have $0 < v(x) < 1$, so the induced valuation on the residue field is nontrivial.

We finally check that (f) implies (d), by establishing the contrapositive. Let I be a nontrivial ideal of A such that A/I is discrete for the quotient topology. Then the trivial valuation on the residue field of any maximal ideal of A/I gives rise to a point of X on whose residue field the induced valuation is trivial. $\square$

Corollary 1.1.4. *If (A, A^+) is an analytic Huber pair, then $\mathrm{Spa}(A, A^+) \to \mathrm{Spa}(A^+, A^+)$ is injective. (We will show later that it is also a homeomorphism onto its image; see Lemma 1.6.5.)*

Proof. For $v \in \mathrm{Spa}(A, A^+)$, by Lemma 1.1.3 there exists a topologically nilpotent element x of A such that $0 < v(x) < 1$. For $w \in \mathrm{Spa}(A, A^+)$ agreeing with v on A^+, for any $y, z \in A$, any sufficiently large positive integer n has the property that $x^n y, x^n z \in A^+$; it follows that the order relations in the pairs

$$(v(y), v(z)), (v(x^n y), v(x^n z)), (w(x^n y), w(x^n z)), (w(y), w(z))$$

all coincide, yielding $v = w$. $\square$

Remark 1.1.5. Lemma 1.1.3 shows that a Huber pair (A, A^+) is analytic if and only if $\mathrm{Spa}(A, A^+)$ is analytic in the sense of Huber. It also shows that if (A, A^+) is analytic, then $\mathrm{Spa}(A, A^+)$ is covered by rational subspaces (see Definition 1.2.1) which are the adic spectra of Tate rings. Consequently, from the point of view of adic spaces, escalating the level of generality of Huber pairs from Tate to analytic does not create any new geometric objects. However, it does improve various statements about acyclicity of sheaves, as in the rest of this lecture.

Exercise 1.1.6. Let A be a Huber ring. If there exists a finite, faithfully flat morphism $A \to B$ such that B is Tate (resp. analytic) under its natural topology as an A-module (see Definition 1.1.11), then A is Tate (resp. analytic).

Exercise 1.1.7. A (continuous) morphism $f : A \to B$ of general Huber rings is *adic* if one can choose rings of definition A_0, B_0 of A, B and an ideal of definition A such that $f(A_0) \subseteq B_0$ and $f(I)B_0$ is an ideal of definition of B_0. Prove that this condition is always satisfied when A is analytic.

From now on, assume (unless otherwise indicated) that A is analytic. In the classical theory of Banach spaces, the *open mapping theorem* of Banach plays a fundamental role in showing that topological properties are often controlled by algebraic properties. The same theorem is available in the nonarchimedean setting for analytic rings.

Definition 1.1.8. A morphism of topological abelian groups is *strict* if the subspace and quotient topologies on its image coincide. For a surjective morphism, this is equivalent to the map being open.

Theorem 1.1.9 (Open mapping theorem)**.** *Let $f : M \to N$ be a continuous morphism of topological A-modules which are Hausdorff, first-countable (i.e., 0 admits a countable neighborhood basis), and complete (which implies Hausdorff). If f is surjective, then f is open. (Note that A itself is first-countable.)*

Proof. As in the archimedean case, this comes down to an application of Baire's theorem that every complete metric space is a Baire space (i.e., the union of countably many nowhere dense subsets is never open). The case where A is a nonarchimedean field can be treated in parallel with the archimedean case, as in Bourbaki [23, I.3.3, Théorème 1]; see also [141, Proposition 8.6]. It was observed by Huber [86, Lemma 2.4(i)] that the argument carries over to the case where A is Tate; this was made explicit by Henkel [83]. The analytic case is similar; see Problem A.3.1. □

Remark 1.1.10. Theorem 1.1.9 is in fact a characterization of analytic Huber rings: if A is not analytic, there exists a morphism $f : M \to N$ of complete first-countable topological A-modules which is continuous but not open. For example, let I be a nontrivial open ideal and take M, N to be two copies of $\prod_{n \in \mathbb{Z}}(A/I)$ equipped with the discrete topology and the product topology, respectively. (Thanks to Zonglin Jiang for this example.)

Before stating an immediate corollary of Theorem 1.1.9, we need a definition.

Definition 1.1.11. Let M be a finitely generated A-module. For any A-linear surjective morphism $F \to M$, we may form the quotient topology of M; the resulting topology does not depend on the choice. (It suffices to compare with a second surjection $F \oplus F' \to M$ by factoring the map $F' \to M$ through $F \oplus F'$.) This topology is called the *natural topology* on A.

If A is noetherian, then M is always complete for its natural topology (see Corollary 1.1.15 below). In general, M need not be complete for the natural topology, but the only way for completeness to fail is for M to fail to be Hausdorff. Namely, if M is Hausdorff, then $\ker(F \to M)$ is closed, so quotienting by it gives a complete A-module.

Even if M is complete for its topology, that does not mean that its image under a morphism of finitely generated A-modules must be complete (unless A is noetherian). For example, for $f \in A$, it can happen that $\times f : A \to A$ is injective but its image is not closed; see Remark 1.8.4.

Corollary 1.1.12. *Suppose that A is analytic. Let M be a finitely generated A-module. If M admits the structure of a complete first-countable topological A-module for some topology, then that topology must be the natural topology.*

Proof. Apply Theorem 1.1.9 to an A-linear surjection $F \to M$ with F finite free. $\square$

Let us now see some examples of this theorem in action. The following argument is essentially [22, Proposition 3.7.2/1] or [64, Lemma 1.2.3].

Lemma 1.1.13. *Let M be a finitely generated A-module which is complete for the natural topology. Then any dense A-submodule of M equals M itself. (This argument does not require A to be analytic, but the following corollary does.)*

Proof. We may lift the problem to the case where M is free on the basis $\mathbf{e}_1, \dots, \mathbf{e}_n$. Let N be a dense submodule of M; we may then choose $\mathbf{e}'_1, \dots, \mathbf{e}'_n \in N$ such that $\mathbf{e}'_j = \sum_i B_{ij}\mathbf{e}_i$ with B_{ij} being topologically nilpotent if $i \neq j$ and $B_{ii} - 1$ being topologically nilpotent if $i = j$. Then the matrix B is invertible (its determinant equals 1 plus a topological nilpotent), so $N = M$. $\square$

Corollary 1.1.14. *Let M be a finitely generated A-module which is complete for the natural topology. Then any A-submodule of M whose closure is finitely generated is itself closed.*

Proof. Let N be an A-submodule whose closure $\widehat{N}$ is finitely generated. By Corollary 1.1.12, the subspace topology on $\widehat{N}$ coincides with the natural topology, so Lemma 1.1.13 may be applied to see that $N = \widehat{N}$. $\square$

Corollary 1.1.15. *The following statements hold.*

(a) *If A is noetherian, then every finitely generated A-module is complete for the natural topology, and every submodule of such a module is closed.*
(b) *Conversely, if every ideal of A is closed, then A is noetherian.*

Proof. Suppose first that A is noetherian. For M a finitely generated A-module and $F \to M$ an A-linear surjection with F finite free, Corollary 1.1.14 implies that $\ker(F \to M)$ is closed, so M is complete. Applying Corollary 1.1.14 again shows that every submodule of M is closed, yielding (a).

Conversely, suppose that every ideal of A is closed. To prove (b), we will obtain a contradiction under the hypothesis that there exists an ascending chain of ideals $I_1 \subseteq I_2 \subseteq \cdots$ which does not stabilize, by showing that the union I of the chain is not closed. In fact this already follows from Baire's theorem, but we give a more elementary argument below.

Since A is analytic, we can find some finite set $x_1, \dots, x_n$ of topologically nilpotent units which generate the unit ideal in A. For each m, choose an element $y_m \in I_m - I_{m-1}$. We can then choose an index $i_m \in \{1, \dots, n\}$ such that $x_{i_m}^j y_m \notin I_{m-1}$ for all positive integers j.

Let $V_1, V_2, \dots$ be a cofinal sequence of neighborhoods of 0 in A. We now choose positive integers $j_1, j_2, \dots$ subject to the following conditions (by choosing j_m sufficiently large given the choice of $j_1, \dots, j_{m-1}$).

(a) For each positive integer m, $x_{i_m}^{j_m} y_m \in V_m$.
(b) For each positive integer m, there exists an open subgroup U_m of A such that $(x_{i_m}^{j_m} y_m + U_m) \cap I_{m-1} = 0$ and $x_{i_{m'}}^{j_{m'}} y_{m'} \in U_m$ for all $m' > m$.

Then $\sum_{m=1}^{\infty} x_{i_m}^{j_m} y_m$ converges to a limit y which is in the closure of I by (a), but not in I by (b) (for each m we have $y \in I_{m-1} + x_{i_m}^{j_m} + U_m$ and hence $y \notin I_{m-1}$), a contradiction. $\square$

As a concrete example of what happens when A is not noetherian, we offer the following exercise.

Definition 1.1.16. For A a Huber ring, let $A\langle T \rangle$ be the completion of $A[T]$ for the topology with a neighborhood basis given by $U[T] = \{\sum_{n=0}^{\infty} a_n T^n : a_n \in U \text{ for all } n\}$ as U runs over neighborhoods of 0 in A. We may similarly define $A\langle T_1, \dots, T_m \rangle$, or even the analogue with infinitely many variables. When the topology on A is induced by a norm, this can be interpreted in terms of a Gauss[1] norm; see Definition 1.5.3.

Exercise 1.1.17. Let p be a prime. Let A be the quotient of the infinite Tate algebra $\mathbb{Q}_p\langle T, U_1, V_1, U_2, V_2, \dots \rangle$ by the closure of the ideal $(TU_1 - pV_1, TU_2 - p^2V_2, \dots)$.

(a) Show that A is uniform (see Definition 1.2.12).
(b) Show that T is not a zero-divisor in A.
(c) Show that the ideal TA is not closed in A.

The following argument can be found in [86, II.1], [87, Lemma 1.7.6].

Lemma 1.1.18. *Let M be an A-module which is the cokernel of a strict morphism between finite projective A-modules. Equivalently by Theorem 1.1.9, M is finitely presented and complete for the natural topology.*

(a) *Let $M\langle T \rangle$ be the set of formal sums $\sum_{n=0}^{\infty} x_n T^n$ with $x_n \in M$ forming a null sequence. Then the natural map $M \otimes_A A\langle T \rangle \to M\langle T \rangle$ is an isomorphism.*
(b) *Let $M\langle T^{\pm} \rangle$ be the set of formal sums $\sum_{n \in \mathbb{Z}} x_n T^n$ with $x_n \in M$ forming a null sequence in each direction. Then the natural map $M \otimes_A A\langle T^{\pm} \rangle \to M\langle T^{\pm} \rangle$ is an isomorphism.*

Proof. We treat only (a), since (b) is similar. If M is finitely generated and complete for the natural topology, then it is apparent that $M \otimes_A A\langle T \rangle \to M\langle T \rangle$ is surjective. Suppose now that as in the statement of the lemma, M is the cokernel of a strict morphism $F_1 \to F_0$ between finite projective A-modules. Put $N := \ker(F_0 \to M)$; then N is finitely generated and complete for the natural topology. We thus have a commutative diagram

$$\begin{tikzcd} & N \otimes_A A\langle T \rangle \arrow[r] \arrow[d] & F_0 \otimes_A A\langle T \rangle \arrow[r] \arrow[d] & M \otimes_A A\langle T \rangle \arrow[r] \arrow[d] & 0 \\ 0 \arrow[r] & N\langle T \rangle \arrow[r] & F\langle T \rangle \arrow[r] & M\langle T \rangle \arrow[r] & 0 \end{tikzcd}$$

with exact rows in which the middle vertical arrow is an isomorphism and both vertical arrows are surjective. By the five lemma, it follows that the right vertical arrow is injective. $\square$

Lemma 1.1.19. *Suppose that A is noetherian.*

(a) *The homomorphism $A \to A\langle T \rangle$ is flat.*
(b) *If $A\langle T \rangle$ is also noetherian, then $A[T] \to A\langle T \rangle$ is also flat.*

[1] Correctly spelled "Gauß", but I'll stick to the customary English transliteration.

Proof. Let $0 \to M \to N \to P \to 0$ be an exact sequence of finite A-modules; by Corollary 1.1.15, it is also a strict exact sequence for the natural topologies. Consequently, the exact sequence

$$0 \to M\langle T\rangle \to N\langle T\rangle \to P\langle T\rangle \to 0$$

is the base extension of the previous sequence from A to $A\langle T\rangle$. This proves (a).

Suppose now that $A\langle T\rangle$ is noetherian. To prove (b), by [152, Tag 00MP] it suffices to check that for every prime ideal $\mathfrak{p}$ of A, the map $A[T] \otimes_A \kappa(\mathfrak{p}) \to A\langle T\rangle \otimes_A \kappa(\mathfrak{p})$ is flat. Since $A\langle T\rangle$ is noetherian (and analytic because A is), Corollary 1.1.15 implies that $\mathfrak{p}A\langle T\rangle$ is a closed ideal; we may thus identify $\mathfrak{p}A\langle T\rangle$ with the subset $\mathfrak{p}\langle T\rangle$ of $A\langle T\rangle$ (again as in Lemma 1.1.18. In particular, as a module over the principal ideal domain $A[T] \otimes_A \kappa(\mathfrak{p}) = \kappa(\mathfrak{p})[T]$, $A\langle T\rangle \otimes_A \kappa(\mathfrak{p}) = A\langle T\rangle/\mathfrak{p}\langle T\rangle$ is torsion-free and hence flat. □

1.2. **The structure sheaf.** We continue with the definition and analysis of the structure presheaf. As in the theory of affine schemes, we have in mind a formula for certain distinguished open subsets, in this case the rational subspaces; the shape of the general definition is meant to enforce this formula. However, we will almost immediately hit a serious difficulty which echoes throughout the entire theory.

We recall some facts about rational subsets of X from the previous lecture [163, Lecture 1].

Definition 1.2.1. A *rational subspace* of X is one of the form

$$X\left(\frac{f_1, \dots, f_n}{g}\right) := \{v \in X : v(f_i) \le v(g) \ne 0 \quad (i = 1, \dots, n)\}$$

where $f_1, \dots, f_n, g \in A$ are some elements which generate an open ideal in A; such subspaces form a neighborhood basis in X. Since we are assuming that A is analytic, by Lemma 1.1.3 any open ideal is in fact the trivial ideal; in particular, we may rewrite the previous formula as

$$X\left(\frac{f_1, \dots, f_n}{g}\right) := \{v \in X : v(f_i) \le v(g) \quad (i = 1, \dots, n)\}.$$

There is a morphism $(A, A^+) \to (B, B^+)$ of (complete) Huber pairs which is initial among morphisms for which $\mathrm{Spa}(B, B^+)$ maps into $X\left(\frac{f_1,\dots,f_n}{g}\right)$; this morphism induces a map $\mathrm{Spa}(B, B^+) \cong X\left(\frac{f_1,\dots,f_n}{g}\right)$ which not only is a homeomorphism, but matches up rational subspace of $\mathrm{Spa}(B, B^+)$ with rational subspaces of X contained in $X\left(\frac{f_1,\dots,f_n}{g}\right)$. We call any such morphism "the" *rational localization* corresponding to $X\left(\frac{f_1,\dots,f_n}{g}\right)$, using the definite article since the morphism is unique up to unique isomorphism.

Since $f_1, \dots, f_n, g$ generate the unit ideal, the ring B in the pair (B, B^+) may be identified explicitly as the quotient of $A\langle T_1, \dots, T_n\rangle$ by the closure of the ideal $(gT_1 - f_1, \dots, gT_n - f_n)$; we denote this ring by $A\left\langle\frac{f_1,\dots,f_n}{g}\right\rangle$. (We will see later that when the structure presheaf on X is a presheaf, it is not necessarily to take the closure; see Theorem 1.2.7.) The ring B^+ may be identified as the integral closure of the image of $A^+\langle T_1, \dots, T_n\rangle$ in B; we denote this ring by $A^+\left\langle\frac{f_1,\dots,f_n}{g}\right\rangle$.

Exercise 1.2.2. Given $f_1, \dots, f_n, g \in A$ which generate the unit ideal, there exists a neighborhood W of 0 in A such that any $f'_1, \dots, f'_n, g' \in A$ satisfying $f'_1 - f_1, \dots, f'_n - f_n, g' - g \in W$

generate the unit ideal and define the same rational subspace as do $f_1, \dots, f_n, g$. (See [142, Remark 2.8], [107, Remark 2.4.7].)

Definition 1.2.3. Define the *structure presheaf* $\mathcal{O}$ on X as follows: for $U \subseteq X$ open, let $\mathcal{O}(U)$ be the inverse limit of B over all rational localizations $(A, A^+) \to (B, B^+)$ with $\mathrm{Spa}(B, B^+) \subseteq U$. In particular, if $U = \mathrm{Spa}(B, B^+)$ then $\mathcal{O}(U) = B$.

Let $\mathcal{O}^+$ be the subsheaf of $\mathcal{O}$ defined as follows: for $U \subseteq X$ open, let $\mathcal{O}(U)$ be the inverse limit of B^+ over all rational localizations $(A, A^+) \to (B, B^+)$ with $\mathrm{Spa}(B, B^+) \subseteq U$. Equivalently,

$$\mathcal{O}^+(U) = \{f \in \mathcal{O}(U) : v(f) \leq 1 \text{ for all } v \in U\}.$$

In particular, if $U = \mathrm{Spa}(B, B^+)$ then $\mathcal{O}(U) = B^+$.

Remark 1.2.4. For any open subset U of X, the ring $\mathcal{O}(U)$ is complete for the inverse limit topology, but in general it is not a Huber ring. A typical example is the open unit disc inside the closed unit disc, which is *Fréchet complete* with respect to the supremum norms over all of the closed discs around the origin of radii less than 1. (This ring cannot be Huber because the topologically nilpotent elements do not form an open set.)

Remark 1.2.5. For each $x \in X$, the stalk $\mathcal{O}_{X,x}$ is a direct limit of complete rings, and hence is a henselian local ring; in particular, the categories of finite étale algebras over $\mathcal{O}_{X,x}$ and over its residue field are equivalent. Compare [107, Lemma 2.4.17].

Remark 1.2.6. In order to follow the theory of affine schemes, one would next expect to prove that the presheaf $\mathcal{O}$ is a sheaf. This is indeed true when A is an affinoid algebra over a nonarchimedean field, as this follows (after a small formal argument; see Lemma 1.6.3) from Tate's acyclicity theorem in rigid analytic geometry [155, Theorem 8.2], [22, Theorem 8.2.1/1].

Unfortunately, there exist examples where $\mathcal{O}$ is not a sheaf. This remains true if we assume that A is Tate, as shown by an example of Huber [86, §1]; or even if we assume that A is Tate and uniform, as shown by examples of Buzzard–Verberkmoes [25, Proposition 18] and Mihara [126, Theorem 3.15].

A conceptual explanation for the previous examples is given by the following result.

Theorem 1.2.7 (original)**.** *Suppose that $\mathcal{O}$ is a sheaf. Then for any $f_1, \dots, f_n, g \in A$ which generate the unit ideal, the ideal $(gT_1 - f_1, \dots, gT_n - f_n)$ in $A\langle T_1, \dots, T_n\rangle$ is closed.*

Proof. Let $(A, A^+) \to (B, B^+)$ be the rational localization defined by the parameters $f_1, \dots, f_n$; then the kernel of the map $A\langle T_1, \dots, T_n\rangle \to B$ taking T_i to f_i/g is the closure of the ideal in question. By Corollary 1.1.14, it thus suffices to check that this kernel is finitely generated; this will follow from Theorem 1.4.19. □

In light of the previous remarks, we are forced to introduced and study the following definition.

Definition 1.2.8. We say that (A, A^+) is *sheafy* if $\mathcal{O}$ is a sheaf. Although it is not immediately obvious from the definition, we will see shortly that this property depends only on A, not on A^+ (Remark 1.6.9).

Definition 1.2.9. When (A, A^+) is sheafy, we may equip X in a natural way with the structure of a *locally v-ringed space*, i.e., a locally ringed space in which the stalk of the structure sheaf at each point is equipped with a distinguished valuation (with morphisms required to correctly pull back these valuations). By considering locally v-ringed spaces which are locally of this form, we obtain Huber's notion of an *analytic adic space*.

As explained in [163, Lecture 1], Huber's theory also allows the use of rings A which are not analytic; this for example allows ordinary schemes and formal schemes to be treated as adic spaces. In addition, Huber shows that a Huber ring A which need not be analytic, but which admits a noetherian ring of definition, is sheafy [86, Theorem 2.5]. However, allowing nonanalytic Huber rings creates some extra complications which are not pertinent to the examples we have in mind (with a small number of exceptions), e.g., the distinction between continuous and adic morphisms (see Exercise 1.1.7). For expository treatments of adic spaces without the analytic restriction, see [28] or [161].

We will establish sheafiness for two primary classes of Huber rings. The first includes the class of affinoid algebras.

Definition 1.2.10. The ring A is *strongly noetherian* if for every nonnegative integer n, the ring $A\langle T_1, \dots, T_n\rangle$ is noetherian. For example, if A is an affinoid algebra over a nonarchimedean field K, then A is strongly noetherian: this reduces to the fact that $K\langle T_1, \dots, T_n\rangle$ is noetherian, for which see [155, Theorem 4.5] or [22, Theorem 5.2.6/1].

When A is Tate, the following result is due to Huber [86, Theorem 2.5]. The general case incorporates an observation of Gabber to treat the case where A is analytic but not Tate; see §1.7 for the proof.

Theorem 1.2.11 (Huber plus Gabber's method)**.** *If A is strongly noetherian, then A is sheafy.*

The second class of sheafy rings we consider includes the class of perfectoid rings.

Definition 1.2.12. Recall that A is said to be *uniform* if the ring of power-bounded elements of A is a bounded subset. (A subset S of A is *bounded* if for every neighborhood U of 0 in A, there exists a neighborhood V of 0 in A such that $S \cdot V \subseteq U$. If A is topologized using a norm, this corresponds to boundedness in the usual sense.) For example, if K is a nonarchimedean field, then $K\langle T\rangle/(T^2)$ is not uniform because the K-line spanned by T is unbounded, but consists of nilpotent and hence power-bounded elements; by the same token, any uniform (analytic) Huber ring is reduced, and conversely for affinoid algebras (see Remark 1.2.16).

The pair (A, A^+) is *stably uniform* if for every rational localization $(A, A^+) \to (B, B^+)$, the ring B is uniform. Again, this depends only on A, not on A^+: one may quantify over all rational localizations by running over finite sequences $f_1, \dots, f_n, g$ of parameters which generate the unit ideal, rather than over rational subspaces; and in this formulation A^+ does not appear. (What is affected by the choice of A^+ is whether or not two different sets of parameters define the *same* rational subspace.)

The case of the following result where A is Tate is due to Buzzard–Verberkmoes [25, Theorem 7] (see also [107, Theorem 2.8.10]). The general case is again obtained by modifying the argument slightly using Gabber's method; see again §1.7 for the proof.

Theorem 1.2.13 (Buzzard–Verberkmoes plus Gabber's method)**.** *If A is stably uniform, then (A, A^+) is sheafy.*

Remark 1.2.14. If A is uniform, then the natural map from A to the ring $H^0(X, \mathcal{O})$ of global sections of $\mathcal{O}$ is automatically injective (Remark 1.5.25); if A is stably uniform, then the analogous map for any rational subspace is also injective. The content of Theorem 1.2.13 is to show that these maps are all surjective.

Let us now discuss the previous two definitions in more detail.

Remark 1.2.15. Unfortunately, it is rather difficult to exhibit examples of strongly noetherian Banach rings, in part because there is no general analogue of the Hilbert basis theorem: it is unknown is general whether or not A being noetherian and Tate implies that $A\langle T \rangle$ is noetherian. See Remark 1.2.17 for further discussion.

For K a discretely valued field, one can build another class of strongly noetherian rings by considering *semiaffinoid algebras*, i.e., the quotients of rings of the form

$$\mathfrak{o}_K[\![T_1, \dots, T_m]\!]\langle U_1, \dots, U_n \rangle \otimes_{\mathfrak{o}_K} K;$$

these rings, and the *uniformly rigid spaces* associated to them, have been studied by Kappen [91]. Beware that the identification of rigid spaces with certain adic spaces does not extend to uniformly rigid spaces, and as a result certain phenomena do not exhibit the same behavior.

A third class of strongly noetherian rings will arise from studying Fargues–Fontaine curves in a subsequent lecture. See Remark 3.1.10.

Remark 1.2.16. Every reduced affinoid algebra over a nonarchimedean field is stably uniform; this follows from the facts that any reduced affinoid algebra is uniform [22, Theorem 6.2.4/1], [64, Theorem 3.4.9] and any rational localization of a reduced affinoid algebra is again reduced [22, Corollary 7.3.2/10], [107, Lemma 2.5.9]. However, this argument does not apply to reduced strongly noetherian rings; see Remark 1.2.17 for further discussion.

Additionally, every perfectoid ring is stably uniform; this is because any rational localization is again perfectoid. These examples are genuinely separate from the strongly noetherian case, because a perfectoid ring is noetherian if and only if it is a finite direct product of perfectoid fields (Corollary 2.9.3).

Remark 1.2.17. One can construct examples where (the underlying ring of) A is a field but is not uniform (see [104]). In particular, any such A is neither discrete nor a nonarchimedean field; in particular, A is Tate. The underlying ring of A is obviously noetherian and reduced.

In no such example do we know whether or not A is strongly noetherian. If so, this would provide an example of a reduced, strongly noetherian, Tate ring which is not even uniform, let alone stably uniform (Remark 1.2.16). If not, this would provide an example of the failure of the Hilbert basis theorem for Huber rings (Remark 1.2.15).

It is not straightforward to check that a given uniform Huber ring A is stably uniform. Most known examples which are not strongly noetherian are derived from perfectoid algebras (to be introduced in the next lecture) using the following observation. (See Exercise 2.5.8 for an exception.)

Lemma 1.2.18. *Suppose that there exist a stably uniform Huber ring B and a continuous A-linear morphism $A \to B$ which splits in the category of topological A-modules. Then A is stably uniform.*

Proof. The existence of the splitting implies that $A \to B$ is strict, so A is uniform. Moreover, the existence of the splitting is preserved by taking the completed tensor product over A with a rational localization. It follows that A is stably uniform. $\square$

Remark 1.2.19. Rings satisfying the hypothesis of Lemma 1.2.18 with B being a perfectoid ring (as in Corollary 2.5.5 below) are called *sousperfectoid* rings in [76], where their basic properties are studied in some detail. This refines the concept of a *preperfectoid* ring considered in [148].

The following question is taken from [107, Remark 2.8.11].

Problem 1.2.20. Is it possible for A to be uniform and sheafy without being stably uniform?

Remark 1.2.21. At this point, it is natural to ask whether the inclusion functor from sheafy Huber rings to arbitrary Huber rings admits a spectrum-preserving left adjoint. This would be clear if $H^0(X, \mathcal{O})$ were guaranteed to be a sheafy Huber ring; however, it is not even clear that it is complete, due to the implicit direct limit in the definition of $H^0(X, \mathcal{O})$. By contrast, if X admits a single covering by the spectra of sheafy rings, then the subspace topology gives $H^0(X, \mathcal{O})$ the structure of a Huber ring, and it turns out (but not trivially) that this ring is sheafy; see Theorem 1.2.22.

Another approach to working around the failure of sheafiness for general Huber rings is to use techniques from the theory of algebraic stacks. For this approach, see §1.11.

For the proof of the following result, see §1.9.

Theorem 1.2.22 (original)**.** *Suppose that there exists a finite covering $\mathfrak{V}$ of X by rational subspaces such that $\mathcal{O}|_V$ is a sheaf for each $V \in \mathfrak{V}$. Put*

$$\tilde{A} := H^0(X, \mathcal{O}), \qquad \tilde{A}^+ := H^0(X, \mathcal{O}^+);$$

note that these rings constitute a Huber pair for the subspace topology on $\tilde{A}$.

(a) *The map $A \to \tilde{A}$ induces a homeomorphism* $\mathrm{Spa}(\tilde{A}, \tilde{A}^+) \cong \mathrm{Spa}(A, A^+)$ *of topological spaces such that rational subspaces pull back to rational subspaces (but possibly not conversely) and on each $V \in \mathfrak{V}$, the structure presheaf pulls back to the structure presheaf.*

(b) *The ring $\tilde{A}$ is sheafy.*

In particular, by Theorem 1.3.4, $\mathcal{O}$ is acyclic.

Remark 1.2.23. In Theorem 1.2.22, it is obvious that if $\mathcal{O}(V)$ is stably uniform for each $V \in \mathfrak{V}$, then so is $\tilde{A}$. The analogous statement for the strongly noetherian property is true but much less obvious; see Corollary 1.4.18.

Remark 1.2.24. One of the examples of Buzzard–Verberkmoes [25, Proposition 13] is a construction in which there exists a finite covering $\mathfrak{V}$ of X by rational subspaces such that $\mathcal{O}(V)$ is a perfectoid (and hence stably uniform and sheafy) Huber ring for each $V \in \mathfrak{V}$, so Theorem 1.2.22 applies, but the map $A \to H^0(X, \mathcal{O})$ is not injective. (In this example, one has $A^+ = A^\circ$.) See Remark 2.5.11 for further discussion.

Another example of Buzzard–Verberkmoes [25, Proposition 16] is a construction in which $A \to H^0(X, \mathcal{O})$ is injective but not surjective. However, in this example, we do not know whether A is uniform (injectivity is instead established using Corollary 1.5.24), or whether $H^0(X, \mathcal{O})$ is a Huber ring (because the construction does not immediately yield local sheafiness).

1.3. **Cohomology of sheaves.** Recall that Tate's acyclicity theorem asserts more than the fact that $\mathcal{O}$ is a sheaf: it also asserts the vanishing of higher cohomology of $\mathcal{O}$ on rational subspaces, and makes similar assertions for the presheaves associated to finitely generated A-modules. We turn next to generalizing these statements to more general Huber rings.

Definition 1.3.1. We say that a sheaf $\mathcal{F}$ on X is *acyclic* if $H^i(U, \mathcal{F}) = 0$ for every rational subspace U of $\mathcal{F}$ and every positive integer i.

Definition 1.3.2. For any A-module M, let $\tilde{M}$ be the presheaf on X such that for $U \subseteq X$ open, $\tilde{M}(U)$ is the inverse limit of $M \otimes_A B$ over all rational localizations $(A, A^+) \to (B, B^+)$ with $\mathrm{Spa}(B, B^+) \subseteq U$. In particular, if $U = \mathrm{Spa}(B, B^+)$ then $\tilde{M}(U) = M \otimes_A B$.

Remark 1.3.3. Beware that the definition of $\tilde{M}$ uses the ordinary tensor product, and makes no reference to any topology on M. However, if M is finitely generated and both M and its base extension are complete for the natural topology (Definition 1.1.11), then the ordinary tensor product coincides with the completed tensor product. Note that the condition on completeness of the base extension cannot be omitted; see Exercise 1.4.9.

In the Tate case, the following result is due to Kedlaya–Liu [107, Theorem 2.4.23]; this again generalizes results of Tate and Huber for affinoid algebras and strongly noetherian rings, respectively. For the proof, see §1.8.

Theorem 1.3.4 (Kedlaya–Liu plus Gabber's method)**.** *If A is sheafy, then for any finite projective A-module M, the presheaf $\tilde{M}$ is an acyclic sheaf.*

Remark 1.3.5. One serious impediment to extending Theorem 1.3.4 to more general modules is that it is not known that rational localization maps are flat. This is true in rigid analytic geometry [155, Lemma 8.6], [22, Corollary 7.3.2/6]; the same result extends to strongly noetherian Tate rings, as shown by Huber [86, II.1], [87, Lemma 1.7.6]. It is not at all clear whether flatness should hold in general; however, one can prove a weaker result which nonetheless implies all of the previously asserted flatness results, and is useful in applications. See Theorem 1.4.13.

Remark 1.3.6. Even in rigid analytic geometry, there is no cohomological criterion for detecting affinoid spaces among quasicompact rigid spaces: it is possible to exhibit a two-dimensional rigid analytic space X over a field such that X is not affinoid, but $H^i(X, \mathcal{F}) = 0$ for every coherent sheaf $\mathcal{F}$ on X and every $i > 0$. This was originally established by Q. Liu [123].

1.4. **Vector bundles and pseudocoherent sheaves.** To further continue the analogy with affine schemes, one would now like to define coherent sheaves (or pseudocoherent sheaves, in the absence of noetherian hypotheses) and verify that they are precisely the sheaves arising from pseudocoherent modules. In rigid analytic geometry, this is a theorem of Kiehl [111, Theorem 1.2]; however, here we are hampered by a lack of flatness (Remark 1.3.5). Before remedying this in a way that leads to a full generalization of Kiehl's result, let us consider separately the case of vector bundles.

Definition 1.4.1. Let $\mathbf{FPMod}_A$ denote the category of finite projective A-modules. A *vector bundle* on X is a sheaf $\mathcal{F}$ of $\mathcal{O}$-modules on X which is locally of the form $\tilde{M}$ for M finite projective. In other words, there exists a finite covering $\{U_i\}_{i=1}^n$ of X by rational subspaces such that for each i, $M_i := \mathcal{F}(U_i) \in \mathbf{FPMod}_{\mathcal{O}(U_i)}$ and the canonical morphism $\tilde{M}_i \to \mathcal{F}|_{U_i}$ of sheaves of $\mathcal{O}|_{U_i}$-modules is an isomorphism. Let $\mathbf{Vec}_X$ denote the category of vector bundles on X; the functor $\mathbf{FPMod}_A \to \mathbf{Vec}_X$ taking M to $\tilde{M}$ is exact. (Note that this exactness comes from the flatness of finite projective modules, not the flatness of rational localizations, which is not known; see Remark 1.3.5.)

When A is Tate, the following result is due to Kedlaya–Liu [107, Theorem 2.7.7]; again, the Tate hypothesis can be removed using Gabber's method. See §1.9 for the proof.

Theorem 1.4.2 (Kedlaya–Liu plus Gabber's method)**.** *If A is sheafy, then the functor $\mathbf{FPMod}_A \to \mathbf{Vec}_X$ taking M to $\tilde{M}$ is an equivalence of categories, with quasi-inverse taking $\mathcal{F}$ to $\mathcal{F}(X)$. In particular, by Theorem 1.3.4, every sheaf in $\mathbf{Vec}_X$ is acyclic.*

Remark 1.4.3. If one restricts attention to finite étale A-algebras and finite étale $\mathcal{O}_X$-modules, then the functor $M \mapsto \tilde{M}$ is an equivalence of categories even if A is not sheafy. See for example [107, Theorem 2.6.9] in the case where A is Tate.

Remark 1.4.4. Theorem 1.4.2 may be reformulated as the statement that the functor $\mathbf{Vec}_{\mathrm{Spec}(A)} \to \mathbf{Vec}_X$ given by pullback along the canonical morphism $X \to \mathrm{Spec}(A)$ of locally ringed spaces (coming from the adjunction property of affine schemes) is an equivalence of categories. It also implies that $\mathbf{Vec}_X$ depends only on A, not on A^+. (The same will be true for $\mathbf{PCoh}_X$ by Theorem 1.4.17.)

We now turn to more general (but still finitely generated) modules; here we give a streamlined presentation of material from [107].

Definition 1.4.5. An A-module M is *pseudocoherent* if it admits a projective resolution (possibly of infinite length) by finite projective A-modules (which may even be taken to be free modules); when A is noetherian, this is equivalent to A being finitely generated. (The term *pseudocoherent* appears to have originated in SGA 6 [90, Exposé I], and used systematically in the paper of Thomason–Trobaugh [157].)

Write $\mathbf{PCoh}_A$ for the category of pseudocoherent A-modules which are complete for the natural topology. If A is noetherian, by Corollary 1.1.15 this is exactly the category of finitely generated A-modules.

Here are some typical examples.

Remark 1.4.6. Let R be a noetherian ring, let $R \to A$ be a flat ring homomorphism with A analytic, and let M be a finitely generated R-module. If $M \otimes_R A$ is complete for the natural topology, then it belongs to $\mathbf{PCoh}_A$.

Remark 1.4.7. For any $f \in A$ which is not a zero-divisor, if the ideal fA is closed in A, then $A/fA \in \mathbf{PCoh}_A$. For some explicit examples, take $A = A_0\langle T \rangle$ with A_0 uniform and choose $f = \sum_{n=0}^{\infty} f_n T^n \in A$ such that the f_n generate the unit ideal in A_0; then f is not a zero-divisor and fA is closed (see Lemma 1.5.26).

Remark 1.4.8. By contrast, if $f \in A$ is not a zero-divisor and fA is not closed in A (which can occur; see Exercise 1.1.17 or Remark 1.8.4), then A/fA is pseudocoherent but not complete for the natural topology, and hence not an object of $\mathbf{PCoh}_A$.

Going further, we have the following example.

Exercise 1.4.9. Set notation as in Exercise 1.1.17.

(a) Show that A is uniform. I do not know whether A is stably uniform.
(b) Show that the natural map $\mathbb{Q}_p\langle T\rangle \to A$ is flat. Recall that $\mathbb{Q}_p\langle T\rangle$ is a principal ideal domain (see [64, Theorem 2.2.9] or [99, Proposition 8.3.2]), so this amounts to checking that no nonzero element of $\mathbb{Q}_p\langle T\rangle$ maps to a zero-divisor in A (the case of T itself having been checked in Exercise 1.1.17.
(c) Deduce that in Remark 1.4.6, the condition that $M \otimes_R A$ be complete for the natural topology cannot be omitted. (Take $R := \mathbb{Q}_p\langle T\rangle$, $M := R/TR$.)

Remark 1.4.10. An easy fact about pseudocoherent A-modules is the "two out of three" property: in a short exact sequence

$$0 \to M_1 \to M \to M_2 \to 0$$

of A-modules, if any two of M, M_1, M_2 are pseudocoherent, then so is the third.

The "two out of three" property is not true as stated for $\mathbf{PCoh}_A$: if $M_1, M \in \mathbf{PCoh}_A$, then M_2 need not be complete for its natural topology (as in Definition 1.1.11, this can occur for $M_1 = M = A$). However, if $M_1, M_2 \in \mathbf{PCoh}_A$, then this easily implies that $M \in \mathbf{PCoh}_A$ (see Exercise 1.4.11); while if $M, M_2 \in \mathbf{PCoh}_A$, then $M_1 \in \mathbf{PCoh}_A$ because M_1 is Hausdorff for the subspace topology and hence also for its natural topology.

Exercise 1.4.11. Let

$$0 \to M_1 \to M \to M_2 \to 0$$

be an exact sequence of topological A-modules in which M_1, M_2 are complete and M is finitely generated over some Huber ring B over A. Then M is complete for its natural topology as a B-module. (Hint: Choose a B-linear surjection $F \to M$ and apply the open mapping theorem to the composition $F \to M \to M_2$ as a morphism of topological A-modules. This implies that the surjection $M \to M_2$ has a bounded set-theoretic section; using this section, separate the problem of summing a null sequence in M to analogous problems in M_1 and M_2.)

Remark 1.4.12. Note that a pseudocoherent module is not guaranteed to have a *finite* projective resolution by finite projective modules, even over a noetherian ring; this is the stronger property of being of *finite projective dimension*. For example, for any field k, over the local ring $k[T]/(T^2)$, the residue field is a module which is pseudocoherent but not of finite projective dimension. More generally, every pseudocoherent module over a noetherian local ring is of finite projective dimension if and only if the ring is regular [152, Tag 0AFS]. (Modules of finite projective dimension are sometimes called *perfect modules*, as in [152, Tag 0656], since they are the ones whose associated singleton complexes are perfect.)

When A is Tate, the following result is due to Kedlaya–Liu [108, Theorem 2.4.7]. See again §1.9 for the proof.

Theorem 1.4.13 (Kedlaya–Liu plus Gabber's method)**.** *If A is sheafy, then for any rational localization $(A, A^+) \to (B, B^+)$, base extension from A to B defines an exact functor $\mathbf{PCoh}_A \to \mathbf{PCoh}_B$. In particular, if A is noetherian, then $A \to B$ is flat (because every finitely generated module belongs to $\mathbf{PCoh}_A$ by Corollary 1.1.15).*

A sample corollary is the following.

Corollary 1.4.14. *Suppose that A is sheafy. Let $f \in A$ be a non-zero-divisor such that fA is closed in A. Then for any rational localization $(A, A^+) \to (B, B^+)$, f is not a zero-divisor in B either (with the proviso that 0 is not a zero-divisor in the zero ring) and fB is closed in B.*

Theorem 1.4.13 makes it possible to consider sheaves constructed from pseudocoherent modules, starting with the following statement which in the Tate case is [108, Theorem 2.5.1]. See again §1.9 for the proof.

Theorem 1.4.15 (Kedlaya–Liu plus Gabber's method)**.** *If A is sheafy, then for any $M \in \mathbf{PCoh}_A$, the presheaf $\tilde{M}$ is an acyclic sheaf.*

Definition 1.4.16. A *pseudocoherent sheaf* on X is a sheaf $\mathcal{F}$ of $\mathcal{O}$-modules on X which is locally of the form $\tilde{M}$ for M pseudocoherent and complete for the natural topology. In other words, there exists a finite covering $\{U_i\}_{i=1}^n$ of X by rational subspaces such that for each i, $M_i := \mathcal{F}(U_i) \in \mathbf{PCoh}_{\mathcal{O}(U_i)}$ and the canonical morphism $\tilde{M}_i \to \mathcal{F}|_{U_i}$ of sheaves of $\mathcal{O}|_{U_i}$-modules is an isomorphism. Let $\mathbf{PCoh}_X$ denote the category of pseudocoherent sheaves on X; by Theorem 1.4.13, the functor $\mathbf{PCoh}_A \to \mathbf{PCoh}_X$ taking M to $\tilde{M}$ is exact.

In case A is strongly noetherian, we refer to pseudocoherent sheaves also as *coherent sheaves*, and denote the category of them also by $\mathbf{Coh}_X$.

When A is Tate, the following result is due to Kedlaya–Liu [108, Theorem 2.5.6]. Somewhat surprisingly, the strongly noetherian case cannot be found in Huber's work. See again §1.9 for the proof.

Theorem 1.4.17 (Kedlaya–Liu plus Gabber's method)**.** *If A is sheafy, then the functor $\mathbf{PCoh}_A \to \mathbf{PCoh}_X$ taking M to $\tilde{M}$ is an exact (by Theorem 1.4.13) equivalence of categories, with quasi-inverse taking $\mathcal{F}$ to $\mathcal{F}(X)$. In particular, by Theorem 1.4.15, every sheaf in $\mathbf{PCoh}_X$ is acyclic.*

Corollary 1.4.18. *In Theorem 1.2.22, if $\mathcal{O}(V)$ is strongly noetherian for each $V \in \mathfrak{V}$, then so is $\tilde{A}$.*

Proof. It suffices to check that if $\tilde{A}$ is noetherian, as we may then apply the same logic to the pullback coverings of the spectra of $A\langle T_1, \dots, T_n\rangle$ for all n. We may further assume that $\tilde{A} = A$.

Let I be any ideal of A and put $M := A/IA$. For $V \in \mathfrak{V}$, $\mathcal{O}(V)$ is noetherian and so $M \otimes_A \mathcal{O}(V) \in \mathbf{PCoh}_{\mathcal{O}(V)}$; this means that $\tilde{M} \in \mathbf{PCoh}_X$. By Theorem 1.4.15 and Theorem 1.4.17, we have $M = H^0(X, \tilde{M}) \in \mathbf{PCoh}_A$. Hence I is finitely generated; since I was arbitrary, it follows that A is noetherian. $\square$

Another corollary of Theorem 1.4.17 is the following result which is used to prove Theorem 1.2.7. See again §1.9 for the proof.

Theorem 1.4.19 (original)**.** *Suppose that A is sheafy. Then for any rational localization $(A, A^+) \to (B, B^+)$ and any factorization of $A \to B$ through a surjective homomorphism $A\langle T_1, \dots, T_n\rangle \to B$, we have $B \in \mathbf{PCoh}_{A\langle T_1, \dots, T_n\rangle}$.*

We also mention the following result. See again §1.9 for the proof.

Theorem 1.4.20 (original)**.** *Suppose that A is sheafy. Then for any closed ideal I of A which is an object of $\mathbf{PCoh}_A$, the ring A/I is sheafy. In other words, if $A \to B$ is a surjective ring homomorphism and $B \in \mathbf{PCoh}_A$, then B is sheafy.*

Remark 1.4.21. In algebraic geometry, one knows that the theory of quasicoherent sheaves on affine schemes is "the same" whether one uses the Zariski topology or the étale topology, in that both the category of sheaves and their cohomology groups are the same. Roughly speaking, the same is true for adic spaces, but one needs to be careful about technical hypotheses. See §1.10 for a detailed discussion.

At this point, we have completed the statements of the main results of this lecture. The notes continue with some technical tools needed for the proofs; the reader impatient to get to the main ideas of the proofs is advised to skip ahead to §1.6 at this point, coming back as needed later.

1.5. **Huber versus Banach rings.** Although most of our discussion will be in terms of Huber rings, which play the starring role in the study of rigid analytic spaces and adic spaces, it is sometimes useful to translate certain statements into the parallel language of Banach rings, which underlie the theory of Berkovich spaces. We explain briefly how these two points of view interact, as in [107, 108] where most of the local theory is described in terms of Banach rings. A key application will be to show that certain multiplication maps on Tate algebras are strict; see Lemma 1.5.26.

Definition 1.5.1. By a *Banach ring* (more precisely, a *nonarchimedean commutative Banach ring*), we will mean a ring B equipped with a function $|\bullet| : B \to \mathbb{R}_{\geq 0}$ satisfying the following conditions.

(a) On the additive group of B, $|\bullet|$ is a norm (i.e., a nonarchimedean absolute value, so that $|x - y| \leq \max\{|x|, |y|\}$ for all $x, y \in B$) with respect to which B is complete.
(b) The norm on B is *submultiplicative*: for all $x, y \in B$, we have $|xy| \leq |x|\,|y|$.

A ring homomorphism $f : B \to B'$ of Banach rings is *bounded* if there exists $c \geq 0$ such that $|f(x)|' \leq c\,|x|$ for all $x \in B$; the minimum such c is called the *operator norm* of f.

We view Banach rings as a category with the morphisms being the bounded ring homomorphisms. In particular, if two norms on the same ring differ by a bounded multiplicative factor on either side, then they define isomorphic Banach rings.

As for Huber rings, we say that a Banach ring B is *analytic* if its topologically nilpotent elements of B generate the unit ideal. (In [107], the corresponding condition is for B to be *free of trivial spectrum*.)

Remark 1.5.2. In condition (b) of Definition 1.5.1, one could instead insist that there exist some constant $c > 0$ such that for all $x, y \in B$, we have $|xy| \leq c\,|x|\,|y|$. However, this adds no essential generality, as replacing $|x|$ with the operator norm of $y \mapsto xy$ gives an isomorphic Banach ring which does satisfy (b).

In the category of Banach rings, we have the following analogue of Tate algebras.

Definition 1.5.3. For B a Banach ring and $\rho > 0$, let $B\langle \frac{T}{\rho} \rangle$ be the completion of $B[T]$ for the *weighted Gauss norm*

$$\left| \sum_{n=0}^{\infty} x_n T^n \right|_{\rho} = \max\{|x_n|\,\rho^n\}.$$

For $\rho = 1$, this coincides with the usual Tate algebra $B\langle T \rangle$.

If we define the associated graded ring

$$\operatorname{Gr} B := \bigoplus_{r>0} \frac{\{x \in B : |x| \leq r\}}{\{x \in B : |x| < r\}},$$

then $\operatorname{Gr} B\langle \frac{T}{\rho} \rangle$ is the graded ring $(\operatorname{Gr} B)[\overline{T}]$ with $\overline{T}$ placed in degree ρ. One consequence of this (which generalizes the usual Gauss's lemma) is that if the norm on B is multiplicative, then $\operatorname{Gr} B$ is an integral domain, as then is $(\operatorname{Gr} B)[\overline{T}]$, so the weighted Gauss norm on $B\langle \frac{T}{\rho} \rangle$ is multiplicative. (See Lemma 1.8.1 for another use of the graded ring construction.)

Remark 1.5.4. As usual, let A be an analytic Huber ring. Choose a ring of definition A_0 of A (which must be open in A) and an ideal of definition I (which must be finitely generated). Using these choices, we can *promote* A to a Banach ring as follows: for $x \in A$, let $|x|$ be the infimum of e^{-n} over all nonnegative integers n for which $xI^m \subseteq I^{m+n}$ for all nonnegative integers m.

Note that this works even if A is not analytic; in particular, any Huber ring is metrizable and in particular first-countable. However, even analytic Huber rings need not be second-countable; consider for example $\mathbb{Q}_p\langle T_1, T_2, \dots \rangle$.

Remark 1.5.5. In the other direction, starting with a Banach ring B, it is not immediately obvious that its underlying topological ring is a Huber ring; the difficulty is to find a finitely generated ideal of definition. However, this is always possible if B is analytic: if $x_1, \dots, x_n$ are topologically nilpotent elements of B generating the unit ideal, then for any ring of definition A_0, for any sufficiently large m the elements $x_1^m, \dots, x_n^m$ of A belong to A_0 and generate an ideal of definition.

Example 1.5.6. The infinite polynomial ring $\mathbb{Q}[T_1, T_2, \dots]$ admits a submultiplicative norm where for $x \neq 0$, $|x| = e^{-n}$ where n is the largest integer such that $x \in (T_1, T_2, \dots)^n$. Let B be the Banach ring obtained by taking the completion with respect to this norm. Then the underlying ring of B is not a Huber ring.

As an example of viewing a Banach ring as a Huber ring, we give an example of a Huber ring which is analytic but not Tate. Of course this example can be described perfectly well without Banach rings, but we find the presentation using norms a bit more succinct.

Example 1.5.7. Choose any $\rho > 1$ and equip

$$A := \mathbb{Z}\left\langle \frac{a}{\rho}, \frac{b}{\rho}, \frac{x}{\rho^{-1}}, \frac{y}{\rho^{-1}} \right\rangle / (ax + by - 1)$$

with the quotient norm; this is an analytic Banach ring (because x and y are topologically nilpotent), so it may be viewed as a Huber ring. If we view A as a filtered ring, the associated graded ring is $\mathbb{Z}[a, b, x, y]/(ax + by - 1)$ with a, b placed in degree -1 and x, y placed in degree $+1$. Since the graded ring is an integral domain and its only units are ± 1, it follows that the norm on A is multiplicative and every unit in A has norm 1. In particular, A is not Tate.

In order to explain the extent to which passage between Huber and Banach rings can be made functorial, we need to introduce the notion of a Banach module.

Remark 1.5.8. Let B be an analytic Banach ring and let M be a complete metrizable topological B-module. Then M may be equipped with the structure of a *Banach module* over B, i.e., a module complete with respect to a norm $|\bullet|_M$ satisfying

$$|bm| \leq |b|\,|m| \qquad (b \in B, m \in M). \tag{1.5.8.1}$$

Namely, if one chooses an open neighborhood M_0 of 0 in M, one can define a surjective morphism $N \to M$ of topological B-modules by taking N to be the completed direct sum of B^{M_0} for the supremum norm, then mapping the generator of N corresponding to $m \in M_0$ to $m \in M$. By Theorem 1.1.9, this morphism is a strict surjection, so the quotient norm from N defines the desired topology on M.

In case B is a nonarchimedean field equipped with a multiplicative norm, one can say more: for $b \in B$ nonzero, we also have

$$|m| \leq |bm|\,\left|b^{-1}\right| = |bm|\,|b|^{-1},$$

which upgrades (1.5.8.1) to an equality

$$|bm| = |b|\,|m| \qquad (b \in B, m \in M). \tag{1.5.8.2}$$

Remark 1.5.9. Let B be an analytic Banach ring, and let M_1, M_2 be Banach modules over B. Then a morphism $f : M_1 \to M_2$ of B-modules is continuous if and only if it is *bounded* if there exists $c > 0$ such that $|f(m)| \leq c\,|m|$ for all $m \in M$. Namely, it is obvious that bounded implies continuous, while the reverse implication follows from Theorem 1.1.9.

Remark 1.5.10. Let B be an analytic Banach ring, and let $B \to A$ be a (continuous) morphism of Huber rings. We may then take $M = A$ in Remark 1.5.8; this amounts to promoting A to a Banach ring using an ideal of definition extended from B. By Remark 1.5.9, the map $B \to A$ is bounded; this remains true if we replace the norm on A with its associated operator norm as per Remark 1.5.2, since the norm topology does not change. To summarize, the category of Banach rings over B is equivalent to the category of Huber rings over A.

We now continue to introduce basic structures associated to Banach rings, keeping an eye on the relationship with Huber rings.

Definition 1.5.11. For B a Banach ring, the *spectral seminorm* on B is the function $|\bullet|_{\mathrm{sp}} : B \to \mathbb{R}_{\geq 0}$ given by

$$|x|_{\mathrm{sp}} = \lim_{n \to \infty} |x^n|^{1/n} \qquad (x \in B).$$

(Using submultiplicativity, it is an elementary exercise in real analysis to show that the limit exists.) In general, the spectral seminorm is not a norm; for example, it maps all nilpotent elements to 0. The spectral seminorm need not be multiplicative, but it is *power-multiplicative*: for any $x \in B$ and any positive integer n, $|x^n|_{\mathrm{sp}} = |x|_{\mathrm{sp}}^n$.

Even if the spectral seminorm is a norm, it need not define the same topology as the original norm. This does however hold if B satisfies the equivalent conditions of Exercise 1.5.13 below; in this case, we say that B is *uniform*. If B is uniform, then it is reduced. See Remark 1.5.4 for the relationship with uniformity of Huber rings.

Exercise 1.5.12. Let B be a Banach ring. Then $x \in B$ is topologically nilpotent if and only if $|x|_{\mathrm{sp}} < 1$.

Exercise 1.5.13. For any integer $m > 1$, the following conditions on a Banach ring B are equivalent.

(a) There exists $c > 0$ such that $|x|_{\mathrm{sp}} \geq c\,|x|$ for all $x \in B$.
(b) There exists $c > 0$ such that $|x^m| \geq c\,|x|^m$ for all $x \in B$.

These conditions also imply the following equivalent conditions, and conversely if B is analytic.

(c) The spectral seminorm defines the same topology as the original norm (and in particular is a norm).
(d) The underlying Huber ring of B is uniform (in the sense of Definition 1.5.11).

Definition 1.5.14. For B a Banach ring, let $\mathcal{M}(B)$ denote the *Gelfand spectrum* of B, which as a set consists of the multiplicative seminorms on B which are bounded by the given norm. Under the evaluation topology (i.e., the subspace topology from the product topology on $\mathbb{R}^B$), $\mathcal{M}(B)$ is compact.

Remark 1.5.15. For (A, A^+) a Huber ring with A promoted to a Banach ring B as per Remark 1.5.4, there is a natural map $\mathcal{M}(B) \to \mathrm{Spa}(A, A^+)$ obtained by viewing a multiplicative seminorm as a valuation; however, this map is not continuous. If A is analytic, this map is a section of a continuous morphism $\mathrm{Spa}(A, A^+) \to \mathcal{M}(B)$ which takes a valuation v to the bounded multiplicative seminorm defining the topology on the residue field (whose underlying valuation is the maximal generization of v). This map identifies $\mathcal{M}(B)$ with the *maximal Hausdorff quotient* of $\mathrm{Spa}(A, A^+)$.

The following is [13, Theorem 1.2.1], with essentially the same proof.

Lemma 1.5.16. *For B a Banach ring, $B = 0$ if and only if $\mathcal{M}(B) = \emptyset$.*

Proof. The content is that if $B \neq 0$, then $\mathcal{M}(B) \neq \emptyset$. Note that for any maximal ideal $\mathfrak{m}$ of B, $\mathfrak{m}$ is closed (see Remark 1.1.1) and $\mathcal{M}(B/\mathfrak{m})$ may be identified with a subset of $\mathcal{M}(B)$; we may thus assume that B is a field. (This does not by itself imply that B is complete for a multiplicative norm; see Remark 1.2.17.)

By Zorn's lemma, we may construct a minimal bounded seminorm β on B; it will suffice to check that β is multiplicative. Note that β must already be power-multiplicative, or else we could replace it with its spectral seminorm and violate minimality.

We can now finish in (at least) two different ways.

- Here is the approach taken in [13, Theorem 1.2.2]. Suppose $x \in B$ is nonzero. For any $\rho < \beta(x)$, the map $B \to B\langle \frac{T}{\rho} \rangle/(T - x)$ must be zero: otherwise, we could restrict the quotient norm on the target to get a seminorm on B bounded by β and strictly smaller at x, contradicting minimality. Since B is nonzero, the map can only be zero if the target is the zero ring, or equivalently if $T - x = -x(1 - x^{-1}T)$ has an inverse in $B\langle \frac{T}{\rho} \rangle$, or equivalently if the unique inverse $\sum_{n=0}^{\infty} -x^{-n-1}T^n$ in $B[\![T]\!]$ converges in $B\langle \frac{T}{\rho} \rangle$. That is, we must have
$$\lim_{n \to \infty} \beta(x^{-n})\rho^n = 0 \qquad \text{for all } \rho \in (0, \beta(x)),$$
and in particular $\beta(x^{-n}) < \rho^{-n}$ for n sufficiently large. By power-multiplicativity, this implies that $\beta(x^{-1}) \leq \beta(x)^{-1}$. For all $y \in B$, we now have
$$\beta(xy) \leq \beta(x)\beta(y) \leq \beta(x^{-1})^{-1}\beta(y) \leq \beta(xy);$$
hence β is multiplicative, as needed.

- Another approach (suggested by Zonglin Jiang) is to use Exercise 1.5.17 below to show that for any nonzero $x \in B$, the formula
$$\beta'(y) = \lim_{n \to \infty} \frac{\beta(x^n y)}{\beta(x^n)}$$
defines a power-multiplicative seminorm β' on B. For all $y \in B$, we have $\beta'(y) \leq \beta(y)$ and $\beta(xy) = \beta(x)\beta(y)$; by minimality, we must have $\beta = \beta'$, proving multiplicativity.

Using either approach, the proof is complete. $\square$

Exercise 1.5.17. Prove [22, Proposition 1.3.2/2]: for any uniform Banach ring B equipped with its spectral norm and any nonzero $x \in B$, the limit
$$|y|_x := \lim_{n \to \infty} \frac{|yx^n|}{|x^n|}$$
exists and defines a power-multiplicative seminorm $|\bullet|_x$ on B.

We now recover [85, Proposition 3.6(i)].

Corollary 1.5.18. *For (A, A^+) a (not necessarily analytic) Huber pair with $A \neq 0$, we have $\mathrm{Spa}(A, A^+) \neq \emptyset$.*

Proof. Promote A to a Banach ring as per Remark 1.5.4, then apply Lemma 1.5.16. $\square$

Corollary 1.5.19. *Let B be a Banach ring. If the* uniform completion *of B (i.e., the separated completion of B with respect to the spectral seminorm) is zero, then so is B itself.*

Proof. The given condition implies that $\mathcal{M}(B) = \emptyset$, at which point Lemma 1.5.16 implies that $B = 0$. $\square$

Corollary 1.5.20. *For B a nonzero Banach ring, an ideal I of B is trivial if and only if for each $\beta \in \mathcal{M}(B)$, there exists $x \in I$ with $\beta(x) > 0$. In particular, an element x of B is invertible if and only if $\beta(x) \neq 0$ for all $\beta \in \mathcal{M}(B)$.*

Proof. By Remark 1.1.1, we may assume that I is closed. If I is trivial, then obviously $x = 1$ satisfies $\beta(x) > 0$ for all $\beta \in \mathcal{M}(B)$. Otherwise, B/I is a nonzero Banach ring (since I is now closed), $\mathcal{M}(B/I)$ is nonempty by Lemma 1.5.16, and any element of $\mathcal{M}(B/I)$ restricts to an element $\beta \in \mathcal{M}(B)$ with $\beta(x) = 0$ for all $x \in I$. $\square$

Corollary 1.5.21. *For (A, A^+) a (not necessarily analytic) Huber pair, an ideal I of A is trivial if and only if for each $v \in \mathrm{Spa}(A, A^+)$, there exists $x \in I$ with $v(x) > 0$. In particular, $x \in A$ is invertible if and only if $v(x) > 0$ for all $v \in \mathrm{Spa}(A, A^+)$.*

Proof. Promote A to a Banach ring as per Remark 1.5.4, then apply Corollary 1.5.20. $\square$

The following is an analogue of the maximum modulus principle in rigid analytic geometry [22, Proposition 6.2.1/4]. The proof is taken from [13, Theorem 1.3.1].

Lemma 1.5.22. *For B a Banach ring, the spectral seminorm of B equals the supremum over $\mathcal{M}(B)$.*

Proof. In one direction, it is obvious that any multiplicative seminorm bounded by the original norm is also bounded by the spectral seminorm. In the other direction, we must check that if $x \in B$, $\rho > 0$ satisfy $\beta(x) < \rho$ for all $\beta \in \mathcal{M}(B)$, then $|x|_{\mathrm{sp}} < \rho$. The input condition implies that $1-xT$ vanishes nowhere on the spectrum of $B\langle \frac{T}{\rho^{-1}} \rangle$, hence is invertible by Corollary 1.5.20. In the larger ring $B[\![T]\!]$, the inverse of $1-xT$ equals $1+xT+x^2T^2+\cdots$; the fact that this belongs to $B\langle \frac{T}{\rho^{-1}} \rangle$ implies that $|x|_{\mathrm{sp}} < \rho$ as in the proof of Lemma 1.5.16. $\square$

Corollary 1.5.23. *For B a Banach ring, an element $x \in B$ is topologically nilpotent if and only if $\beta(x) < 1$ for all $\beta \in \mathcal{M}(B)$.*

Proof. This is immediate from Exercise 1.5.12 and Lemma 1.5.22. $\square$

This immediately yields the following corollary of Lemma 1.5.22; see also [25, Lemma 5] for a purely topological proof.

Corollary 1.5.24. *Let (A, A^+) be a (not necessarily analytic) Huber pair. Then the kernel of $A \to H^0(X, \mathcal{O})$ contains only topologically nilpotent elements.*

Proof. Promote A to a Banach ring as per Remark 1.5.4. Then any $x \in A$ mapping to zero in $H^0(X, \mathcal{O})$ satisfies $\alpha(x) = 0$ for all $\alpha \in \mathcal{M}(A)$; by Corollary 1.5.23, x is topologically nilpotent. $\square$

Remark 1.5.25. In light of Remark 1.5.4, Lemma 1.5.22 implies that for any covering $\mathfrak{V}$ of X, the map $A \to \bigoplus_{V \in \mathfrak{V}} \mathcal{O}(V)$ is an isometry for the spectral seminorms on all terms. In particular, if A is uniform, then $A \to \mathcal{O}(X)$ is injective; by Remark 1.5.4, the same statement holds for Huber rings.

This can be used to prove the following key lemma (compare [25, Lemma 3]).

Lemma 1.5.26. *Suppose that A is uniform. Choose $x = \sum_{n=0}^{\infty} x_n T^n \in A\langle T \rangle$ such that the x_n generate the unit ideal. Then multiplication by x defines a strict inclusion $A\langle T \rangle \to A\langle T \rangle$. (The analogous statement for $A\langle T^{\pm} \rangle$ also holds, with an analogous proof.)*

Proof. Using Remark 1.5.4, we may reduce to considering the analogous problem where A is a uniform Banach ring. For $\alpha \in \mathcal{M}(A)$, write $\tilde{\alpha} \in \mathcal{M}(A\langle T \rangle)$ for the Gauss extension; note that the latter is the maximal seminorm on $A\langle T \rangle$ restricting to β on $\mathcal{M}(A)$. By Lemma 1.5.22, we may then compute the spectral seminorm on $A\langle T \rangle$ as the supremum of $\tilde{\alpha}$ as α runs over $\mathcal{M}(A)$.

Choose $n \geq 0$ such that $x_0, \dots, x_n$ generate the unit ideal in A; then the quantity

$$c := \inf_{\alpha \in \mathcal{M}(B)} \{\alpha(x_0), \dots, \alpha(x_n)\}$$

is positive. For all $y \in A\langle T \rangle$, we have

$$\sup_{\alpha \in \mathcal{M}(A)} \{\tilde{\alpha}(xy)\} = \sup_{\alpha \in \mathcal{M}(A)} \{\tilde{\alpha}(x)\tilde{\alpha}(y)\} \geq c \sup_{\alpha \in \mathcal{M}(A)} \{\tilde{\alpha}(y)\};$$

this proves the claim. $\square$

Exercise 1.5.27. Let $\{A_i\}_{i \in I}$ be a filtered direct system of uniform Banach rings, equipped with their spectral norms, and let A be the completed direct limit of the A_i. Prove that every finite projective module on A is the base extension of some finite projective module over some A_i. (See [108, Lemma 5.6.8].)

Remark 1.5.28. It is possible to construct a version of the theory of adic spaces in which Banach rings, rather than Huber rings, form the building blocks; this is the theory of *reified adic spaces* described in [103]. The main structural distinction is that valuations do not map into totally arbitrary value groups; rather, each value group must be normalized using a fixed inclusion of the positive real numbers. In the reified theory, many occurrences of Tate or analytic hypotheses can be relaxed, because the normalization of the value group can be used in the place of topologically nilpotent elements. However, the open mapping theorem, being a purely topological statement, provides a stumbling block.

1.6. **A strategy of proof: variations on Tate's reduction.** We next collect some general observations that will be used to complete the omitted proofs from earlier in the lecture. The reader is again reminded to keep in mind the analogy with affine schemes, as many of the ideas are similar.

To begin, we reduce the sheafy property, and the acyclicity of sheaves, to a statement about sufficiently fine coverings of and by basic open subsets.

Definition 1.6.1. By a *cofinal family of rational coverings*, we will mean a function C assigning to each rational subspace U of X a set of finite coverings of U by rational subspaces which is *cofinal*: every covering of U by open subspaces is refined by some covering in $C(U)$. For example, since U is quasicompact, one obtains a cofinal family of rational coverings by taking $C(U)$ to be all finite coverings by rational subspaces.

Definition 1.6.2. We use the following notation for Čech cohomology groups. For $\mathcal{F}$ a presheaf on X, $U \subseteq X$ open, $\mathfrak{V} = \{V_i\}_{i \in I}$ a covering of U by open subspaces, and j a nonnegative integer, let $\check{C}^j(U, \mathcal{F}; \mathfrak{V})$ be the product of $\mathcal{F}(V_{i_0} \cap \cdots \cap V_{i_j})$ over all distinct $i_0, \dots, i_j \in I$. Let $d^j : \check{C}^j(U, \mathcal{F}; \mathfrak{V}) \to \check{C}^{j+1}(U, \mathcal{F}; \mathfrak{V})$ be the map given by the formula

$$(s_{i_0,\dots,i_j})_{i_0,\dots,i_j \in I} \mapsto \left(\sum_{k=0}^{j+1} (-1)^k s_{i_0,\dots,\widehat{i_k},\dots,i_{j+1}} \right)_{i_0,\dots,i_{j+1} \in I};$$

with these differentials, $\check{C}^\bullet(U, \mathcal{F}; \mathfrak{V})$ form a complex whose cohomology groups we denote by $\check{H}^\bullet(U, \mathcal{F}; \mathfrak{V})$.

The following is the same standard argument used to establish the basic properties of the structure sheaf on affine schemes; compare [152, Tag 01EW].

Lemma 1.6.3. *Let C be a cofinal family of rational coverings. Let $\mathcal{F}$ be a presheaf on X with the property that for any open subset U, $\mathcal{F}(U)$ is the inverse limit of $\mathcal{F}(V)$ over all rational subspaces $V \subseteq U$.*

(a) *Suppose that for every rational subspace U of X and every covering $\mathfrak{V} \in C(U)$, the natural map*
$$\mathcal{F}(U) \to \check{H}^0(U, \mathcal{F}; \mathfrak{V})$$
is an isomorphism. Then $\mathcal{F}$ is a sheaf.

(b) *Suppose that $\mathcal{F}$ is a sheaf, and that for every rational subspace U of X and every covering $\mathfrak{V} \in C(U)$, we have $\check{H}^i(U, \mathcal{F}; \mathfrak{V}) = 0$ for all $i > 0$. Then $\mathcal{F}$ is acyclic.*

Proof. We start with (a). To show that $\mathcal{F}$ is a sheaf, we must check that $\mathcal{F}(U) \to \check{H}^0(U, \mathcal{F}; \mathfrak{V})$ is an isomorphism for every open subspace U of X and every open covering $\mathfrak{V}$ of U. We will check injectivity, then surjectivity; note that each of these assertions follows formally from the case where U is rational.

Suppose that U is rational and that $\mathfrak{V}$ is a covering of U. Let $\mathfrak{V}' \in C(U)$ be a refinement of $\mathfrak{V}$. The map $\mathcal{F}(U) \to \check{H}^0(U, \mathcal{F}; \mathfrak{V}')$ then factors as the map $\mathcal{F}(U) \to \check{H}^0(U, \mathcal{F}; \mathfrak{V})$ followed by the restriction map $\check{H}^0(U,\mathcal{F};\mathfrak{V}) \to \check{H}^0(U,\mathcal{F};\mathfrak{V}')$. Since $\mathcal{F}(U) \to \check{H}^0(U,\mathcal{F};\mathfrak{V}')$ is injective, the map $\mathcal{F}(U) \to \check{H}^0(U,\mathcal{F};\mathfrak{V})$ is injective for U rational, and hence for arbitrary U.

By the previous paragraph, the map $\check{H}^0(U,\mathcal{F};\mathfrak{V}) \to \check{H}^0(U,\mathcal{F};\mathfrak{V}')$ is also injective. Consequently, the surjectivity of $\check{H}^0(U,\mathcal{F};\mathfrak{V}) \to \check{H}^0(U,\mathcal{F};\mathfrak{V}')$ implies the surjectivity of $\mathcal{F}(U) \to \check{H}^0(U,\mathcal{F};\mathfrak{V})$ for U rational, and hence for arbitrary U. This completes the proof of (a).

To establish (b), we will show that $H^i(U,\mathcal{F}) = 0$ for all rational subspaces U and all $i > 0$; we do this by induction on i. Given that $\mathcal{F}$ is a sheaf and that $H^j(U,\mathcal{F}) = 0$ for all rational subspaces U and all $j < i$, a standard spectral sequence argument [22, Corollary 8.1.4/3] produces a canonical morphism $H^i(U,\mathcal{F}) \to \check{H}^i(U,\mathcal{F};\mathfrak{V})$ for any open covering $\mathfrak{V}$ of U, with the property that if $\mathfrak{V}'$ is a refinement of $\mathfrak{V}$ then the morphism $H^i(U,\mathcal{F}) \to \check{H}^i(U,\mathcal{F};\mathfrak{V}')$ factors as $H^i(U,\mathcal{F}) \to \check{H}^i(U,\mathcal{F};\mathfrak{V})$ followed by the natural morphism $\check{H}^i(U,\mathcal{F};\mathfrak{V}) \to \check{H}^i(U,\mathcal{F};\mathfrak{V}')$. With this in mind, we may imitate the proof of (a) to conclude. □

In order to maximally exploit this argument, we construct some special finite coverings of rational subspaces, so as to cut down the required number of explicit calculations of Čech cohomology.

Definition 1.6.4. For $f_1,\dots,f_n \in A$ which generate the unit ideal, the sets

$$X\left(\frac{f_1,\dots,f_n}{f_i}\right) \qquad (i = 1,\dots,n)$$

form a covering of X by rational subspaces; this covering is called the *standard rational covering* defined by the parameters $f_1,\dots,f_n$. A standard rational covering with $n = 2$ will be called a *standard binary rational covering*.

Although we have generically been assuming that A is analytic, the previous definition makes sense without this hypothesis. However, in order for the sets $X\left(\frac{f_1,\dots,f_n}{f_i}\right)$ to form a covering, the elements $f_1,\dots,f_n$ must still generate the unit ideal in A, not an arbitrary open ideal (because the condition that $v \in X$ belongs to $X\left(\frac{f_1,\dots,f_n}{f_i}\right)$ includes the requirement that $v(f_i) \neq 0$).

We record the following statement for use in a subsequent lecture.

Lemma 1.6.5. *The map* $X = \operatorname{Spa}(A,A^+) \to \operatorname{Spa}(A^+,A^+)$ *(which by Corollary 1.1.4 is injective) is an open immersion (i.e., a homeomorphism onto an open subset of the target).*

Proof. Let $x_1,\dots,x_n \in A^+$ be topologically nilpotent elements which generate the unit ideal in A. The image of X in $\operatorname{Spa}(A^+,A^+)$ can then be written as the union of the open subsets $\operatorname{Spa}(A^+,A^+)(\frac{x_1,\dots,x_n}{x_i})$ for $i = 1,\dots,n$; it is thus open. To complete the argument, we need only check that each of these open subsets is itself homeomorphic to the rational subspace $X(\frac{x_1,\dots,x_n}{x_i})$ of X; that is, we may reduce to the case where A is Tate.

Now suppose that $x \in A^+$ is a topologically nilpotent unit in A. In this case, we must check that for any $f_1,\dots,f_n,g \in A$ which generate the unit ideal, the rational subspace $X\left(\frac{f_1,\dots,f_n}{g}\right)$ of X is the pullback of a rational subspace of $\operatorname{Spa}(A^+,A^+)$. To see this, let m be an integer which is large enough that $x^m f_1,\dots,x^m f_n, x^m g \in A^+$; these parameters then generate an open ideal in A^+, and so define a rational subspace of $\operatorname{Spa}(A^+,A^+)$ of the desired form. □

Definition 1.6.6. It will be useful to single out some special types of standard binary rational coverings. The covering with parameters $f_1 = f$, $f_2 = 1$ will be called the *simple Laurent covering* defined by f. The covering with parameters $f_1 = f$, $f_2 = 1 - f$ will be called the *simple balanced covering* defined by f; note that the terms in this covering can be rewritten as $X(\frac{1}{f}), X(\frac{1}{1-f})$.

Remark 1.6.7. The concept of a standard rational covering of X is the closest analogue in this context to a covering of an affine scheme by distinguished open affine subschemes. That is because for R a ring and $f_1, \dots, f_n, g \in R$ generating the unit ideal, the ring obtained from R by adjoining $f_1/g, \dots, f_n/g$ is precisely $R[1/g]$; consequently, for $f_1, \dots, f_n \in R$ generating the unit ideal, the "rational covering" defined by $f_1, \dots, f_n$ is nothing but the covering of $\operatorname{Spec} R$ by $\operatorname{Spec} R[1/f_1], \dots, \operatorname{Spec} R[1/f_n]$.

The previous remark suggests the following lemma, due in this form to Huber [86, Lemma 2.6]; see also [22, Lemma 8.2.2/2] for the case where A is an affinoid algebra over a nonarchimedean field, or [107, Lemma 2.4.19(a)] for the case where A is Tate.

Lemma 1.6.8 (Huber)**.** *Every open covering of a rational subspace of X can be refined by some standard rational covering.*

Proof. Since every rational subspace of X is itself the spectrum of a Huber pair, it suffices to consider coverings of X itself. Since X is quasicompact, we may start with a finite covering $\mathfrak{V} = \{V_i\}_{i \in I}$ of X by rational subspaces. For $i \in I$, write

$$V_i = X\left(\frac{f_{i1}, \dots, f_{in_i}}{g_i}\right) \qquad (i \in I)$$

for some $f_{i1}, \dots, f_{in_i}, g_i$ which generate the unit ideal in A. Let S_0 be the set of products $\prod_{i \in I} s_i$ with $s_i \in \{f_{i1}, \dots, f_{in_i}, g_i\}$; then S_0 generates the unit ideal.

Let S be the subset of S_0 consisting of those products $\prod_{i \in I} s_i$ where $s_i = g_i$ for at least one $i \in I$. These also generate the unit ideal: to see this, by Corollary 1.5.21 it suffices to check that for each $v \in X$ there exists some $s \in S$ for which $v(s) \neq 0$. To see this, choose an index $i \in I$ for which $v \in V_i$, put $s_i = g_i$, and for each $j \neq i$ choose $s_j \in \{f_{j1}, \dots, f_{jn_j}, g_j\}$ to maximize $v(s_j)$. Since $f_{j1}, \dots, f_{jn_j}, g_j$ generate the unit ideal, we must have $v(s_j) \neq 0$; it follows that $v(s) \neq 0$.

We may thus form the standard covering by S. This refines the original covering: if $s \in S$ with $s_i = g_i$, then $X(\frac{S}{s}) \subseteq V_i$. $\square$

Remark 1.6.9. Using Lemma 1.6.8, we may see that the property of (A, A^+) being sheafy depends only on A, not on A^+: both the collection of standard rational coverings, and the assertions of the sheaf axiom for these coverings, depend only on A.

Remark 1.6.10. For A not necessarily analytic, a rational subspace of X is defined by parameters which generate an open ideal of A, rather than the trivial ideal. Recall that these two conditions coincide if and only if A is analytic (Lemma 1.1.3). For this reason, the proof of Lemma 1.6.8 does not extend to the case where A is not analytic.

To see just how different the nonanalytic case is, we consider the following example. The ring $\mathbf{A}_{\mathrm{inf}}$ to be introduced later exhibits similar behavior; see Remark 3.1.10.

Example 1.6.11. Let k be a field, equip $A := k[\![x, y]\!]$ with the (x, y)-adic topology, and put $A^+ := A$. The space X then contains a unique valuation v with $v(x) = v(y) = 0$. The only rational subspace of X containing v is X itself: for $f_1, \dots, f_n, g \in A$ generating an open ideal, we have $v \in X\left(\frac{f_1,\dots,f_n}{g}\right)$ if and only if $v(g) \neq 0$, which forces g to be a unit. Thus in this case the conclusion of Lemma 1.6.8 does turn out to be correct: any covering of X is refined by the trivial covering of X by itself, whereas any proper rational subspace of X is the spectrum of an analytic ring and so is subject to Lemma 1.6.8.

Continuing the analogy with affine schemes, we may further reduce the cofinal family consisting of the standard rational coverings by considering compositions of special coverings. The following argument is due to Gabber–Ramero (taken from [65]).

Lemma 1.6.12 (Gabber–Ramero)**.** *Every open covering of a rational subspace of X can be refined by some composition of standard binary rational coverings.*

Proof. Again, we need only consider coverings of X itself. By Lemma 1.6.8, there is no harm in starting with the standard rational covering defined by some $f_1, \dots, f_n \in A$ generating the unit ideal. We induct on the smallest value of m for which some m-element subset of $\{f_1, \dots, f_n\}$ generates the unit ideal; there is nothing to check unless $m \geq 3$. Without loss of generality, we may assume that $f_1, \dots, f_m$ generate the unit ideal, and choose $g_1, \dots, g_m \in A$ for which $f_1 g_1 + \cdots + f_m g_m = 1$. Now define

$$h = \sum_{i=1}^{\lfloor m/2 \rfloor} f_i g_i, \qquad h' = \sum_{i=\lfloor m/2 \rfloor + 1}^{m} f_i g_i$$

and form the standard binary rational covering generated by h, h'. On each of $X\langle \frac{h}{h'} \rangle$ and $X\langle \frac{h'}{h} \rangle$, the unit ideal is generated by a subset of $f_1, \dots, f_m$ of size at most $\lceil m/2 \rceil \leq m - 1$; we may thus apply the induction hypothesis to conclude. $\square$

The following refinement of Lemma 1.6.12 will be useful for checking flatness.

Lemma 1.6.13. *Every open covering of a rational subspace of X can be refined by some composition of coverings, each of which is either a simple Laurent covering or a simple balanced covering.*

Proof. By Lemma 1.6.12, it suffices to prove the claim for the standard binary rational covering of X defined by some $f, g \in A$ which generate the unit ideal. Choose $a, b \in A$ with $af + bg = 1$. We then may refine the original covering by taking the simple balanced covering defined by af, and forming the simple Laurent coverings of $X(\frac{1}{af}), X(\frac{1}{bg})$ defined by the respective parameters $g/f, f/g$. $\square$

Remark 1.6.14. Although we will not need this refinement, we note that in the Tate case, one can do even better than Lemma 1.6.13: it is only necessary to use simple Laurent coverings. This was shown for affinoid algebras in [22, Lemma 8.2.2/3, Lemma 8.2.2/4] and in general in [86, Theorem 2.5, (II.1)(iv)], [107, Lemma 2.4.19]. In light of Lemma 1.6.12, one may see this simply by checking that for any $f, g \in A$ generating the unit ideal, the simple balanced covering defined by f is refined by a composition of simple Laurent coverings. To this end, choose a topologically nilpotent unit $x \in X$; then for any sufficiently large integer n, we have

$$\max\{v(f/x^n), v(g/x^n)\} \geq 1 \qquad (v \in X).$$

From the ensuing equality (and its symmetric counterpart)

$$X\left(\frac{f}{g}\right) = X\left(\frac{f/x^n}{1}\right)\left(\frac{1}{g/x^n}\right) \cup X\left(\frac{1}{f/x^n}\right)\left(\frac{g/f}{1}\right),$$

we see that the original covering is refined by a suitable composition of simple Laurent coverings.

Let us now sketch how we will use the preceding lemmas to carry out the various proofs that we are still due to provide.

Remark 1.6.15. To show that some particular A is sheafy (as in Theorem 1.2.11 or Theorem 1.2.13), by Lemma 1.6.3 it suffices to check the isomorphism $\mathcal{O}(U) \cong \check{H}^0(U, \mathcal{O}; \mathfrak{V})$ for every rational subspace U and every finite covering $\mathfrak{V}$ by rational subspaces in some cofinal family. By Lemma 1.6.12, we may take this collection to be the compositions of standard binary rational coverings. Checking the isomorphism for such coverings immediately reduces to checking for a single standard binary rational covering; by Lemma 1.6.13 it is even sufficient to check for a simple Laurent covering and a simple balanced covering. That is, we must show that for every rational localization $(A, A^+) \to (B, B^+)$ and every pair $f, g \in B$ with $g \in \{1, 1-f\}$, the sequence

$$0 \to B \to B\left\langle \frac{f}{g} \right\rangle \oplus B\left\langle \frac{g}{f} \right\rangle \to B\left\langle \frac{f}{g}, \frac{g}{f} \right\rangle \to 0 \tag{1.6.15.1}$$

is exact at the left and middle.

In the same vein, if $\mathcal{O}$ is a sheaf, then the above sequence is known to be exact at the left and middle; to show that $\mathcal{O}$ is acyclic, it suffices to check exactness at the right. Similarly, to prove that $\tilde{M}$ is an acyclic sheaf for some given A-module M (as in Theorem 1.3.4 and Theorem 1.4.15), it suffices to check that tensoring (1.6.15.1) over B with $M \otimes_A B$ gives another exact sequence.

Remark 1.6.16. Given that A is sheafy and $\tilde{M}$ is acyclic for every finite projective A-module M, to show that every vector bundle on X arises from some finite projective A-module (as in Theorem 1.4.2), by Lemma 1.6.13 it suffices to consider a bundle which is specified by modules on each term of a composition of simple Laurent coverings and simple balanced coverings. It then suffices to check that for every rational localization $(A, A^+) \to (B, B^+)$ and every pair $f, g \in B$ with $g \in \{1, 1-f\}$, the functor

$$\mathbf{FPMod}_B \to \mathbf{FPMod}_{B\langle \frac{f}{g} \rangle} \times_{\mathbf{FPMod}_{B\langle \frac{f}{g}, \frac{g}{f} \rangle}} \mathbf{FPMod}_{B\langle \frac{g}{f} \rangle}$$

is an equivalence of categories. (Note that since we are only considering a covering by two open sets, there is no need to impose a cocycle condition on the objects on the right-hand side.) Similarly, given that $\tilde{M}$ is acyclic for every pseudocoherent A-module M, to show that every pseudocoherent sheaf on X arises from some complete pseudocoherent A-module (as in Theorem 1.4.17), we must check that for B, f, g as above, the functor

$$\mathbf{PCoh}_B \to \mathbf{PCoh}_{B\langle \frac{f}{g} \rangle} \times_{\mathbf{PCoh}_{B\langle \frac{f}{g}, \frac{g}{f} \rangle}} \mathbf{PCoh}_{B\langle \frac{g}{f} \rangle}$$

is an equivalence of categories. However, in order to even have such a functor, we must first establish the preservation of complete pseudocoherent modules under base extension along rational localizations; see Remark 1.6.17.

Remark 1.6.17. Given that A is sheafy and acyclic, to show that base extension along a rational localization preserves the category of complete pseudocoherent modules (as in Theorem 1.4.13), we will check this for each term in a simple Laurent covering or a simple balanced covering; note that the Laurent case implies the balanced case because

$$X\left(\frac{f}{1-f}\right) = X\left(\frac{1}{1-f}\right), \qquad X\left(\frac{1-f}{f}\right) = X\left(\frac{1}{f}\right).$$

Using Lemma 1.6.13, we then deduce that for any rational subspace $U = \mathrm{Spa}(B, B^+)$ of X, any pair of finite coverings $\mathfrak{V}, \mathfrak{V}'$ of U by rational subspaces, with $\mathfrak{V}'$ refining $\mathfrak{V}$, and any $M \in \mathbf{PCoh}_B$, the map

$$M \otimes_B \bigoplus_{V \in \mathfrak{V}} \mathcal{O}(V) \to M \otimes_B \bigoplus_{V \in \mathfrak{V}'} \mathcal{O}(V)$$

is a strict inclusion. (Namely, we may replace $\mathfrak{V}'$ by an even finer covering, and so by Lemma 1.6.13 we may get to one which consists of one covering of each $V \in \mathfrak{V}$ by a composition of simple Laurent and balanced coverings.)

Now let $U = \mathrm{Spa}(B, B^+)$ be a rational subspace of X, let $\mathfrak{V}$ be any finite covering by rational subspaces, and apply Lemma 1.6.13 to refine $\mathfrak{V}$ to a finite covering $\mathfrak{V}'$ by rational subspaces in which for each $V \in \mathfrak{V}'$, base extension along $B \to \mathcal{O}(V)$ is known to preserve complete pseudocoherent modules. If we now start with $M \in \mathbf{PCoh}_B$, choose a B-linear surjection $F \to M$ with F finite free, and let N be the kernel, we obtain a commutative diagram

$$\begin{tikzcd}
N \otimes_B \bigoplus_{V \in \mathfrak{V}} \mathcal{O}(V) \arrow[r] \arrow[d] & F \otimes_B \bigoplus_{V \in \mathfrak{V}} \mathcal{O}(V) \arrow[d] \\
N \otimes_B \bigoplus_{V \in \mathfrak{V}'} \mathcal{O}(V) \arrow[r] & F \otimes_B \bigoplus_{V \in \mathfrak{V}'} \mathcal{O}(V)
\end{tikzcd}$$

in which both vertical arrows are strict injective (by the previous paragraph) and the bottom horizontal arrow is strict injective (by the previous sentence). It follows that the top horizontal arrow is strict injective, proving that base extension along $B \to \bigoplus_{V \in \mathfrak{V}} \mathcal{O}(V)$ preserves complete pseudocoherent modules, as then does each individual map $B \to \mathcal{O}(V)$.

These arguments are summarized in [107, Proposition 2.4.20] as follows.

Lemma 1.6.18. *Let $\mathcal{P}_{\mathrm{an}}$ be the set of pairs $(U, \mathfrak{V})$ where U is a rational subspace and $\mathfrak{V}$ is a finite covering of U by rational subspaces. Suppose that $\mathcal{P} \subseteq \mathcal{P}_{\mathrm{an}}$ satisfies the following conditions.*

(i) Locality*: if $(U, \mathfrak{V})$ admits a refinement in $\mathcal{P}$, then $(U, \mathfrak{V}) \in \mathcal{P}$.*
(ii) Transitivity*: Any composition of coverings in $\mathcal{P}$ is in $\mathcal{P}$.*
(iii) *Every standard binary rational covering is in $\mathcal{P}$.*

Then $\mathcal{P} = \mathcal{P}_{\mathrm{an}}$.

Remark 1.6.19. While glueing theorems fit neatly into the framework of Lemma 1.6.18, writing the proofs of sheafiness and acyclicity theorems in this language still requires an argument in the style of Lemma 1.6.3. See [107, Proposition 2.4.21] for a presentation of this form.

1.7. **Proofs: sheafiness.** We now use the formalism we have set up to establish sheafiness when A is strongly noetherian (Theorem 1.2.11) or stably uniform (Theorem 1.2.13).

Hypothesis 1.7.1. Throughout §1.7, let $(A, A^+) \to (B, B^+)$ be a rational localization, and let $f, g \in B$ be elements which generate the unit ideal. We will use frequently and without comment the fact that $B\langle \frac{f}{g} \rangle$ is the quotient of $B\langle T \rangle$ by the closure of the ideal $(f - gT)$, and similarly.

Lemma 1.7.2. *With notation as in Hypothesis 1.7.1, suppose that for each of the pairs*

$$(R, x) = (B\langle T \rangle, f - gT), (B\langle T^{-1} \rangle, g - fT^{-1}), (B\langle T^{\pm} \rangle, f - gT),$$

the ideal xR is closed. Then the sequence (1.6.15.1) *is exact at the middle.*

Proof. By hypothesis, we have a commutative diagram

(1.7.2.1)
$$\begin{tikzcd}
0 \arrow[r] & 0 \arrow[r] \arrow[d] & B\langle T \rangle \oplus B\langle T^{-1} \rangle \arrow[r, "\bullet + T^{-1}\bullet"] \arrow[d, "\times(f-gT, g-fT^{-1})"] & B\langle T^{\pm} \rangle \arrow[r] \arrow[d, "\times(f-gT)"] & 0 \\
0 \arrow[r] & B \arrow[r] \arrow[d] & B\langle T \rangle \oplus B\langle T^{-1} \rangle \arrow[r, "\bullet - \bullet"] \arrow[d] & B\langle T^{\pm} \rangle \arrow[r] \arrow[d] & 0 \\
0 \arrow[r] & B \arrow[r] \arrow[d] & B\left\langle \frac{f}{g} \right\rangle \oplus B\left\langle \frac{g}{f} \right\rangle \arrow[r] \arrow[d] & B\left\langle \frac{f}{g}, \frac{g}{f} \right\rangle \arrow[r] \arrow[d] & 0 \\
 & 0 & 0 & 0 &
\end{tikzcd}$$

in which all three columns, and the first two rows, are exact. By diagram-chasing, or applying the snake lemma to the first two rows, we deduce the claim. □

At this point, it is easy to finish the proof of sheafiness in the stably uniform case.

Lemma 1.7.3. *With notation as in Hypothesis 1.7.1, if B is uniform, then* (1.6.15.1) *is exact at the left and middle.*

Proof. From Lemma 1.5.26, we see that on one hand, the criterion of Lemma 1.7.2 applies, so (1.6.15.1) is exact in the middle; on the other hand, the middle and right columns in (1.7.2.1) may be augmented to short exact sequences, so (1.6.15.1) is exact at the left. (To obtain exactness at the left, one can also invoke Remark 1.5.25.) □

Proof of Theorem 1.2.13. By Remark 1.6.15, this reduces immediately to Lemma 1.7.3. □

In the strongly noetherian case, exactness at the middle is no issue, but we must do a bit of work to check exactness at the left. Here we must essentially give Huber's proof that rational localization maps are flat in the strongly noetherian case [86, II.1], [87, Lemma 1.7.6]. We warn the reader that [64, Lemma 4.2.5] is somewhat sketchy on this point.

Lemma 1.7.4. *Suppose that A is strongly noetherian. With notation as in Hypothesis 1.7.1, the maps $B \to B\langle \frac{f}{g} \rangle, B \to B\langle \frac{g}{f} \rangle$ are flat.*

Proof. By symmetry, we need only check the first claim. By Lemma 1.1.19, the map $B[T] \to B\langle T\rangle$ is flat; we thus obtain a flat map

$$B = \frac{B[T]}{(f-gT)} \to \frac{B\langle T\rangle}{(f-gT)} = B\left\langle \frac{f}{g}\right\rangle,$$

using Corollary 1.1.15 to make the last identification. □

Lemma 1.7.5. *With notation as in Hypothesis 1.7.1, if the map $B \to B\langle \frac{f}{g}\rangle \oplus B\langle \frac{g}{f}\rangle$ is flat, then it is faithfully flat.*

Proof. By [152, Tag 00HQ], it suffices to show that the image of the map

$$\mathrm{Spec}\left(B\left\langle \frac{f}{g}\right\rangle \oplus B\left\langle \frac{g}{f}\right\rangle\right) \to \mathrm{Spec}(B)$$

includes every maximal ideal $\mathfrak{m}$ of B. To see this, note that since $\mathfrak{m}$ is necessarily closed (see Remark 1.1.1), $B/\mathfrak{m}$ is again a nonzero Huber ring and so has nonzero spectrum (by Lemma 1.5.16); we can thus choose $v \in \mathrm{Spa}(B, B^+)$ containing $\mathfrak{m}$ in its kernel. The point v must appear in the spectra of one of $B\langle \frac{f}{g}\rangle$ or $B\langle \frac{g}{f}\rangle$; taking the kernel of the resulting valuation gives a prime ideal of the corresponding ring which contracts to $\mathfrak{m}$. □

Lemma 1.7.6. *Suppose that A is strongly noetherian. With notation as in Hypothesis 1.7.1,* (1.6.15.1) *is exact at the left and middle.*

Proof. By Corollary 1.1.15, every ideal of $B\langle T\rangle$ is closed; by Lemma 1.7.2, this means that (1.6.15.1) is exact at the middle. To show exactness at the left, note that the map in question is flat by Lemma 1.7.4, and hence faithfully flat by Lemma 1.7.5. □

Proof of Theorem 1.2.11. By Remark 1.6.15, this reduces immediately to Lemma 1.7.6. □

1.8. **Proofs: acyclicity.** We next turn to acyclicity of sheaves associated to finite projective A-modules (Theorem 1.3.4). Throughout §1.8, continue to set notation as in Hypothesis 1.7.1.

Lemma 1.8.1. *With notation as in Hypothesis 1.7.1, the map $B\langle \frac{f}{g}\rangle \oplus B\langle \frac{g}{f}\rangle \to B\langle \frac{f}{g}, \frac{g}{f}\rangle$ is strict surjective, i.e., the sequence* (1.6.15.1) *is always strict exact at the right.*

Proof. In the commutative diagram

$$\begin{tikzcd} B\langle T\rangle \oplus B\langle T^{-1}\rangle \arrow[r] \arrow[d] & B\langle T^{\pm}\rangle \arrow[d] \\ B\left\langle \frac{f}{g}\right\rangle \oplus B\left\langle \frac{g}{f}\right\rangle \arrow[r] & B\left\langle \frac{f}{g}, \frac{g}{f}\right\rangle \end{tikzcd}$$

both vertical arrows and the top horizontal arrow are strict surjections; this yields the claim. □

Proof of Theorem 1.3.4. Since M is a direct summand of a finite free A-module, we may assume without loss of generality that $M = A$. By Remark 1.6.15, this reduces immediately to Lemma 1.8.1. □

To obtain acyclicity for $\tilde{M}$ for more general M, we must study the sequence (1.6.15.1) a bit more closely. A key step is the following converse of sorts to Lemma 1.7.2.

Lemma 1.8.2. *With notation as in Hypothesis 1.7.1, suppose that* (1.6.15.1) *is exact at the left and middle (e.g., because A is sheafy). Then multiplication by $f - gT$ defines injective maps $B\langle T\rangle \to B\langle T\rangle, B\langle T^{\pm}\rangle \to B\langle T^{\pm}\rangle$ with closed image.*

Proof. We treat the case of $B\langle T\rangle$, the case of $B\langle T^{\pm}\rangle$ being similar. We first argue that we may check the claim after replacing B with each of $B\langle \frac{f}{g}\rangle$, $B\langle \frac{g}{f}\rangle$, $B\langle \frac{f}{g}, \frac{g}{f}\rangle$. Given these cases, for $x \in B\langle T\rangle$, we may recover x from $x(f - gT)$ by doing so in each of $B\langle \frac{f}{g}, T\rangle$ and $B\langle \frac{g}{f}, T\rangle$ and noting that the answers must agree in $B\langle \frac{f}{g}, \frac{g}{f}, T\rangle$. Since (1.6.15.1) is strict exact (by our hypothesis plus Lemma 1.8.1 and Theorem 1.1.9), it follows that the map $x(f - gT) \mapsto x$ is continuous, as desired.

Now promote B to a Banach ring as per Remark 1.5.4, and let $\operatorname{Gr} B$ denote the associated graded ring (Definition 1.5.3), so that $\operatorname{Gr} B\langle T\rangle = (\operatorname{Gr} B)[\overline{T}]$ with $\overline{T}$ placed in degree 1. By our initial reduction, we may assume that either $g = 1$ and $|f| \leq 1$, or $f = 1$ and $|g| \leq 1$. (Namely, when passing from B to $B\langle \frac{f}{g}\rangle$ or $B\langle \frac{g}{f}\rangle$, we replace f, g with $\frac{f}{g}, 1$ or with $1, \frac{g}{f}$.) Consequently, we may assume that the image of $g - fT$ in $\operatorname{Gr} B\langle T\rangle$ has the form $\overline{x} = \overline{x}_0 + \overline{x}_1\overline{T}$ with $1 \in \{\overline{x}_0, \overline{x}_1\}$. It follows easily from this (by examining the effect of multiplication by $\overline{x}$ on constant and leading coefficients; see also Exercise 1.8.3) that $\overline{x}$ is not a zero-divisor in $\operatorname{Gr} B\langle T\rangle$. This in turn implies that multiplication by $g - fT$ defines an isometric (hence strict) inclusion of $B\langle T\rangle$ into itself, thus proving the claim. $\square$

We mention a purely algebraic fact related to the proof of Lemma 1.8.2.

Exercise 1.8.3. Let R be a ring. Let $f \in R[T_1, \dots, T_n]$ be a polynomial whose coefficients generate the unit ideal in R. Prove that f is not a zero-divisor in $R[T_1, \dots, T_n]$.

Remark 1.8.4. Lemma 1.8.2 provides a key special case of Theorem 1.2.7: if either of the ideals $(T - f)$ or $(1 - fT)$ in $A\langle T\rangle$ is not closed for some $f \in A$, then A is not sheafy. This is precisely the mechanism of Mihara's example [126, Proposition 3.14]; we have not checked whether the same criterion applies directly to the example of Buzzard–Verberkmoes.

Remark 1.8.5. At this point, it is tempting to try to emulate the proof of [64, Lemma 4.2.5] to show that the sequence (1.6.15.1), if it is exact, also splits in the category of topological B-modules. However, this is not true in general; the proof of [64, Lemma 4.2.5] is correspondingly incorrect, although the result stated there does turn out to be correct for other reasons. A related phenomenon is that for R a ring and $f \in R$, in general the exact sequence

$$0 \to R \to R\left[\frac{1}{f}\right] \oplus R\left[\frac{1}{1-f}\right] \to R\left[\frac{1}{f}, \frac{1}{1-f}\right] \to 0$$

does not necessarily split in the category of A-modules.

1.9. **Proofs: vector bundles and pseudocoherent sheaves.** We finally establish the local nature of sheafiness (Theorem 1.2.22), the base change property for pseudocoherent modules (Theorem 1.4.13), the acyclicity of sheaves associated to pseudocoherent modules (Theorem 1.4.15), the glueing theorems for vector bundles (Theorem 1.4.2) and pseudocoherent sheaves (Theorem 1.4.17), and corollaries (Theorem 1.4.19 and Theorem 1.4.20). Throughout §1.9, continue to set notation as in Hypothesis 1.7.1; in order to address Theorem 1.2.22, we refrain from globally assuming that A is sheafy.

In order to treat the two glueing theorems in parallel, we start with the base change and acyclicity arguments.

Remark 1.9.1. We will use in a couple of places the fact that the analogue of the sequence (1.6.15.1) with B, f, g replaced by $B\langle \frac{f}{g} \rangle, \frac{f}{g}, 1$ is the sequence

$$0 \to B\left\langle \frac{f}{g} \right\rangle \to B\left\langle \frac{f}{g} \right\rangle \oplus B\left\langle \frac{f}{g}, \frac{g}{f} \right\rangle \to B\left\langle \frac{f}{g}, \frac{g}{f} \right\rangle \to 0,$$

which is trivially exact.

Lemma 1.9.2. *With notation as in Hypothesis 1.7.1, suppose that $g \in \{1, 1-f\}$. Then for any $M \in \mathbf{PCoh}_B$, we have $\operatorname{Tor}_1^B(M, B\langle \frac{g}{f} \rangle) = 0$.*

Proof. As in Remark 1.6.17, we may use the equality $B\left\langle \frac{1-f}{f} \right\rangle = B\left\langle \frac{1}{f} \right\rangle$ to reduce to the case $g = 1$. By Lemma 1.1.18, for $M \in \mathbf{PCoh}_B$, we may identify $M \otimes_B B\langle T \rangle$ with $M\langle T \rangle$; in particular, we have an exact base extension functor $\mathbf{PCoh}_B \to \mathbf{PCoh}_{B\langle T \rangle}$. Now let $M[\![T]\!]$ be the set of formal sums $\sum_{n=0}^{\infty} x_n T^n$ with $x_n \in M$ with no convergence condition on the sequence $\{x_n\}$; on $M[\![T]\!]$, multiplication by $1 - fT$ has an inverse given by multiplication by $1 + fT + f^2T^2 + \cdots$. It follows that multiplication by $1 - fT$ on $M\langle T \rangle$ is injective, so $\operatorname{Tor}_1^B(M, B\langle \frac{1}{f} \rangle) = 0$ as desired. □

Lemma 1.9.3. *With notation as in Hypothesis 1.7.1, suppose that $g \in \{1, 1-f\}$ and* (1.6.15.1) *is exact (e.g., because A is sheafy). Then we have exact base extension functors*

$$\mathbf{PCoh}_B \to \mathbf{PCoh}_{B\langle \frac{f}{g} \rangle}, \qquad \mathbf{PCoh}_B \to \mathbf{PCoh}_{B\langle \frac{g}{f} \rangle}, \qquad \mathbf{PCoh}_B \to \mathbf{PCoh}_{B\langle \frac{f}{g}, \frac{g}{f} \rangle}.$$

Proof. Given $M \in \mathbf{PCoh}_B$, choose a short exact sequence $0 \to N \to F \to M \to 0$ of B-modules with F finite free; we then have $N \in \mathbf{PCoh}_B$ by Remark 1.4.10. In the diagram

$$\begin{tikzcd} N\langle T \rangle \arrow[r] \arrow[d, "\times g - fT"] & F\langle T \rangle \arrow[d, "\times g - fT"] \\ N\langle T \rangle \arrow[r] & F\langle T \rangle \end{tikzcd}$$

we may see that the left vertical arrow is a strict inclusion by tracing around the diagram: both horizontal arrows are strict inclusions by Lemma 1.1.18, while the right vertical arrow is a strict inclusion by Theorem 1.1.9 plus Lemma 1.8.2. It follows that $M\langle T \rangle / (g - fT)M\langle T \rangle \cong M \otimes_B B\langle \frac{g}{f} \rangle$ is complete for the natural topology; combining this with Lemma 1.9.2, we obtain the exact base extension functor $\mathbf{PCoh}_B \to \mathbf{PCoh}_{B\langle \frac{g}{f} \rangle}$.

Using Remark 1.9.1, we may repeat the argument to obtain the exact base extension functor $\mathbf{PCoh}_{B\langle \frac{g}{f} \rangle} \to \mathbf{PCoh}_{B\langle \frac{f}{g}, \frac{g}{f} \rangle}$ and the equality $\operatorname{Tor}_1^B(M, B\langle \frac{f}{g}, \frac{g}{f} \rangle) = 0$. Using the latter, we may tensor (1.6.15.1) with M to obtain an exact sequence

$$0 \to M \to M \otimes_B \left(B\left\langle \frac{f}{g} \right\rangle \oplus B\left\langle \frac{g}{f} \right\rangle \right) \to M \otimes B\left\langle \frac{f}{g}, \frac{g}{f} \right\rangle \to 0$$

from which we see (using Exercise 1.4.11) that $M \otimes_B B\langle \frac{f}{g} \rangle$ is complete for the natural topology. We thus obtain the exact base extension functor $\mathbf{PCoh}_B \to \mathbf{PCoh}_{B\langle \frac{f}{g} \rangle}$. □

Proof of Theorem 1.4.13. By Remark 1.6.17, this reduces immediately to Lemma 1.9.3. □

Lemma 1.9.4. *With notation as in Hypothesis 1.7.1, suppose that $g \in \{1, 1-f\}$ and* (1.6.15.1) *is exact (e.g., because A is sheafy). Then for any $M \in \mathbf{PCoh}_B$, tensoring* (1.6.15.1) *over B with M yields another exact sequence.*

Proof. Let $0 \to N \to F \to M \to 0$ be an exact sequence of B-modules with F finite free; we then have $N \in \mathbf{PCoh}_A$ by Remark 1.4.10. By Lemma 1.9.3, tensoring this sequence over B with $B\langle \frac{f}{g}, \frac{g}{f} \rangle$ yields another exact sequence; it follows that $\mathrm{Tor}_1^B(M, B\langle \frac{f}{g}, \frac{g}{f} \rangle) = 0$. This proves the claim. $\square$

Proof of Theorem 1.4.15. By Remark 1.6.15, this reduces immediately to Lemma 1.9.4. $\square$

With Theorems 1.4.13 and 1.4.15 in hand, we now begin to work on glueing. The following lemma is essentially [107, Lemma 2.7.2]; compare [64, Lemma 4.5.3].

Lemma 1.9.5. *With notation as in Hypothesis 1.7.1, there exists a neighborhood U of 0 in $B\langle \frac{f}{g}, \frac{g}{f} \rangle$ such that for every positive integer n, every matrix $V \in \mathrm{GL}_n(B\langle \frac{f}{g}, \frac{g}{f} \rangle)$ for which $V - 1$ has entries in U can be factored as $V_1 \cdot V_2$ with $V_1 \in \mathrm{GL}_n(B\langle \frac{f}{g} \rangle)$, $V_2 \in \mathrm{GL}_n(B\langle \frac{g}{f} \rangle)$.*

Proof. This is a direct consequence of Lemma 1.8.1 via a contraction mapping argument. $\square$

The following lemma combines [107, Lemma 1.3.8, Lemma 2.7.4], together with some minor modifications to work around the fact that we are not limiting ourselves to simple Laurent coverings. (The relevant special feature of a Laurent covering is that the map $B\langle \frac{1}{f} \rangle \to B\langle f, \frac{1}{f} \rangle$ has dense image.)

Lemma 1.9.6. *With notation as in Hypothesis 1.7.1, let M_1, M_2, M_{12} be finitely generated modules over $B\langle \frac{f}{g} \rangle$, $B\langle \frac{g}{f} \rangle$, $B\langle \frac{f}{g}, \frac{g}{f} \rangle$, respectively, and let $\psi_1 : M_1 \otimes_{B\langle \frac{f}{g} \rangle} B\langle \frac{f}{g}, \frac{g}{f} \rangle \to M_{12}$, $\psi_2 : M_2 \otimes_{B\langle \frac{g}{f} \rangle} B\langle \frac{f}{g}, \frac{g}{f} \rangle \to M_{12}$ be isomorphisms.*

(a) *The map $\psi : M_1 \oplus M_2 \to M_{12}$ taking $(\mathbf{v}, \mathbf{w})$ to $\psi_1(\mathbf{v}) - \psi_2(\mathbf{w})$ is strict surjective.*

(b) *For $M = \ker(\psi)$, the induced maps*

$$M \otimes_B B\left\langle \frac{f}{g} \right\rangle \to M_1, \qquad M \otimes_B B\left\langle \frac{g}{f} \right\rangle \to M_2$$

are strict surjective.

Proof. Let $\mathbf{v}_1, \dots, \mathbf{v}_n$ and $\mathbf{w}_1, \dots, \mathbf{w}_n$ be generating sets of M_1 and M_2, respectively, of the same cardinality. We may then choose $n \times n$ matrices V and W over $B\langle \frac{f}{g}, \frac{g}{f} \rangle$ such that $\psi_2(\mathbf{w}_j) = \sum_i V_{ij} \psi_1(\mathbf{v}_i)$ and $\psi_1(\mathbf{v}_j) = \sum_i W_{ij} \psi_2(\mathbf{w}_i)$.

Choose U as in Lemma 1.9.5. Since $B\langle \frac{f}{g} \rangle [f^{-1}]$ is dense in $B\langle \frac{f}{g}, \frac{g}{f} \rangle$, we can choose a nonnegative integer m and an $n \times n$ matrix W' over $B\langle \frac{g}{f} \rangle$ so that $V(f^{-m} W' - W)$ has entries in U. We may thus write $1 + V(f^{-m} W' - W) = X_1 X_2^{-1}$ with $X_1 \in \mathrm{GL}_n(B\langle \frac{f}{g} \rangle)$, $X_2 \in \mathrm{GL}_n(B\langle \frac{g}{f} \rangle)$.

We now define elements $\mathbf{x}_j \in M_1 \oplus M_2$ by the formula

$$\mathbf{x}_j = (\mathbf{x}_{j,1}, \mathbf{x}_{j,2}) = \left(\sum_i f^m (X_1)_{ij} \mathbf{v}_i, \sum_i (W' X_2)_{ij} \mathbf{w}_i \right) \qquad (j = 1, \dots, n).$$

Then

$$\psi_1(\mathbf{x}_{j,1}) - \psi_2(\mathbf{x}_{j,2}) = \sum_i (f^m X_1 - V W' X_2)_{ij} \psi_1(\mathbf{v}_i) = \sum_i f^m ((1 - VW) C_2)_{ij} \psi_1(\mathbf{v}_i) = 0,$$

so $\mathbf{x}_j \in M$. Since $C_1 \in \mathrm{GL}_n(B\langle \frac{f}{g}, \frac{g}{f} \rangle)$, we deduce that the map $M \otimes_B B\langle \frac{f}{g} \rangle \to M_1$ induces a strict surjection onto $f^m M_1$.

The induced map $M \otimes_B B\langle \frac{f}{g}, \frac{g}{f} \rangle \to M_{12}$ is strict surjective (because f is invertible in $B\langle \frac{f}{g}, \frac{g}{f} \rangle$), so using Lemma 1.8.1 we obtain a strict surjection $M \otimes_B (B\langle \frac{f}{g} \rangle \oplus B\langle \frac{g}{f} \rangle) \to M_{12}$. Since this map factors through ψ, we obtain (a).

For each $\mathbf{v} \in M_2$, $\psi_2(\mathbf{v})$ lifts to $M \otimes_B (B\langle \frac{f}{g} \rangle \oplus B\langle \frac{g}{f} \rangle)$ as above, so we can find $\mathbf{w}_1 \in M_1, \mathbf{w}_2 \in M_2$ in the images of the base extension maps from M with $\psi_1(\mathbf{w}_1) - \psi_2(\mathbf{w}_2) = \psi_2(\mathbf{v})$. Then $(\mathbf{w}_1, \mathbf{v} + \mathbf{w}_2) \in M$, so both $\mathbf{w}_2$ and $\mathbf{v} + \mathbf{w}_2$ are elements of M_2 in the image of the base extension map. This proves that $M \otimes_B B\langle \frac{g}{f} \rangle \to M_2$ is strict surjective; we may reverse the roles of f and g to deduce (b). $\square$

Lemma 1.9.7. *With notation as in Hypothesis 1.7.1, the image of the map* $\mathrm{Spec}(B\langle \frac{f}{g} \rangle \oplus B\langle \frac{g}{f} \rangle) \to \mathrm{Spec}(B)$ *includes all maximal ideals.*

Proof. By Remark 1.1.1 and Corollary 1.5.18, every maximal ideal $\mathfrak{m}$ of B occurs as the kernel of some valuation v. That valuation extends to one of $B\langle \frac{f}{g} \rangle$ or $B\langle \frac{g}{f} \rangle$, and the kernel of that extension is a prime ideal contracting to $\mathfrak{m}$. $\square$

Lemma 1.9.8. *With notation as in Hypothesis 1.7.1, suppose that* (1.6.15.1) *is exact (e.g., because A is sheafy).*

(a) *There is an exact functor*

$$\mathbf{FPMod}_{B\langle \frac{f}{g} \rangle} \times_{\mathbf{FPMod}_{B\langle \frac{f}{g}, \frac{g}{f} \rangle}} \mathbf{FPMod}_{B\langle \frac{g}{f} \rangle} \to \mathbf{FPMod}_B$$

given by taking equalizers. Moreover, the composition of this functor with the base extension functor in the opposite direction is naturally isomorphic to the identity.

(b) *There is an exact, fully faithful functor*

$$\mathbf{PCoh}_{B\langle \frac{f}{g} \rangle} \times_{\mathbf{PCoh}_{B\langle \frac{f}{g}, \frac{g}{f} \rangle}} \mathbf{PCoh}_{B\langle \frac{g}{f} \rangle} \to \mathbf{PCoh}_B$$

given by taking equalizers. Moreover, the composition of this functor with the base extension functor in the opposite direction is well-defined (that is, for $M \in \mathbf{PCoh}_B$ in the essential image, we have $M \otimes_B B\langle * \rangle \in \mathbf{PCoh}_{B\langle * \rangle}$) and naturally isomorphic to the identity.

Proof. Set notation as in Lemma 1.9.6. To prove (a), we must check that if M_1, M_2, M_{12} are finite projective over their respective base rings, then M is finite projective over B and the maps

$$M \otimes_B B\left\langle \frac{f}{g} \right\rangle \to M_1, \qquad M \otimes_B B\left\langle \frac{g}{f} \right\rangle \to M_2 \tag{1.9.8.1}$$

are isomorphisms. Similarly to prove (b), we must check that if M_1, M_2, M_{12} are pseudocoherent over their respective base rings, then M is pseudocoherent over B and the maps in (1.9.8.1) are again isomorphisms.

Let us treat both cases in parallel for the moment. By Lemma 1.9.6, we can choose a finite free B-module F and a (not necessarily surjective) B-linear map $F \to M$ such that for F_1, F_2, F_{12} the respective base extensions of F, the induced maps

$$F_1 \to M_1, \qquad F_2 \to M_2, \qquad F_{12} \to M_{12}$$

are surjective. Let N_1, N_2, N_{12} be the kernels of these maps and put $N = \ker(N_1 \oplus N_2 \to N_{12})$; by applying the snake lemma to the second and third columns in the diagram

$$\begin{tikzcd}
 & 0 \arrow[d] & 0 \arrow[d] & 0 \arrow[d] & \\
0 \arrow[r] & N \arrow[r] \arrow[d] & N_1 \oplus N_2 \arrow[r] \arrow[d] & N_{12} \arrow[r, dashed] \arrow[d] & 0 \\
0 \arrow[r] & F \arrow[r] \arrow[d] & F_1 \oplus F_2 \arrow[r] \arrow[d] & F_{12} \arrow[r] \arrow[d] & 0 \\
0 \arrow[r] & M \arrow[r] \arrow[d, dashed] & M_1 \oplus M_2 \arrow[r] \arrow[d] & M_{12} \arrow[r] \arrow[d] & 0 \\
 & 0 & 0 & 0 &
\end{tikzcd}$$

we obtain the first column and its exactness (minus the dashed arrows). In particular, $N = \ker(F \to M)$.

It is obvious in case (a), and a consequence of Remark 1.9.1 and Lemma 1.9.4 in case (b), that

$$N_1 \otimes_{B\langle \frac{f}{g} \rangle} B\left\langle \frac{f}{g}, \frac{g}{f} \right\rangle \cong N_{12}, \qquad N_2 \otimes_{B\langle \frac{g}{f} \rangle} B\left\langle \frac{f}{g}, \frac{g}{f} \right\rangle \cong N_{12}.$$

Consequently, in both cases the modules N_1, N_2, N_{12} again form an object of the fiber product category (using in case (b) the "two out of three" property of pseudocoherent modules, as in Remark 1.4.10); hence any general statement we can make about M, M_1, M_2, M_{12} also applies to N, N_1, N_2, N_{12}. This means that we may apply Lemma 1.9.6 to see that the dashed arrows in the previous diagram are surjective. Also, in the diagram

$$\begin{tikzcd}
 & N \otimes_B B\left\langle \frac{f}{g} \right\rangle \arrow[r] \arrow[d] & F \otimes_B B\left\langle \frac{f}{g} \right\rangle \arrow[r] \arrow[d] & M \otimes_B B\left\langle \frac{f}{g} \right\rangle \arrow[r] \arrow[d] & 0 \\
0 \arrow[r] & N_1 \arrow[r] & F_1 \arrow[r] & M_1 \arrow[r] & 0
\end{tikzcd}$$

with exact rows, we know from Lemma 1.9.6 that *both* outside vertical arrows are surjective. Since the middle arrow is an isomorphism, we may apply the five lemma to obtain injectivity of the right vertical arrow; this (and a similar argument with M_1 replaced with M_2) yields the fact that the maps in (1.9.8.1) are isomorphisms.

In case (b), it remains to check that $M \in \mathbf{PCoh}_B$. Since M is the kernel of a strict surjective morphism, it is complete for the subspace topology and hence for the natural topology. To prove that M is pseudocoherent, it will suffice to prove that for every positive integer m, M admits a projective resolution in which the last m terms are finite projective modules. This holds for $m = 1$ by Lemma 1.9.6 as above; however, given the claim for any given m, it applies not only to M but also to N, which formally implies the claim about M for $m + 1$. We may thus conclude by induction on m.

In case (a), it remains to check that $M \in \mathbf{FPMod}_B$. To establish this, note that M is finitely presented (by the previous paragraph) and $M_{\mathfrak{m}}$ is a finite free $B_{\mathfrak{m}}$-module for every maximal ideal $\mathfrak{m}$ of B (by Lemma 1.9.7). By [152, Tag 00NX], M is a finite projective B-module. $\square$

Proof of Theorem 1.4.2. By Remark 1.6.16, this reduces immediately to the statement that with notation as in Hypothesis 1.7.1,

$$\mathbf{FPMod}_B \to \mathbf{FPMod}_{B\langle \frac{f}{g}\rangle} \times_{\mathbf{FPMod}_{B\langle \frac{f}{g}, \frac{g}{f}\rangle}} \mathbf{FPMod}_{B\langle \frac{g}{f}\rangle}$$

is an exact equivalence of categories. This functor is fully faithful by Theorem 1.3.4, exact trivially, and essentially surjective by Lemma 1.9.8(a). $\square$

Proof of Theorem 1.4.17. By Remark 1.6.16, this reduces immediately to the statement that with notation as in Hypothesis 1.7.1 with $g \in \{1, 1-f\}$, there is an exact equivalence of categories

$$\mathbf{PCoh}_B \to \mathbf{PCoh}_{B\langle \frac{f}{g}\rangle} \times_{\mathbf{PCoh}_{B\langle \frac{f}{g}, \frac{g}{f}\rangle}} \mathbf{PCoh}_{B\langle \frac{g}{f}\rangle} \tag{1.9.8.2}$$

To begin with, by Lemma 1.9.3 we obtain exact base extension functors from $\mathbf{PCoh}_B$ to each of $\mathbf{PCoh}_{B\langle \frac{f}{g}\rangle}$, $\mathbf{PCoh}_{B\langle \frac{g}{f}\rangle}$, $\mathbf{PCoh}_{B\langle \frac{f}{g}, \frac{g}{f}\rangle}$. This yields an exact functor as in (1.9.8.2); this functor is fully faithful by Lemma 1.9.4 and essentially surjective by Lemma 1.9.8(b). $\square$

Remark 1.9.9. One source of inspiration for the preceding arguments is the Beauville–Laszlo theorem [11], [152, Tag 0BNI], [14], which asserts (among other things) that if R is a ring, $f \in R$ is not a zero-divisor, and $\widehat{R}$ is the f-adic completion of R, then the functor

$$\mathbf{FPMod}_R \to \mathbf{FPMod}_{R_f} \times_{\mathbf{FPMod}_{\widehat{R}_f}} \mathbf{FPMod}_{\widehat{R}}$$

is an equivalence of categories. See [107, Remark 2.7.9] for further explanation of how this result can be derived in the style of the arguments given above.

Another similarity that should be noted is that the Beauville–Laszlo theorem was originally introduced in order to construct and study affine Grassmannians associated to algebraic groups in the context of geometric Langlands. The glueing results discussed here are themselves relevant for the construction and study of certain mixed-characteristic analogues of affine Grassmannians, to be introduced in a later lecture (§4.6).

Proof of Theorem 1.4.19. Choose parameters $f_1, \dots, f_m, g$ defining the rational localization $(A, A^+) \to (B, B^+)$. By Theorem 1.4.17, we may prove the claim locally on X; using the standard rational covering defined by $f_1, \dots, f_m, g$, we reduce to the situation where one of $f_1, \dots, f_m, g$ is itself a unit. In this case $(A, A^+) \to (B, B^+)$ factors as a composition of rational localizations, each of which occurs in a simple Laurent covering:

- if g is a unit, then the rational subspace is defined by the conditions $v(f_1/g) \leq 1, \dots, v(f_n/g) \leq 1$;
- if f_1 is a unit, then after imposing the condition $v(g/f_1) \geq 1$, g becomes a unit and we may continue as in the previous case.

We are thus reduced to checking the claim in case B is the quotient of $A\langle U\rangle$ by the closure of the ideal generated by either $f - U$ or $1 - fU$ for some $f \in A$. In these cases, we have $B \in \mathbf{PCoh}_{A\langle U\rangle}$ by Lemma 1.8.2: in fact, B is not only pseudocoherent but of projective dimension ≤ 1 as an $A\langle U\rangle$-module. For any surjective homomorphism $g : A\langle T_1, \dots, T_n\rangle \to B$, choose lifts $y_1, \dots, y_n$ of $g(T_1), \dots, g(T_n)$ to $A\langle U\rangle$. Via the morphism $A\langle T_1, \dots, T_n, U\rangle \to A\langle U\rangle$ taking T_i to y_i, we have $A\langle U\rangle \in \mathbf{PCoh}_{A\langle T_1,\dots,T_n,U\rangle}$ and hence $B \in \mathbf{PCoh}_{A\langle T_1,\dots,T_n,U\rangle}$. Now choose a lift z to $A\langle T_1, \dots, T_n \rangle$ of the image of U in B; we may then view $A\langle T_1, \dots, T_n\rangle$ as a quotient of $A\langle T_1, \dots, T_n, U\rangle$ via the map $U \mapsto z$, and then conclude that $B \in \mathbf{PCoh}_{A\langle T_1,\dots,T_n\rangle}$. $\square$

Proof of Theorem 1.2.22. There is no harm in refining the covering $\mathfrak{V}$; by Lemma 1.6.12, we may reduce to the case where $\mathfrak{V}$ is the simple binary rational covering generated by some $f, g \in A$. By hypothesis, the sequence

$$0 \to \tilde{A} \to A\left\langle \frac{f}{g} \right\rangle \oplus A\left\langle \frac{g}{f}\right\rangle \to A\left\langle \frac{f}{g}, \frac{g}{f}\right\rangle \to 0$$

is strict exact, as then is

$$0 \to \tilde{A}\langle T\rangle \to A\left\langle \frac{f}{g}, T \right\rangle \oplus A\left\langle \frac{g}{f}, T\right\rangle \to A\left\langle \frac{f}{g}, \frac{g}{f}, T\right\rangle \to 0.$$

Using Lemma 1.8.2, we obtain a commutative diagram

$$\begin{tikzcd}
& 0 \arrow[d] & 0 \arrow[d] & 0 \arrow[d] & \\
0 \arrow[r] & \tilde{A}\langle T\rangle \arrow[r] \arrow[d] & A\left\langle \frac{f}{g}, T \right\rangle \oplus A\left\langle \frac{g}{f}, T\right\rangle \arrow[r] \arrow[d, "\times g - fT"] & A\left\langle \frac{f}{g}, \frac{g}{f}, T\right\rangle \arrow[r] \arrow[d, "\times g - fT"] & 0 \\
0 \arrow[r] & \tilde{A}\langle T\rangle \arrow[r] \arrow[d] & A\left\langle \frac{f}{g}, T \right\rangle \oplus A\left\langle \frac{g}{f}, T\right\rangle \arrow[r] \arrow[d] & A\left\langle \frac{f}{g}, \frac{g}{f}, T\right\rangle \arrow[r] \arrow[d] & 0 \\
0 \arrow[r] & \tilde{A}\left\langle \frac{f}{g}\right\rangle \arrow[r] \arrow[d] & A\left\langle \frac{f}{g} \right\rangle \oplus A\left\langle \frac{f}{g}, \frac{g}{f}\right\rangle \arrow[r] \arrow[d] & A\left\langle \frac{f}{g}, \frac{g}{f}\right\rangle \arrow[r] \arrow[d] & 0 \\
& 0 & 0 & 0 &
\end{tikzcd}$$

in which the first and second rows are exact, the second and third columns are exact, and the left column is exact at the top and bottom (but not *a priori* in the middle). By diagram chasing, the third row is exact. From this (and the analogous argument with f, g interchanged), we obtain natural isomorphisms

$$A\left\langle \frac{f}{g}\right\rangle \cong \tilde{A}\left\langle \frac{f}{g}\right\rangle, \quad A\left\langle \frac{g}{f}\right\rangle \cong \tilde{A}\left\langle \frac{g}{f}\right\rangle, \quad A\left\langle \frac{f}{g}, \frac{g}{f}\right\rangle \cong \tilde{A}\left\langle \frac{f}{g}, \frac{g}{f}\right\rangle;$$

from this we deduce (a). Note that we cannot say that any rational subspace of $\operatorname{Spa}(\tilde{A}, \tilde{A}^+)$ arises by pullback from $\operatorname{Spa}(A, A^+)$, since such a subspace is defined by parameters in $\tilde{A}$ which we cannot immediately replace with parameters in A.

In light of the previous arguments, to finish the proof of both (a) and (b), it now suffices to check (b) in the case where $\tilde{A} = A$. To this end, let $(A, A^+) \to (B, B^+)$ be the rational localization defined by the parameters $h_1, \dots, h_n, k \in A$, and identify B with the quotient of $A\langle T_1, \dots, T_n\rangle$ by the closure of the ideal generated by $kT_1 - h_1, \dots, kT_n - h_n$. Since we have now established Theorem 1.4.19, it is legitimate to apply Theorem 1.2.7; hence for each nonempty subset $*$ of $\{\frac{f}{g}, \frac{g}{f}\}$, we obtain an exact sequence of pseudocoherent $A\langle *, T_1, \dots, T_n\rangle$-modules of the form

$$A\langle *, T_1, \dots, T_n\rangle^n \to A\langle *, T_1, \dots, T_n\rangle \to B\langle *\rangle \to 0 \tag{1.9.9.1}$$

in which the first map takes the generators to $kT_1 - h_1, \dots, kT_n - h_n$.

Although we do not know[2] that $A\langle T_1, \dots, T_n\rangle$ is sheafy, we do have the exact sequence

$$0 \to A\langle T_1, \dots, T_n\rangle \to A\left\langle \frac{f}{g}, T_1, \dots, T_n\right\rangle \oplus A\left\langle \frac{g}{f}, T_1, \dots, T_n\right\rangle \to A\left\langle \frac{f}{g}, \frac{g}{f}, T_1, \dots, T_n\right\rangle \to 0,$$

using which we may apply Lemma 1.9.8 to the various objects in (1.9.9.1). This yields another exact sequence

$$0 \to (kT_1 - h_1, \dots, kT_n - h_n) \to A\langle T_1, \dots, T_n\rangle \to H^0(\mathrm{Spa}(B, B^+), \mathcal{O}) \to 0$$

of pseudocoherent $A\langle T_1, \dots, T_n\rangle$-modules. In particular, the ideal $(kT_1 - h_1, \dots, kT_n - h_n)$ is closed; it follows that $B = H^0(\mathrm{Spa}(B, B^+), \mathcal{O})$, as desired. $\square$

Lemma 1.9.10. *Let I be a (not necessarily closed) ideal of A. For any $f_1, \dots, f_n \in A$ which generate the unit ideal in A/I, there exists a rational localization $(A, A^+) \to (B, B^+)$ such that $\mathrm{Spa}(B, B^+)$ contains the zero locus of I on $\mathrm{Spa}(A, A^+)$ and $f_1, \dots, f_n$ generate the unit ideal in B.*

Proof. By hypothesis, there exist $b_1, \dots, b_n \in A$ such that $a_1b_1 + \dots + a_nb_n \equiv 1 \pmod{I}$. The rational localization corresponding to the subspace $\{v \in \mathrm{Spa}(A, A^+) : v(a_1b_1 + \dots + a_nb_n) \geq 1\}$ has the desired property. $\square$

Lemma 1.9.11. *Let I be a closed ideal of A which is an object of $\mathbf{PCoh}_A$. Let $(A, A^+) \to (B, B^+)$ be a rational localization.*

(a) *Put $\overline{A} := A/I$ and let $\overline{A}^+$ be the integral closure of the image of A^+ in $\overline{A}$. Let $(\overline{A}, \overline{A}^+) \to (\overline{B}, \overline{B}^+)$ be the base extension of the given rational localization. Then $\overline{B} \cong B/IB$.*

(b) *Suppose that $\mathrm{Spa}(B, B^+)$ contains the zero locus of I on $\mathrm{Spa}(A, A^+)$. Then $A/IA \cong B/IB$.*

Proof. By Theorem 1.4.13, the sequence

$$0 \to I \to A \to A/I \to 0$$

remains exact upon tensoring over A with B. In particular, $IB \cong I \otimes B \in \mathbf{PCoh}_B$ is a closed ideal, so B/IB coincides with the completed tensor product $B \widehat{\otimes}_A A/I$; this proves (a). From (a), (b) is obvious. $\square$

Proof of Theorem 1.4.20. Put $\overline{A} := A/I$ and let $\overline{A}^+$ be the integral closure of the image of A^+ in $\overline{A}$. By Remark 1.6.15, it suffices to check that for any rational localization $(\overline{A}, \overline{A}^+) \to (\overline{B}, \overline{B}^+)$ and any $\overline{f}, \overline{g} \in \overline{B}$ with $\overline{g} \in \{1, 1 - \overline{f}\}$, the sequence

$$0 \to \overline{B} \to \overline{B}\left\langle \frac{\overline{f}}{\overline{g}}\right\rangle \oplus \overline{B}\left\langle \frac{\overline{g}}{\overline{f}}\right\rangle \to B\left\langle \frac{\overline{f}}{\overline{g}}, \frac{\overline{g}}{\overline{f}}\right\rangle \to 0$$

[2] In fact, even knowing that A is sheafy would not imply that $A\langle T_1, \dots, T_n\rangle$ is sheafy!

is exact. By Lemma 1.9.11, there is no harm in replacing (A, A^+) with the rational localization corresponding to a subspace containing the zero locus of I on $\mathrm{Spa}(A, A^+)$. By Lemma 1.9.10, we may thus ensure that some set of parameters in $\overline{A}$ defining the rational localization $(\overline{A}, \overline{A}^+) \to (\overline{B}, \overline{B}^+)$ lift to elements of A which generate the unit ideal. By Lemma 1.9.11 again, there is no harm in replacing (A, A^+) by the corresponding rational localization; that is, we may assume that $(\overline{A}, \overline{A}^+) = (\overline{B}, \overline{B}^+)$.

Lift $\overline{f}, \overline{g}$ to $f, g \in A$ with $g \in \{1, 1-f\}$. In the diagram

$$\begin{tikzcd}
 & 0 \arrow[d] & 0 \arrow[d] & 0 \arrow[d] & \\
0 \arrow[r] & IA \arrow[r] \arrow[d] & IA\left\langle \frac{f}{g} \right\rangle \oplus IA\left\langle \frac{g}{f} \right\rangle \arrow[r] \arrow[d] & IA\left\langle \frac{f}{g}, \frac{g}{f} \right\rangle \arrow[r] \arrow[d] & 0 \\
0 \arrow[r] & A \arrow[r] \arrow[d] & A\left\langle \frac{f}{g} \right\rangle \oplus A\left\langle \frac{g}{f} \right\rangle \arrow[r] \arrow[d] & A\left\langle \frac{f}{g}, \frac{g}{f} \right\rangle \arrow[r] \arrow[d] & 0 \\
0 \arrow[r] & \overline{A} \arrow[r] \arrow[d] & \overline{A}\left\langle \frac{\overline{f}}{\overline{g}} \right\rangle \oplus \overline{A}\left\langle \frac{\overline{g}}{\overline{f}} \right\rangle \arrow[r] \arrow[d] & \overline{A}\left\langle \frac{\overline{f}}{\overline{g}}, \frac{\overline{g}}{\overline{f}} \right\rangle \arrow[r] \arrow[d] & 0 \\
 & 0 & 0 & 0 &
\end{tikzcd}$$

the second row is exact by the sheafiness of A, the first row is exact by Theorem 1.4.13, the first column is exact by definition, and the second and third columns are exact by Lemma 1.9.11. This implies exactness of the third row, as needed. □

1.10. **Remarks on the étale topology.** As promised at the end of §1.4, we include some remarks about the étale topology on X.

Lemma 1.10.1. *Suppose that A is uniform. Then every finite étale A-algebra is uniform for its natural topology as an A-module.*

Proof. In the case where A is Tate, this is [107, Proposition 2.8.16]. The analytic case can be treated similarly, but can also be handled as follows. Extend A to a Huber pair (A, A^+), let B be a finite étale A-algebra of constant rank (it suffices to treat this case), and let B^+ be the integral closure of A^+ in B. Since B is finitely generated as an A-module and the unit ideal of A is generated by topologically nilpotent elements, we can find $b_1, \dots, b_n \in B^+$ which generate B as an A-module. We now see B^+ is contained in the set

$$\{b \in B : \mathrm{Trace}_{B/A}(bb_1), \dots, \mathrm{Trace}_{B/A}(bb_n) \in A^+\},$$

which is bounded (because the trace pairing is nondegenerate); hence B is uniform. □

Remark 1.10.2. Let B be a finite étale A-algebra and let B^+ be the integral closure of A^+ in B. If A is strongly noetherian, then so is B, so there are no technical issues with considering the étale site of X. On the other hand, if A is only sheafy, or even stably uniform, then we cannot immediately infer the same about B: we have the sheaf axiom for coverings of $\mathrm{Spa}(B, B^+)$ arising by pullback from X, but any covering that separates points within some fiber of the projection $\mathrm{Spa}(B, B^+) \to X$ will fail to be refined by such a covering. (That said, we do not have a counterexample in mind.)

In light of the previous remark, we make the following hypothesis.

Hypothesis 1.10.3. For the remainder of §1.10, let X_{et} be the site whose morphisms are compositions of rational localizations and finite étale morphisms. (Note that this gives the "right" definition of the étale site for analytic adic spaces, but not for schemes.) Assume that X_{et} admits a basis $\mathcal{B}$ closed under formation of rational localizations and finite étale covers, and consisting of subspaces of the form $\mathrm{Spa}(B, B^+)$ where B is sheafy. For example, this hypothesis is satisfied if A is perfectoid, or even sousperfectoid (see Remark 1.2.19).

For the étale topology, one has the following analogue of Lemma 1.6.18; however, the proof is somewhat less straightforward, and uses a method introduced by de Jong–van der Put [34, Proposition 3.2.2].

Lemma 1.10.4. *Let $\mathcal{P}_{\mathrm{et}}$ be the collection[3] of pairs $(U, \mathfrak{V})$ where $U \in \mathcal{B}$ and $\mathfrak{V}$ is a finite covering of U in X_{et} by elements of $\mathcal{B}$. Suppose that $\mathcal{P} \subseteq \mathcal{P}_{\mathrm{et}}$ satisfies the following conditions.*

(i) Locality*: if $(U, \mathfrak{V})$ admits a refinement in $\mathcal{P}$, then $(U, \mathfrak{V}) \in \mathcal{P}$.*
(ii) Transitivity*: Any composition of coverings in $\mathcal{P}$ is in $\mathcal{P}$.*
(iii) *Every standard binary rational covering is in $\mathcal{P}$.*
(iv) *Every finite étale surjective morphism, viewed as a covering, is in $\mathcal{P}$.*

Then $\mathcal{P} = \mathcal{P}_{\mathrm{et}}$.

Proof. See [107, Proposition 8.2.20]. □

Remark 1.10.5. Using Lemma 1.10.4, it is straightforward to extend Theorems 1.3.4, 1.4.2, 1.4.13, 1.4.15, and 1.4.17 to X_{et}; we leave the details to the reader.

1.11. **Preadic spaces.** We end with a remark about how to formally build "spaces" out of Huber pairs even when they are not sheafy. This discussion is taken from [148, §2], although our terminology[4] instead follows [107, §8.2]. (To keep within our global context, we build only *analytic* preadic spaces here.)

Definition 1.11.1. For $\mathcal{C}$ a category equipped with a Grothendieck topology (so in particular admitting fiber products), a *sheaf* on $\mathcal{C}$ is by definition a contravariant functor F from $\mathcal{C}$ to some target category satisfying the sheaf axiom: for $\{U_i \to X\}_i$ a covering in $\mathcal{C}$, the sequence

$$F(X) \to \prod_i F(U_i) \rightrightarrows \prod_{i,j} F(U_i \times_X U_j)$$

is an equalizer.

We say that $\mathcal{C}$ is *subcanonical* if for each $X \in \mathcal{C}$, the representable functor $h_X : Y \mapsto \mathrm{Hom}_{\mathcal{C}}(Y, X)$ is a sheaf. For example, the Zariski topology on schemes is subcanonical: this reduces immediately to the corresponding statement for affine schemes, which asserts that if R and S are rings and $f_1, \dots, f_n \in R$ generate the unit ideal in R, then the diagram

$$\mathrm{Hom}(S, R) \to \prod_i \mathrm{Hom}(S, R_{f_i}) \rightrightarrows \prod_{i,j} \mathrm{Hom}(S, R_{f_i f_j})$$

[3] This is not a set in general, but a proper class.

[4] In [148], what we call *preadic spaces* and *adic spaces* are called *adic spaces* and *honest adic spaces*, respectively. We prefer to leave the term *adic spaces* with the original meaning specified by Huber.

is an equalizer; this follows from the sheaf axiom for the structure sheaf. By the same reasoning, the étale, fppf, and fpqc topologies on the category of schemes are subcanonical.

Similarly, if $\mathcal{C}$ is a subcategory of the category of adic spaces admitting fiber products (e.g., locally noetherian spaces, or perfectoid spaces), the analytic topology on $\mathcal{C}$ (i.e., the site coming from the underlying topology on underlying spaces) is subcanonical. By contrast, the analytic topology on the full category of Huber pairs is not subcanonical.

Definition 1.11.2. Let $\mathcal{C}$ be the opposite category of (analytic but not necessarily sheafy) Huber pairs, equipped with the analytic topology. Let $\mathcal{C}^\sim$ be the associated topos. For $(A, A^+) \in \mathcal{C}$, let $\widetilde{\mathrm{Spa}}(A, A^+) \in \mathcal{C}^\sim$ be the sheafification of the representable functor on $\mathcal{C}$ defined by (A, A^+).

By an *open immersion* in $\mathcal{C}^\sim$, we will mean a morphism $f : \mathcal{F} \to \mathcal{G}$ such that for every $(A, A^+) \in \mathcal{C}$ and every morphism $\widetilde{\mathrm{Spa}}(A, A^+) \to \mathcal{G}$ in $\mathcal{C}^\sim$, there is an open subset U of $\mathrm{Spa}(A, A^+)$ such that

$$\mathcal{F} \times_{\mathcal{G}} \widetilde{\mathrm{Spa}}(A, A^+) = \varinjlim_{V \subseteq U, V \text{ rational}} \widetilde{\mathrm{Spa}}(\mathcal{O}_{\mathrm{Spa}(A,A^+)}(V), \mathcal{O}^+_{\mathrm{Spa}(A,A^+)}(V)).$$

A *preadic*[5] *space*, or more precisely an *analytic preadic space*, is an object $\mathcal{F} \in \mathcal{C}^\sim$ such that

$$\mathcal{F} = \varinjlim_{\widetilde{\mathrm{Spa}}(A,A^+) \to \mathcal{F} \text{ open}} \widetilde{\mathrm{Spa}}(A, A^+).$$

The obvious functor from analytic adic spaces to analytic preadic spaces is a full embedding.

A morphism $f : \mathcal{F} \to \mathcal{G}$ is *finite étale* if for every $(A, A^+) \in \mathcal{C}$ and every morphism $\widetilde{\mathrm{Spa}}(A, A^+) \to \mathcal{G}$ in $\mathcal{C}^\sim$, we have $\mathcal{F} \times_{\mathcal{G}} \widetilde{\mathrm{Spa}}(A, A^+) \cong \widetilde{\mathrm{Spa}}(B, B^+)$ for some finite étale morphism $(A, A^+) \to (B, B^+)$. Using finite étale morphisms and open immersions, we may define the *etale topology* on analytic preadic spaces.

Remark 1.11.3. Similar sheaf-theoretic considerations give rise to other types of objects that one might think of as analytic analogues of *algebraic spaces* (or more generally *algebraic stacks*). Notably, they underlie the construction of *diamonds*, which will be introduced in [163, Lecture 3] and used in our subsequent lectures (see Definition 4.3.1 and beyond).

[5]Or *pre-adic space* if you prefer to hyphenate prefixes.

2. Perfectoid rings and spaces

In this lecture, we define perfectoid rings and spaces, picking up from the discussion of perfectoid fields in [163, Lecture 2]. As for fields, there is a "tilting" construction that converts these spaces into related objects in characteristic p; however, this time the "Galois" component of the tilting correspondence is augmented by a "spatial" component.

As in the first lecture, we state all of the main results first, then return to the proofs. Along the way, we attempt to lay flat some of the tangled history[6] surrounding these results.

Hereafter, we fix a prime number p and consider only Huber rings in which p is topologically nilpotent.

2.1. **Perfectoid rings and pairs.**

We now define perfectoid rings and pairs, postponing some proofs for the time being.

Definition 2.1.1. Let (A, A^+) be a uniform analytic Huber pair; by uniformity, A^+ is a ring of definition of A. We say (A, A^+) is *perfectoid* if there exists an ideal of definition $I \subseteq A^+$ such that $p \in I^p$ and $\varphi : A^+/I \to A^+/I^p$ is surjective (but not necessarily injective; see Remark 2.3.16). This turns out to depend only on A (Corollary 2.3.10); we may thus say also that A is a *perfectoid ring*.

Example 2.1.2. If (A, A^+) is a uniform analytic Huber pair of characteristic p, then (A, A^+) is perfectoid if and only if A is perfect: if A is perfect, then A^+ must also be perfect because it is integrally closed in A.

Example 2.1.3. Any algebraically closed nonarchimedean field is a perfectoid ring. More generally, recall that a *perfectoid field* is defined as a nonarchimedean field F which is not discretely valued for which $\varphi : \mathfrak{o}_F/(p) \to \mathfrak{o}_F/(p)$ is surjective. If F is a perfectoid field, then $(F, \mathfrak{o}_F)$ is a perfectoid Huber pair; this is obvious in characteristic p (because F is then perfect), and otherwise we may take $I = (\mu)$ for any topologically nilpotent element μ such that μ^p divides p in $\mathfrak{o}_F$ (which exists because F is not discretely valued). Conversely, if A is a perfectoid ring which is a field, then A is a perfectoid field; see Corollary 2.3.11 and Theorem 2.9.1.

Example 2.1.4. Let (A, A^+) be any perfectoid Huber pair (e.g., $(F, \mathfrak{o}_F)$ for F a perfectoid field as in Example 2.1.3). Then for every nonnegative integer n,

$$(A\langle T_1^{p^{-\infty}}, \dots, T_n^{p^{-\infty}}\rangle, A^+\langle T_1^{p^{-\infty}}, \dots, T_n^{p^{-\infty}}\rangle)$$

and

$$(A\langle T_1^{\pm p^{-\infty}}, \dots, T_n^{\pm p^{-\infty}}\rangle, A^+\langle T_1^{\pm p^{-\infty}}, \dots, T_n^{\pm p^{-\infty}}\rangle)$$

are also perfectoid Huber pairs.

Remark 2.1.5. If (A, A^+) is perfectoid, then $\varphi : A^+/((p)+I) \to A^+/((p)+I^p)$ is surjective for any ideal of definition I as in Definition 2.1.1. By Lemma 2.7.4, the same is true for any ideal of definition I whatsoever. In particular, the criterion of Definition 2.1.1 is satisfied for *every* ideal of definition I for which $p \in I^p$. However, the existence of such an ideal

[6] I am reminded here of a famous quote of David Mumford [128, Preface]: "When I first started doing research in algebraic geometry, I thought the subject attractive... because it was a small, quiet field where a dozen people did not leap on each new idea the minute it became current."

of definition is a genuine condition; for instance, it precludes the case $(\mathbb{Q}_p, \mathbb{Z}_p)$, for which $\varphi : A^+/((p)+I) \to A^+/((p)+I^p)$ is surjective for any ideal of definition I.

Note that the previous paragraph does not immediately imply that $\varphi : A^+/(p) \to A^+/(p)$ is surjective. However, this will follow from the tilting correspondence (Theorem 2.3.9); see Remark 2.3.12.

Remark 2.1.6. Note that for A a perfectoid ring, the ring $A\langle T_1^{p^{-\infty}}, \dots, T_n^{p^{-\infty}}\rangle$ of Example 2.1.4 is evidently not noetherian: the ideal $(T_1^{p^{-n}} : n = 0, 1, \dots)$ is not finitely generated. In fact, perfectoid rings can never be noetherian except in the trivial case where they are finite direct sums of perfectoid fields; see Corollary 2.9.3.

This means that we cannot hope to use noetherian properties to show that perfectoid rings are sheafy. Instead, we will have to show that they are stably uniform, by establishing the preservation of the perfectoid property under rational localizations using the tilting construction (Corollary 2.5.4).

Remark 2.1.7. For (A, A^+) a perfectoid ring, I an ideal of definition as in Definition 2.1.1, and $(A, A^+) \to (B, B^+)$ a morphism of uniform Huber pairs, the pair (B, B^+) is perfectoid if and only if $\varphi : B^+/IB^+ \to B^+/I^pB^+$ is surjective. This is a consequence of Remark 2.1.5, which allows us to use IB^+ as an ideal of definition to check the perfectoid condition.

We record some historical aspects of the definition of perfectoid fields and rings.

Remark 2.1.8. The term *perfectoid* was introduced by Scholze [142], but various aspects of the general concept had appeared several times before then. Here we report on these appearances.

The term *perfectoid field* was introduced by Scholze in [142]. A similar definition was given independently[7] by Kedlaya in [101] and incorporated into the work of Kedlaya–Liu [107]; while much of the work on [107] predates the appearance of [142], some terminology from the latter was adapted in the published version of the former. See Remark 2.5.12 for more details. It was subsequently discovered that Matignon–Reversat [125] had introduced the same concept in 1984 as a *hyperperfect field (corps hyperparfait)*, but the importance of this development seems to have gone unnoticed at the time.

Some examples of perfectoid rings which are not fields appear in the hypothesis of the *almost purity theorem* of Faltings [51, Theorem 3.1], [53, Theorem 4]. These examples served as a key motivation for the general construction.

In [27], Colmez introduces the concept of a *sympathetic algebra* (*algèbre sympathique*), which is equivalent to a perfectoid ring over an algebraically closed perfectoid field. He then uses these rings to define what are commonly known as *Banach-Colmez spaces*, which will be discussed in [163, Lecture 4] and used in our student project.

The term *perfectoid ring* was introduced in [142] to refer to a perfectoid ring in the present sense over an arbitrary perfectoid field. The concept of a perfectoid ring over $\mathbb{Q}_p$ was introduced independently by Kedlaya–Liu [107], with terminology adapted from [142]; this excludes perfectoid rings of characteristic p, which appear separately in [107] as *perfect uniform Banach* $\mathbb{F}_p$*-algebras*. Some alternate characterizations of perfectoid rings over $\mathbb{Q}_p$, phrased in terms of Witt vectors, can be found in [30].

[7]The paper [101] was originally written as a supplement to the lecture notes from the 2009 Clay Mathematics Institute summer school on p-adic Hodge theory. As of this writing, those notes remain unpublished.

In his Bourbaki seminar on the work of Scholze, Fontaine [61] introduced the concept of a Tate perfectoid ring, phrasing the definition in terms of condition (a) of Corollary 2.6.16. As [61] is primarily a survey of [142], the theory of Tate perfectoid rings is not developed in any detail there; this development was subsequently carried out by Kedlaya–Liu [108]. Note that in characteristic p, a perfectoid ring is Tate precisely when it admits the structure of an algebra over some perfectoid field.

The definition of perfectoid rings used here, which allows for arbitrary analytic rings, is original to these notes. For examples that separate the various definitions, see the following references herein:

- for a perfectoid ring over $\mathbb{Q}_p$ which is not an algebra over a perfectoid field, see Exercise 2.4.10;
- for a perfectoid ring which is Tate but not a $\mathbb{Q}_p$-algebra, see the proof of Lemma 3.1.3;
- for a perfectoid ring which is analytic but not Tate, see Exercise 2.4.11.

The massive work-in-progress [65] should ultimately be even more inclusive, in ways we do not attempt to treat here (for instance, it includes some rings which are not analytic).

2.2. **Witt vectors.** In order to say more about perfectoid rings, we need to recall some basic facts about Witt vectors (compare [101, §1.1]).

Definition 2.2.1. A ring of characteristic p is *perfect* if the absolute Frobenius map is a bijection. A *strict p-ring* is a p-adically complete (so in particular p-adically separated) ring S which is flat over $\mathbb{Z}_p$ (that is, p is not a zero-divisor) with the property that S/pS is perfect.

Example 2.2.2. The ring $S := \mathbb{Z}_p$ is a strict p-ring with $S/pS \cong \mathbb{F}_p$. Similarly, for any finite unramified extension F of $\mathbb{Q}_p$ with residue field $\mathbb{F}_q$, the integral closure S of $\mathbb{Z}_p$ in F is a strict p-ring with $S/pS \cong \mathbb{F}_q$. Similarly, for F the maximal unramified extension of $\mathbb{Q}_p$, the completed integral closure S of $\mathbb{Z}_p$ in F is a strict p-ring with $S/pS \cong \overline{\mathbb{F}}_p$.

Example 2.2.3. For n a nonnegative integer, the p-adic completion S of $\mathbb{Z}[T_1^{p^{-\infty}}, \dots, T_n^{p^{-\infty}}]$ is a strict p-ring with $S/pS \cong \mathbb{F}_p[\overline{T}_1^{p^{-\infty}}, \dots, \overline{T}_n^{p^{-\infty}}]$.

Lemma 2.2.4. *For any ring R, any ideal I of R, and any nonnegative integer n, the map $x \mapsto x^{p^n}$ induces a morphism of multiplicative monoids*

$$R/((p) + I) \to R/((p)^{n+1} + (p)^n I + \cdots + (p) I^{p^{n-1}} + I^{p^n}).$$

Proof. This is an immediate consequence of the p-divisibility of binomial coefficients. □

Corollary 2.2.5. *For S a strict p-ring, the map $S \to S/pS$ admits a unique multiplicative section $x \mapsto [x]$, called the* Teichmüller map. *In particular, the element $[x] \in S$ (called the* Teichmüller lift *of x) is the unique lift of x which admits p^n-th roots for all positive integers n.*

Corollary 2.2.6. *For S a strict p-ring, every element x of S has a unique representation as a p-adically convergent series $\sum_{n=0}^{\infty} p^n [\overline{x}_n]$ with $\overline{x}_n \in S/pS$. The $\overline{x}_n$ are called the* Teichmüller coordinates *of x.*

Lemma 2.2.7. *Let S be a strict p-ring. Let S' be any p-adically complete ring. Then every ring homomorphism $\overline{\pi} : S/pS \to S'/pS'$ lifts uniquely to a homomorphism $S \to S'$.*

Proof. By Lemma 2.2.4, $\overline{\pi}$ lifts uniquely to a multiplicative map $\pi : S/pS \to S'$. One then shows that the formula

$$\sum_{n=0}^{\infty} p^n[\overline{x}_n] \mapsto \sum_{n=0}^{\infty} p^n \pi(\overline{x}_n)$$

defines the desired ring homomorphism, by checking additivity modulo p^m by induction on m. For details, see [101, Lemma 1.1.6]. □

Remark 2.2.8. By applying Lemma 2.2.7 in the case where S is as in Example 2.2.3, we may see that arithmetic in a strict p-ring can be expressed in terms of certain universal "polynomials" in the Teichmüller coordinates. For example, if one writes

$$[x] + [y] = \sum_{n=0}^{\infty} p^n[z_n],$$

then z_n is given by a certain homogeneous polynomial over $\mathbb{F}_p$ in $x^{p^{-n}}, y^{p^{-n}}$ of degree 1 (for the convention that $\deg(x) = \deg(y) = 1$) divisible by $x^{p^{-n}}y^{p^{-n}}$.

Remark 2.2.9. A corollary of the previous remark is that if S is a strict p-ring and I is a perfect ideal of S/pS, then the set of $x \in S$ whose Teichmüller coordinates all belong to I is an ideal of S. Note that the quotient by this ideal is again a strict p-ring.

Theorem 2.2.10 (Witt)**.** *The functor $S \mapsto S/pS$ defines an equivalence of categories between strict p-rings and perfect $\mathbb{F}_p$-algebras.*

Proof. Full faithfulness follows from Lemma 2.2.7. To check essential surjectivity, we first lift perfect polynomial rings over $\mathbb{F}_p$ in any (possibly infinite) number of variables as in Example 2.2.3, then use Remark 2.2.9 to lift quotients of such rings. That covers all perfect $\mathbb{F}_p$-algebras. □

Remark 2.2.11. There is no analogue of Theorem 2.2.10 for general R. For example, if R is a field which is not perfect, then the Cohen structure theorem implies that R can be realized as S/pS for some flat p-adically complete $\mathbb{Z}_p$-algebra S (any such S is called a *Cohen ring* for R), but not functorially in R. For example, if $R = \mathbb{F}_p((\overline{T}))$, then the p-adic completion S of $\mathbb{Z}_p((T))$ admits an isomorphism $S/pS \cong R$ taking the class of T to $\overline{T}$, but there are numerous automorphisms of S lifting the identity map on R; in fact, the group of such automorphisms acts simply transitively on the inverse image of $\overline{T}$.

One way to lift imperfect rings is to consider pairs (R, B) in which R is a reduced ring of characteristic p and B is a finite subset of R such that $\{\prod_{b \in B} b^{e_b} : e_b \in \{0, \dots, p-1\}\}$ is a basis for R^p as an R-module. (Such a set B is called a *p-basis* of R; the existence of such a set implies that R^p is a finite projective R-module, which is to say that R is *F-split*.) Then one can functorially lift (R, B) to a pair $(S, \tilde{B})$ in which S is a p-adically complete flat $\mathbb{Z}_p$-algebra and $\tilde{B}$ is a finite subset of S lifting B.

Definition 2.2.12. For R a perfect ring of characteristic p, let $W(R)$ denote the strict p-ring with residue ring R; concretely, $W(R)$ consists of sequences $(\overline{x}_0, \overline{x}_1, \dots)$ in R which are identified with the convergent sums $\sum_{n=0}^{\infty} p^n[\overline{x}_n]$. By functoriality (i.e., by Lemma 2.2.7), the absolute Frobenius φ on R lifts to a unique automorphism of R. For $I \subseteq R$ a perfect ideal, let $W(I)$ denote the ideal of $W(R)$ described in Remark 2.2.9.

Remark 2.2.13. For conceptual purposes, it is sometimes useful to imagine the ring $W(R)$ as "the ring of power series in the variable p with coefficients in R." This point of view must of course be abandoned when one attempts to make any arguments involving calculations in $W(R)$; however, Remark 2.2.8 gives some control over the "carries" that occur in these calculations.

2.3. **Tilting and untilting.** In order to say more about perfectoid rings, we describe a fundamental construction that relates perfectoid rings to rings in characteristic p. This construction has its roots in the foundations of p-adic Hodge theory (see Remark 2.3.18), and the definition of perfectoid rings is in turn motivated by the construction.

Definition 2.3.1. For (A, A^+) a perfectoid pair, define the *tilt* of A, denoted $A^\flat$, as the set $\varprojlim_{x \mapsto x^p} A$; it carries the structure of a monoid under multiplication. We equip $A^\flat$ with the inverse limit topology (i.e., the coarsest topology with respect to which the map $\sharp : A^\flat \to A$ which projects onto the final component is continuous); this gives $A^\flat$ the structure of a topological monoid. (We sometimes write $x^\sharp$ instead of $\sharp(x)$.) Let $A^{\flat+}$ be the submonoid $\varprojlim_{x \mapsto x^p} A^+$ of $A^\flat$.

We will see later (Theorem 2.3.9) that the formula

$$(x_n)_n + (y_n)_n = (z_n)_n, \qquad z_n := \lim_{m \to \infty} (x_{m+n} + y_{m+n})^{p^m} \tag{2.3.1.1}$$

defines a ring structure on $A^\flat$ with respect to which it is a perfectoid ring of characteristic p, in such a way that $A^{\flat+}$ is a ring of integral elements. Another interpretation of the ring structure on $A^{\flat+}$ will come from the bijection

$$A^{\flat+} \cong \varprojlim_{x \mapsto x^p} (A^+/I) \tag{2.3.1.2}$$

for any ideal of definition I as in Definition 2.1.1; note that the right-hand side of (2.3.1.2) is obviously a perfect ring of characteristic p.

In order to fill in the details of the previous construction, we describe an inverse construction using Witt vectors.

Definition 2.3.2. Let (R, R^+) be a perfectoid pair in characteristic p. We will make frequent use of the ring $W(R^+)$, which is commonly denoted $\mathbf{A}_{\mathrm{inf}}(R, R^+)$ (although we will not use this notation until the next lecture).

Let $W^b(R)$ denote the subset of $W(R)$ consisting of series $\sum_{n=0}^\infty p^n [\overline{x}_n]$ for which the set $\{\overline{x}_n : n = 0, 1, \dots\}$ is bounded in R. By Remark 2.2.8, this forms a subring of $W(R)$ containing $W(R^+)$. We equip $W^b(R)$ with the topology of uniform convergence in the coordinates (see Remark 2.6.3); any continuous map $R \to S$ of perfectoid rings in characteristic p induces a homomorphism $W^b(R) \to W^b(S)$.

Remark 2.3.3. If R is Tate, then so is $W^b(R)$: for any pseudouniformizer ϖ in R, $[\varpi]$ is a pseudouniformizer in $W^b(R)$. However, if R is analytic, it is not clear that $W^b(R)$ is analytic; compare Lemma 2.6.13.

The following construction provides something analogous to a Weierstrass-prepared power series over a nonarchimedean field.

Definition 2.3.4. An element $z = \sum_{n=0}^{\infty} p^n[\overline{z}_n] \in W(R^+)$ is *primitive of degree* 1 (or *primitive* for short) if $\overline{z}_0$ is topologically nilpotent and $\overline{z}_1$ is a unit in R^+; in other words, $z = [\overline{z}_0] + pz_1$ where z_1 is a unit in $W(R^+)$. Note that multiplying a primitive element by a unit gives another such element (e.g., using Remark 2.2.8); we say that an ideal of $W(R^+)$ is *primitive (of degree* 1*)* if it is principal with some (hence any) generator being a primitive element.

The primitive elements will play a role analogous to that played by the ideal $(T-p)$ in the isomorphism $\mathbb{Z}[\![T]\!]/(T-p) \cong \mathbb{Z}_p$. In particular, they admit a form of Euclidean division which is quite useful for getting control of elements of perfectoid rings; this will be studied extensively in §2.6.

Remark 2.3.5. If $z_1, z_2 \in W(R^+)$ are primitive elements such that $z_1 = yz_2$ for some $y \in W(R^+)$, then by Remark 2.2.8, $\overline{z}_{1,1} - \overline{y}_0\overline{z}_{2,1}$ is topologically nilpotent. It follows that y is a unit in $W(R^+)$.

Exercise 2.3.6. If R is Tate, then in some sources a primitive element of $W(R^+)$ is assumed to have the form $p + [\varpi]\alpha$ where $\varpi \in R$ is a topologically nilpotent unit and $\alpha \in W(R^+)$ is arbitrary. Show that if z is a primitive element in the sense of Definition 2.3.4, then it has some associate of the form $p + [\varpi]\alpha$ but need not have this form itself.

Before stating a general theorem, let us discuss a couple of key examples.

Example 2.3.7. Put $R := \mathbb{F}_p((\overline{T}^{p^{-\infty}}))$ (i.e., the ring obtained by taking the $\overline{T}$-adic completion of $\mathbb{F}_p[\overline{T}^{p^{-\infty}}]$ and then inverting $\overline{T}$) and $R^+ := R^\circ$. The element $z := p - [\overline{T}]$ is primitive, and $W^b(R)/(z)$ is the completion of $\mathbb{Q}_p(p^{p^{-\infty}})$, which is a perfectoid field.

Example 2.3.8. Put $R := \mathbb{F}_p((\overline{T}^{p^{-\infty}}))$, $R^+ := R^\circ$. The element $z := \sum_{i=0}^{p-1} [1 + \overline{T}]^i$ is primitive (because it maps to p under $W(R^+) \to W(\mathbb{F}_p)$), and $W^b(R)/(z)$ is the completion of $\mathbb{Q}_p(\mu_{p^\infty})$, which is a perfectoid field.

Theorem 2.3.9. *The formula*

$$(R, R^+, I) \mapsto (A := W^b(R)/IW^b(R), A^+ := W(R^+)/I) \tag{2.3.9.1}$$

defines a equivalence of categories from triples (R, R^+, I)*, in which* (R, R^+) *is a perfectoid pair of characteristic* p *and* I *is a primitive ideal of* $W(R^+)$*, to perfectoid pairs* (A, A^+)*. (A morphism* $(R, R^+, I) \to (S, S^+, J)$ *in this category is a morphism* $(R, R^+) \to (S, S^+)$ *of Huber pairs carrying* I *into* J*; in fact, by Remark 2.3.5 the image always equals* J*.) Furthermore, there is a quasi-inverse functor which takes* (A, A^+) *to* $(A^\flat, A^{\flat+}, I)$ *with the ring structure on* $A^\flat$ *given by* (2.3.1.1).

Proof. By Lemma 2.6.14, it will follow that the equation (2.3.9.1) gives a well-defined functor. By Lemma 2.7.9, we will obtain the quasi-inverse functor. □

Corollary 2.3.10. *Let* A *be a perfectoid ring, that is,* A *is a Huber ring such that* (A, A^+) *is a perfectoid pair for* some *ring of integral elements* A^+*. Then* (A, A^+) *is a perfectoid pair for* every *ring of integral elements* A^+*.*

Proof. By Theorem 2.3.9, we can write $A = W^b(R)/I$ for some perfectoid ring R of characteristic p and some ideal I; more precisely, the ideal I admits a primitive generator in $W(R^+)$ for some ring of integral elements R^+ of R, and hence in $W(R^\circ)$. By Example 2.1.2, every ring of integral elements R^+ of R untilts to a ring of integral elements A^+ of A such that (A, A^+) is perfectoid. It thus suffices to check that every A^+ arises in this fashion.

Since A is uniform, every ring of integral elements is contained in the ring A° of power-bounded elements and contains the set $A^{\circ\circ}$ of topologically nilpotent elements, which is an ideal of A°. In fact, the rings of integral elements are in bijection with integrally closed subrings of $A^\circ/A^{\circ\circ}$. Similarly, the rings of integral elements of R are in bijection with integrally closed subrings of $A^\circ/A^{\circ\circ}$. By Lemma 2.7.10, the rings $A^\circ/A^{\circ\circ}$ and $R^\circ/R^{\circ\circ}$ are isomorphic; this completes the proof. (See Remark 2.3.16 for a more refined version of the isomorphism $A^\circ/A^{\circ\circ} \cong R^\circ/R^{\circ\circ}$). □

Corollary 2.3.11. *A nonarchimedean field F is a perfectoid ring if and only if it is a perfectoid field.*

Proof. One direction is Example 2.1.3. In the other direction, if F is a perfectoid ring, then by Corollary 2.3.10, $(F, \mathfrak{o}_F)$ is a perfectoid pair. If F is of characteristic p, then F is perfect (Example 2.1.2) and the valuation on F is nontrivial (hence not discrete by perfectness), so F is a perfectoid field. If F is of characteristic 0, then (p) is an ideal of definition of $\mathfrak{o}_F$, so Remark 2.3.12 applies to show that $\varphi : \mathfrak{o}_F/(p) \to \mathfrak{o}_F/(p)$ is surjective. Since the valuation on $F^\flat$ is not discrete, neither is the valuation on F, so F is a perfectoid field. □

Remark 2.3.12. If (A, A^+) is a perfectoid pair, then the existence of a surjective morphism $W(R^+) \to A^+$ as in Theorem 2.3.9 implies that $\varphi : A^+/(p) \to A^+/(p)$ is surjective: every $x \in A^+/(p)$ lifts to some element $y = \sum_{n=0}^\infty p^n[\overline{y}_n] \in W(R^+)$, and the image of $[\overline{y}_0^{1/p}]$ in $A^+/(p)$ maps to x via φ.

Definition 2.3.13. With notation as in Theorem 2.3.9, the perfectoid pair (A, A^+) corresponding to the triple (R, R^+, I) is called the *untilt* of (R, R^+) corresponding to the primitive ideal I.

Definition 2.3.14. Theorem 2.3.9 implies that for any perfectoid pair (A, A^+), there is a surjective map $W(A^{\flat+}) \to A^+$ whose kernel is primitive (and in particular principal), which extends to a map $W^b(A^\flat) \to A$. These maps are traditionally denoted by θ.

Note that for any $\overline{x} \in R$, the sequence $(\theta([\overline{x}^{p^{-n}}])_n$ forms an element of $\varprojlim_{x \mapsto x^p} A = A^\flat$. In the course of proving Theorem 2.3.9, we will see that the identification of $A^\flat$ with R identifies this sequence with $\overline{x}$. In other words,

$$\sharp = \theta \circ [\bullet].$$

Example 2.3.15. With notation as in Theorem 2.3.9, the pair

$$(A\langle T_1^{p^{-\infty}}, \dots, T_n^{p^{-\infty}} \rangle, A^+\langle T_1^{p^{-\infty}}, \dots, T_n^{p^{-\infty}} \rangle)$$

is the untilt of

$$(R\langle \overline{T}_1^{p^{-\infty}}, \dots, \overline{T}_n^{p^{-\infty}} \rangle, R^+\langle \overline{T}_1^{p^{-\infty}}, \dots, \overline{T}_n^{p^{-\infty}} \rangle)$$

corresponding to the primitive ideal generated by I, with $\sharp(\overline{T}_i) = T_i$. Similarly, the pair

$$(A\langle T_1^{\pm p^{-\infty}}, \dots, T_n^{\pm p^{-\infty}} \rangle, A^+\langle T_1^{\pm p^{-\infty}}, \dots, T_n^{\pm p^{-\infty}} \rangle)$$

is the untilt of

$$(R\langle \overline{T}_1^{\pm p^{-\infty}}, \dots, \overline{T}_n^{\pm p^{-\infty}} \rangle, R^+\langle \overline{T}_1^{\pm p^{-\infty}}, \dots, \overline{T}_n^{\pm p^{-\infty}} \rangle)$$

corresponding to the primitive ideal generated by I, again with $\sharp(\overline{T}_i) = T_i$.

Remark 2.3.16. With notation as in Theorem 2.3.9, let z be a generator of I. For J an ideal (resp. ideal of definition) of A^+ containing p, the inverse image of J in $W(R^+)$ is an ideal containing p and z, and so is also the inverse image of an ideal (resp. ideal of definition) $J^\flat$ of $A^{\flat+}$ containing $\overline{z}$. Conversely, every ideal (resp. ideal of definition) of $A^{\flat+}$ containing $\overline{z}$ arises in this fashion. From the construction, we have a canonical isomorphism

$$A^+/J \cong R^+/J^\flat.$$

We mention some related facts here.

- A perfectoid ring A is Tate if and only if $A^\flat$ is Tate (Corollary 2.6.16). This implies that if A is Tate, then A^+ admits a principal ideal of definition J satisfying $p \in J^p$.
- For J an ideal of definition of A^+ with $p \in J^p$, let $J^{(p)}$ be the ideal of A^+ generated by p together with x^p for each $x \in J$; this ideal is contained in J^p, but may be strictly smaller unless J is principal. Then the surjective map $\varphi : A^+/J \to A^+/J^p$ induces a bijective map $A^+/J \to A^+/J^{(p)}$.

Remark 2.3.17. For A, B two perfectoid rings, it is not true in general that any homomorphism $f^\flat : A^\flat \to B^\flat$ lifts to a homomorphism $f : A \to B$, because the image of $\ker(\theta : W^b(A^\flat) \to A)$ need not be contained in $\ker(\theta : W^b(B^\flat) \to B)$. However, this image is always a primitive ideal, so given $A, B^\flat, f^\flat$ there is a unique choice of an untilt B of $B^\flat$ for which $f^\flat$ does lift to a homomorphism $f : A \to B$. In other words, the categories of perfectoid A-algebras and perfectoid $A^\flat$-algebras are equivalent. For example, this comment applies to the setting of [142], in which the only perfectoid rings are considered are algebras over some fixed perfectoid field.

Again, we collect some historical notes.

Remark 2.3.18. The operation $A \mapsto A^\flat$ was originally introduced by Fontaine–Wintenberger [62, 63, 164, 165] in the case where A is the completion of an algebraic extension of $\mathbb{Q}_p$ having a certain property (of being *strictly arithmetically profinite*) which implies that A is perfectoid; by a theorem of Sen [149], this includes the completion of any infinitely ramified Galois extension of $\mathbb{Q}_p$ whose Galois group is a p-adic Lie group. (See [110] for further discussion of this implication.) This construction is a key step in the Fontaine's construction of the de Rham and crystalline period rings [59, §2], [60].

The same construction for somewhat more general rings appeared in the work on Faltings on the crystalline comparison isomorphism [52, §2], [53, §2b], as a key step in constructive relative analogues of Fontaine's rings; this direction was further pursued by Andreatta [7]. The terminology of *tilting* and *untilting*, and the notations $\flat$ and $\sharp$, were introduced by Scholze in [142]; previously these constructions did not have commonly used names (the term *inverse perfection* for the construction $A^+ \mapsto \varprojlim A^+/I$ is used in [107]).

Like other concepts in the theory of perfectoid spaces, the notion of a *primitive element* can be found implicitly throughout the literature of p-adic Hodge theory, and somewhat more explicitly in [57] and [100]. However, it does not appear at all in [142] because no reference is made therein to Witt vectors: since only perfectoid algebras over a perfectoid field are considered, one implicitly untilts using the primitive ideal coming from the base field. In [107], the primitive elements considered (as per [107, Definition 3.3.4]) are those for which $\overline{z}_0$ is a unit in R; this level of generality corresponds to the restriction that perfectoid rings be $\mathbb{Q}_p$-algebras. The definition of primitive elements used here is the one introduced by Fontaine in [61], and adopted by Kedlaya–Liu in [108, Definition 3.2.3].

The theta map (Definition 2.3.14) first appeared in the case where A is a completed algebraic closure of $\mathbb{Q}_p$, in Fontaine's construction of the ring $\mathbf{B}_{\mathrm{dR}}$ of de Rham periods [59, §2] (see also Definition 4.6.5); it appears again in the work of Andreatta (see above).

Theorem 2.3.9 was established by Scholze [142, Theorem 5.2] for perfectoid rings over a perfectoid field, (but without reference to Witt vectors; see above) and independently by Kedlaya–Liu [107, Theorem 3.6.5] for perfectoid rings over $\mathbb{Q}_p$. It was extended to Tate rings by Kedlaya–Liu [108, Theorem 3.3.8]. The extension to analytic rings is original to these notes.

2.4. **Algebraic aspects of tilting.** We next describe the extent to which tilting is compatible with certain algebraic properties of morphisms of perfectoid rings.

Remark 2.4.1. Let $f : A \to B$ be a morphism of perfectoid rings. If f is injective, then $f^\flat : A^\flat \to B^\flat$ is obviously injective. However, if $f^\flat$ is injective, it does not follow that f is injective; see Example 2.4.2. (The ideal $I := \ker(A)$ is closed, but A/I is not necessarily uniform because it need not be closed in B.)

Example 2.4.2. Let $\mathbb{C}_p$ be the completion of an algebraic closure of $\mathbb{Q}_p$. Take $A := \mathbb{C}_p\langle T^{\pm p^{-\infty}}\rangle$ and let I be the ideal $(1 + T^{1/p} + \cdots + T^{(p-1)/p})$ of A. The quotient A/I is not uniform; let B denote the uniform completion (i.e., the completion with respect to the spectral seminorm). Explicitly, B is isomorphic to the ring of continuous functions from $\mathbb{Z}_p^\times$ to $\mathbb{C}_p$, whereas A/I corresponds to a certain subring of locally analytic functions. Let $f : A \to B$ be the natural map, which by construction is not injective. Then $B^\flat$ is isomorphic to the ring of continuous functions from $\mathbb{Z}_p^\times$ to $\mathbb{C}_p^\flat$; more precisely, this isomorphism depends on the choice of an element $\epsilon = (\dots, \zeta_p, 1) \in A^\flat$ in which ζ_{p^n} is a primitive p^n-th root of unity. In terms of this choice, $f^\flat$ can be described as the map from $A^\flat \cong \mathbb{C}_p^\flat\langle \overline{T}^{\pm p^{-\infty}}\rangle$ to $B^\flat$ taking $\overline{T}$ to the function $\gamma \mapsto \epsilon^\gamma \overline{T}$. From this description, it can be shown (with some effort) that $f^\flat$ is injective.

Theorem 2.4.3. *Let $f : A \to B$ be a morphism of perfectoid rings. Then f is a strict inclusion if and only if $f^\flat$ is a strict inclusion.*

Proof. From the definition of $A^\flat$ as a topological space, it is obvious that if f is a strict inclusion, then so is $f^\flat$. The reverse implication will follow from Lemma 2.8.1. □

Theorem 2.4.4. *Let $f : A \to B$ be a morphism of Huber rings in which A is perfectoid.*

(a) *If B is uniform and f has dense image, then B is perfectoid.*
(b) *If B is perfectoid, then f has dense image if and only if $f^\flat$ has dense image.*

Proof. Part (a) will follow from Lemma 2.8.4. To check (b) in one direction, note that if $f^\flat$ has dense image, then the composition $W^b(A^\flat) \to W^b(B^\flat) \to B$ has dense image, so f has dense image. The other direction will again follow from Lemma 2.8.4. □

Theorem 2.4.5. *Let $f : A \to B$ be a morphism of Huber rings in which A is perfectoid.*

(a) *If B is uniform and f is surjective, then B is perfectoid.*
(b) *If B is perfectoid, then f is surjective if and only if $f^\flat$ is surjective.*

Proof. Part (a) is a consequence of Theorem 2.4.4(a). To check (b) in one direction, note that if $f^\flat$ is surjective, then the composition $W^b(A^\flat) \to W^b(B^\flat) \to B$ is surjective, so f is surjective. In the other direction, if f is surjective, then Lemma 2.8.6 implies that $f^\flat$ is surjective. $\square$

Corollary 2.4.6. *For A a perfectoid ring, the map $I \mapsto I^\flat$ defines a bijection between closed ideals of A with A/I uniform and closed perfect ideals of $A^\flat$.*

Proof. In light of Theorem 2.4.5, it suffices to check that if R is a perfectoid ring of characteristic p and I is a perfect ideal of R, then R/I is uniform (this being the case of the desired statement where $A = A^\flat = R$). To this end, promote R to a uniform Banach ring as per Remark 1.5.4. Then note that if $\tilde{x} \in R$ lifts $x \in R/I$, then $\tilde{x}^{1/p}$ lifts $x^{1/p} \in R/I$, so $\left|x^{1/p}\right| \leq |x|^{1/p}$. From Definition 1.5.11, it follows that R/I is uniform. $\square$

Corollary 2.4.7. *For A a perfectoid ring and I a closed ideal of A, A/I is uniform (and hence perfectoid) if and only if there exists some subset S of $A^\flat$ such that I is the closure of the ideal generated by $\theta(x^{p^{-n}})$ for all $x \in S$ and all nonnegative integers n.*

Proof. Immediate from Corollary 2.4.6. $\square$

Example 2.4.8. Let K be a perfectoid field. The quotient of $K\langle T^{p^{-\infty}} \rangle$ by the closed ideal (T) is not uniform (or even reduced), and hence not perfectoid. By contrast, the quotient by the closure of the ideal $(T^{p^{-n}} : n = 0, 1, \dots)$ is the field K again.

Remark 2.4.9. If A is a perfectoid ring and I is a maximal ideal of A, then I is automatically closed (see Remark 1.1.1) but we do not know in general whether A/I is uniform. In all cases where this holds, it will follow from Theorem 2.4.5(a) and Theorem 2.9.1 that A/I is a perfectoid field. For example, by Corollary 2.4.6, this holds if A is of characteristic p, as then I is necessarily perfect.

Exercise 2.4.10. In this exercise, we exhibit a perfectoid ring over $\mathbb{Q}_p$ which is not an algebra over a perfectoid field.

(a) Prove that the completions of $\mathbb{Q}_p(p^{p^{-\infty}})$ and $\mathbb{Q}_p(\mu_{p^\infty})$ have no common subfield larger than $\mathbb{Q}_p$. One way to do this is to use the Ax–Sen–Tate theorem [10] to show that any complete subfield of either of these two fields is itself the completion of an algebraic extension of $\mathbb{Q}_p$.

(b) Let F be the completed perfect closure of $\mathbb{F}_p((\overline{T}_1))$. Put $R := F\langle \overline{T}_2^{p^{-\infty}} \rangle$, $R^+ := R^\circ$. Let R_1 be the quotient of R by the closure of the ideal $(\overline{T}_2^{p^{-n}} : n = 0, 1, \dots)$ and put $R_1^+ = R_1^\circ$. Let R_2 be the quotient of R by the closure of the ideal $(\overline{T}_2^{p^{-n}} - 1 : n = 0, 1, \dots)$ and put $R_2^+ = R_2^\circ$. Prove that there exists a primitive element $z \in W(R^+)$ which maps to $p - [\overline{T}_1]$ in $W(R_1^+)$ and to $\sum_{i=0}^{p-1} [1 + \overline{T}_1]^i$ in $W(R_2^+)$.

(c) Combine (a), (b), Example 2.3.7, and Example 2.3.8 to obtain the desired example.

Exercise 2.4.11. Using Corollary 2.4.7, adapt Example 1.5.7 to give an example of a perfectoid ring in characteristic p which is analytic but not Tate. Note that no such example can exist over $\mathbb{Q}_p$, but one can construct mixed-characteristic examples by untilting.

Remark 2.4.12. A morphism $f : (A, A^+) \to (B, B^+)$ of uniform Huber pairs with dense image behaves in many ways like a closed immersion: in particular, the induced map $\mathrm{Spa}(B, B^+) \to$

$\mathrm{Spa}(A, A^+)$ is a homeomorphism of the source onto a closed subspace of the target, and moreover identifies rational subspaces on both sides (because rational subspaces in $\mathrm{Spa}(B, B^+)$ can always be defined using parameters in the dense subset $f(A)$ of B). In [146], geometric maps arising in this way are called *closed immersions* and closed immersions in the usual sense are called *strongly closed immersions*. For further discussion, see the project of Weinstein [163].

The following argument shows that the category of perfectoid spaces admits fiber products, which is not known for the full category of adic spaces (because a completed tensor product of sheafy Huber rings is not known to again be sheafy).

Theorem 2.4.13. *Let $A \to B, A \to C$ be two morphisms of perfectoid rings. Then $B\widehat{\otimes}_A C$ is again a perfectoid ring, and its formation commutes with tilting.*

Proof. See Lemma 2.8.7. □

Remark 2.4.14. Theorem 2.4.4 and Theorem 2.4.5 were proved for perfectoid rings over $\mathbb{Q}_p$ in [107, Theorem 3.6.17], and for Tate rings in [108, Theorem 3.3.18]. The extensions to analytic rings are original to these notes.

Theorem 2.4.13 was proved by Scholze [142, Proposition 6.18] for perfectoid rings over a perfectoid field, and independently by Kedlaya–Liu [107, Theorem 3.6.11] for perfectoid rings over $\mathbb{Q}_p$. This was extended to Tate perfectoid rings by Kedlaya–Liu [108, Theorem 3.3.13]. The extension to analytic rings is original to these notes.

2.5. **Geometric aspects of tilting.** We now describe the interaction of the perfectoid condition with rational and étale localization. This will allow us to define perfectoid spaces and the étale topology on them.

Theorem 2.5.1. *For (A, A^+) a perfectoid pair, the formula $v \mapsto v \circ \sharp$ defines a bijection $\mathrm{Spa}(A, A^+) \cong \mathrm{Spa}(A^\flat, A^{\flat+})$ which identifies rational subspaces on both sides; in particular, this map is a homeomorphism.*

Proof. By Theorem 2.3.9, (A, A^+) is the untilt of $(A^\flat, A^{\flat+})$ corresponding to some primitive ideal. We may thus apply Lemma 2.6.12 to show that the map is well-defined, and Corollary 2.6.15 to show that it is bijective. It is easy to see that the rational subspace of $\mathrm{Spa}(A^\flat, A^{\flat+})$ defined by the parameters $\overline{f}_1, \dots, \overline{f}_n, \overline{g}$ corresponds to the rational subspace of $\mathrm{Spa}(A, A^+)$ defined by the parameters $\sharp(\overline{f}_1), \dots, \sharp(\overline{f}_n), \sharp(\overline{g})$; the converse implication follows from Lemma 2.6.17. □

Remark 2.5.2. One of the remarkable features of Theorem 2.5.1 is the fact that $\sharp : A^\flat \to A$ is not a ring homomorphism (it is multiplicative but not additive), and yet pullback by $\sharp$ defines a morphism of spectra. We like to think of $\sharp$ as defining a "homotopy equivalence" between $A^\flat$ and A instead of a true morphism.

Theorem 2.5.3. *Let (A, A^+) be a perfectoid pair.*

(a) *For $(A, A^+) \to (B, B^+)$ a rational localization, (B, B^+) is again a perfectoid pair.*
(b) *The functor $(B, B^+) \to (B^\flat, B^{\flat+})$ defines an equivalence of categories between rational localizations of (A, A^+) and of $(A^\flat, A^{\flat+})$.*

Proof. In light of Theorem 2.5.1, it suffices to check that the untilt of any rational localization is again a rational localization; this requires some argument because the universal property of a rational localization is quantified over arbitrary Huber pairs, not just perfectoid pairs or uniform pairs. See Lemma 2.8.8. □

Corollary 2.5.4. *Any perfectoid ring is stably uniform, and hence sheafy by Theorem 1.2.13.*

Corollary 2.5.5. *Let A be a Huber ring admitting a continuous homomorphism $A \to B$ to a perfectoid ring which splits in the category of topological A-modules (a/k/a a* sousperfectoid ring*; see Remark 1.2.19). Then A is stably uniform.*

Proof. Combine Corollary 2.5.4 with Lemma 1.2.18. □

Definition 2.5.6. In light of Corollary 2.5.4, for any perfectoid pair (A, A^+) the space $\mathrm{Spa}(A, A^+)$ admits the structure of an adic space. An adic space locally of this form is called a *perfectoid space.*

Corollary 2.5.7. *For (A, A^+) a perfectoid pair, the residue field of every point of* $\mathrm{Spa}(A, A^+)$ *is a perfectoid field.*

Proof. By Theorem 2.5.3, any such residue field is a completed direct limit of perfectoid rings, hence itself a perfectoid ring, and hence (by Corollary 2.3.11) a perfectoid field. □

Exercise 2.5.8. Here is a rare example of a ring which can be shown to be stably uniform despite not being (directly) susceptible to Corollary 2.5.5. Let K be an algebraically closed perfectoid field of characteristic $p > 2$. Put

$$A_0 := K\langle T^{p^{-\infty}} \rangle, \qquad A := A_0[T^{1/2}], \qquad A' := K\langle (T^{1/2})^{p^{-\infty}} \rangle.$$

Equip these rings with the Gauss norm.

(a) Show that for $i \in \frac{1}{2}\mathbb{Z}[p^{-1}]$, if $i \geq \frac{1}{2}$ then $T^i \in A$. Deduce that the natural map $A \to A'$ does not split in the category of A-modules.

(b) Show that the map $\mathrm{Spa}(A', A'^\circ) \to \mathrm{Spa}(A, A^\circ)$ is a homeomorphism and identifies rational subspaces on both sides.

(c) Show that $\mathrm{Spa}(A, A^\circ)$ contains a unique point v_0 with $v_0(T) = 0$, whose complement is a perfectoid space. Note that the residue field of v_0 is K, so A satisfies the conclusion of Corollary 2.5.7 but not the hypothesis.

(d) Using (b), show that a general rational subspace of $\mathrm{Spa}(A, A^\circ)$ containing v_0 has the form

$$\{v \in \mathrm{Spa}(A, A^\circ) : v(\lambda_0 T^{1/2}) \leq 1, v(\lambda_1 T^{1/2} - \mu_1) \geq 1, \dots, v(\lambda_n T^{1/2} - \mu_n) \geq 1\}$$

for some nonnegative integer n and some $\lambda_0, \dots, \lambda_n, \mu_1, \dots, \mu_n \in K$ with $\lambda_i \geq 1$ for $i \geq 0$ and $\lambda_i \geq \mu_i \geq 1$ for $i > 0$.

(e) Let $(A, A^\circ) \to (B, B^+)$ be the rational localization corresponding to a rational subspace as in (d) with $\lambda_0 = 1$. (It turns out that $B^+ = B^\circ$, but this isn't crucial for what follows.) A general element of B can be written in the form

$$a_0 + \sum_{i=1}^{n} \sum_{j=1}^{\infty} a_{i,j} (\lambda_i T^{1/2} - \mu_i)^{-j}$$

for some $a_0, a_{i,j} \in A_0$. Prove that each $a_{i,j}$ can be replaced by an element with all exponents in $[0, 1]$ without increasing the quotient norm (i.e., the maximum of the norms of a_0 and all of the $a_{i,j}$).

(f) Put $B' = B\widehat{\otimes}_A A'$. Show that for

$$x = a_0 + \sum_{i=1}^{n} \sum_{j \in \mathbb{Z}[p^{-1}], j>0} a_{i,j}(\lambda_i T^{1/2} - \mu_i)^{-j} \in B'$$

with $a_0 \in A_0, a_{i,j} \in K$, the spectral norm of x is equal to the maximum of the norms of a_0 and the $a_{i,j}$.

(g) Show that $B \to B\widehat{\otimes}_A A'$ is a strict inclusion. Deduce that A is stably uniform.

Theorem 2.5.9. *Let A be a perfectoid ring.*

(a) *For $A \to B$ a finite étale morphism, B is again a perfectoid ring for its natural topology as an A-module. (Note that Lemma 1.10.1 implies that B is uniform.)*

(b) *The functor $B \mapsto B^\flat$ defines an equivalence of categories between finite étale algebras over A and over $A^\flat$.*

Proof. See Lemma 2.8.11. □

Corollary 2.5.10. *For (A, A^+) a perfectoid pair, there is a functorial homeomorphism* $\mathrm{Spa}(A, A^+)_{\mathrm{et}} \cong \mathrm{Spa}(A^\flat, A^{\flat+})_{\mathrm{et}}$.

Remark 2.5.11. It was conjectured in [144, Conjecture 2.16] that if (A, A^+) is a Huber pair over a perfectoid field and $\mathrm{Spa}(A, A^+)$ is a perfectoid space, then A is a perfectoid ring. This is refuted by the first example of Buzzard–Verberkmoes cited in Remark 1.2.24.

However, it is possible that a similar question with slightly different hypotheses does admit an affirmative answer. For an example of a partial result in this direction, Theorem 1.2.22 implies that if (A, A^+) is a Huber pair such that $\mathrm{Spa}(A,^+)$ is a perfectoid space and $A = H^0(\mathrm{Spa}(A, A^+), \mathcal{O})$, then A is sheafy.

Remark 2.5.12. For A a perfectoid ring promoted to a Banach ring, the homeomorphism $\mathcal{M}(A) \cong \mathcal{M}(A^\flat)$ induced by Theorem 2.5.1 (by restricting to valuations of height 1) was first described in [100, Corollary 7.2] (and alluded to in [98]).

Theorem 2.5.3 was proved for perfectoid rings over a perfectoid field by Scholze [142, Theorem 6.3], and independently for perfectoid rings over $\mathbb{Q}_p$ by Kedlaya–Liu [107, Theorem 3.6.14]. It was generalized to Tate rings by Kedlaya–Liu [108, Theorem 3.3.18]. The extension to analytic rings is original to these notes, but uses similar methods.

For perfectoid fields, Theorem 2.5.9 generalizes the *field of norms correspondence* of Fontaine–Wintenberger [62, 63]. The result in this case originated from a private communication between this author and Brian Conrad after the 2009 Clay Mathematics Institute summer school on p-adic Hodge theory; this argument was subsequently reproduced in [101, Theorem 1.5.6] and [107, Theorem 3.5.6]. (The key special case of algebraically closed perfectoid fields amounts to an argument we learned from Robert Coleman in 1998, as documented in [92, §4].) The result was obtained independently by Scholze using a different approach based on almost ring theory; see [142, Theorem 3.7] for a side-by-side treatment of both approaches.

Theorem 2.5.3 is a generalization of (part of) the *almost purity theorem* of Faltings, which appears implicitly in [51] and somewhat more explicitly in [53]. It was proved for perfectoid rings over a perfectoid field by Scholze [142, Theorem 7.9], and independently for perfectoid rings over $\mathbb{Q}_p$ by Kedlaya–Liu [107, Theorem 3.6.21]. It was generalized to Tate rings by

Kedlaya–Liu [108, Theorem 3.3.18]. The extension to analytic rings is original to these notes, but uses similar methods. (See [107, Remark 5.5.10] for some additional discussion.)

2.6. **Euclidean division for primitive ideals.** In order to prove most of our main results, we need to establish a version of Euclidean divisor for primitive elements. Our presentation of this construction follows [100].

Hypothesis 2.6.1. Throughout §2.6, let (R, R^+) be a perfectoid pair of characteristic p, and let $z \in W(R^+)$ be a primitive element. Promote R to a uniform Banach ring as per Remark 1.5.4.

Note that by definition, these hypotheses imply that R is analytic. However, we will only need that hypothesis starting with Lemma 2.6.13; the results before that require only that R be perfect and uniform. This will be important in §2.7, where we must make some calculations with a putative perfectoid ring of characteristic p before establish its analyticity.

Definition 2.6.2. Define the *Gauss norm* on $W^b(R)$ by the formula

$$\left|\sum_{n=0}^{\infty} p^n[\overline{x}_n]\right| = \sup\{|\overline{x}_n| : n = 0, 1, \ldots, \};$$

note that the supremum is in general not achieved. Using Remark 2.2.8, it can be shown that the Gauss norm is a power-multiplicative norm, or even a multiplicative norm in case the norm on R is multiplicative [100, Lemma 4.2]; moreover, $W(R^+)$ and $W^b(R)$ are both complete with respect to this norm.

For z a primitive element, for the Gauss norm we have $|pz_1| = 1 > |z - pz_1|$, so for all $x \in W^b(R)$, we have $|zx| = |x|$. Consequently, the ideals $zW(R^+)$ and $zW^b(R)$ are closed in their respective rings, and using the quotient norms we may equip $W(R^+)/(z)$ and $W^b(R)/(z)$ with the structure of Banach (and Huber) rings. Note also that $zW^b(R) \cap W(R^+) = zW(R^+)$, so the map $W(R^+)/(z) \to W^b(R)/(z)$ is injective.

Remark 2.6.3. The topology induced by the Gauss norm on $W^b(R)$ can be interpreted as the topology of uniform convergence in the Teichmüller coordinates. On $W(R^+)$, this can also be interpreted as the I-adic topology for $I = ([\overline{x}_1], \ldots, [\overline{x}_n])$ where $\overline{x}_1, \ldots, \overline{x}_n \in R^+$ generate the unit ideal in R, although it requires some care to show this when $n > 1$.

As in [100, 107, 108], one can also consider *weighted Gauss norms* on $W^b(R)$ given by formulas of the form

$$\left|\sum_{n=0}^{\infty} p^n[\overline{x}_n]\right|_\rho = \max\{\rho^{-n}\,|\overline{x}_n| : n = 0, 1, \ldots, \}$$

for some $\rho \in (0, 1)$. (Note that the supremum becomes a maximum as soon as $\rho < 1$.) On $W(R^+)$, the topology induced by a weighted Gauss norm can be interpreted as the (p, I)-adic topology for I as above.

For z a primitive element, the quotient norms on $W^b(R)/(z)$ induced by the Gauss norm and any weighted Gauss norm coincide. This will follow from the fact that every nonzero element of the quotient admits a prepared representative (Lemma 2.6.9).

For primitive elements, we have a useful analogue of Euclidean division. The following discussion is taken from [100, §5].

Definition 2.6.4. Let $z = [\overline{z}_0] + pz_1 \in W(R^+)$ be primitive. For $x = \sum_{n=0}^{\infty} p^n[\overline{x}_n] \in W^b(R)$, define the *Euclidean quotient and remainder* of x modulo z as the pair (q, r) where

$$x_1 := p^{-1}(x - [\overline{x}_0]), \quad q := z_1^{-1}x_1, \quad r := x - qz = [\overline{x}_0] - [\overline{z}_0]\sum_{n=0}^{\infty} z_1^{-1}p^n[\overline{x}_{n+1}].$$

Definition 2.6.5. An element $x = \sum_{n=0}^{\infty} p^n[\overline{x}_n] \in W^b(R)$ is *prepared* if $|\overline{x}_0| \geq |\overline{x}_n|$ for all $n > 0$. (This corresponds to the definition of *stable* in [100, §5], but we need to save that term for another meaning later; this terminology is meant to suggest the *Weierstrass preparation theorem.*)

Lemma 2.6.6. *For $z \in W(R^+)$ primitive, no nonzero multiple of z in $W^b(R)$ is prepared.*

Proof. Suppose $x = \sum_{n=0}^{\infty} p^n[\overline{x}_n] \in W^b(R)$ is such that zx is prepared. Then on one hand $|x| = |zx|$, so the reduction of zx has norm $|\overline{x}_0|$; on the other hand, this reduction is $\overline{z}_0\overline{x}_0$ and we know that $|\overline{z}_0| < 1$. These two statements can only be consistent when $\overline{x}_0 = 0$; since zx is prepared, this forces $zx = 0$ and hence $x = 0$. □

Lemma 2.6.7. *Let $z \in W(R^+)$ be primitive. If $x \in W^b(R)$ is prepared, then the quotient norm of the class of x in $W^b(R)/(z)$ is equal to $|x|$.*

Proof. Suppose to the contrary that there exists $y \in x + zW^b(R)$ with $|y| < |x|$. Then $x - y$ is a nonzero prepared multiple of z, so Lemma 2.6.6 yields a contradiction. □

Corollary 2.6.8. *Let $z \in W(R^+)$ be primitive. For $\overline{x} \in R$, the quotient norm of the class of $[\overline{x}]$ in $W^b(R)/(z)$ equals $|\overline{x}|$.*

Lemma 2.6.9. *For z primitive and $x \in W^b(R)$ not divisible by z, form the sequence $x_0, x_1, \dots$ in which $x_0 = x$ and x_{m+1} is the Euclidean remainder of x_m modulo z. Then for every sufficiently large m, x_m is prepared.*

Proof. If $|x_{m+1}| > |\overline{z}_0|\,|x_m|$ for some m, then x_{m+1} is prepared; in addition, $|x_{m+2}| = |x_{m+1}|$, so $x_{m+2}, x_{m+3}, \dots$ are also prepared. Otherwise, for q_m the Euclidean quotient of x_m modulo z, the sum $\sum_{m=0}^{\infty} q_m$ converges to a limit q satisfying $x = qz$, so x represents the zero class in the quotient ring, contradiction. □

Corollary 2.6.10. *For z primitive, the quotient norm on $W^b(R)/(z)$ is power-multiplicative. If in addition the norm on R is multiplicative, then the quotient norm on $W^b(R)/(z)$ is multiplicative.*

Proof. Combine Lemma 2.6.7 with Lemma 2.6.9. □

Remark 2.6.11. In the proof of Corollary 2.6.10, we are implicitly using the fact that under certain circumstances, a finite product of prepared elements is prepared, namely if

- all of the terms in the product are the same, *or*
- the norm on R is multiplicative.

However, in general a product of prepared elements need not be prepared.

Lemma 2.6.12. *For z primitive, $A := W^b(R)/(z)$, $A^+ = W(R^+)/(z)$, and $v \in \mathrm{Spa}(A, A^+)$ arbitrary, we have $v \circ \pi \circ [\bullet] \in \mathrm{Spa}(R, R^+)$.*

Proof. Let $\pi : W^b(R) \to A$ denote the quotient map. The nontrivial point is that $v \circ \pi \circ [\bullet]$ satisfies the strong triangle inequality; this follows from Remark 2.2.8. □

Up to now, none of the arguments have required the hypothesis that R be analytic. We add that hypothesis now.

Lemma 2.6.13. *For R analytic and z primitive, $W^b(R)/(z)$ is analytic.*

Proof. Put $A = W^b(R)/(z)$, $A^+ = W(R^+)/(z)$. Let $\pi : W^b(R) \to A$ denote the quotient map. For $v \in \mathrm{Spa}(A, A^+)$, by Lemma 2.6.12 the formula $x \mapsto v(\pi([x]))$ defines a valuation $w \in \mathrm{Spa}(R, R^+)$. By Lemma 1.1.3, there exists $x \in R$ such that $w(x) \neq 0$; hence $v(\pi([x])) \neq 0$. By Lemma 1.1.3 again, A is analytic. $\square$

Lemma 2.6.14. *The formula* (2.3.9.1) *defines a functor from triples (R, R^+, I), in which (R, R^+) is a perfectoid pair of characteristic p and I is a primitive ideal of $W(R^+)$, to perfectoid pairs (A, A^+).*

Proof. By Corollary 2.6.10 and Lemma 2.6.13, A is uniforrm and analytic. Let z be a generator of I. Let J be an ideal of definition of R^+ such that $\overline{z}_0 \in J^p$ (this exists because R is perfect). Then the set of $x = \sum_{n=0}^{\infty} p^n[\overline{x}_n] \in W(R^+)$ with $\overline{x}_0 \in J$ maps to an ideal of definition $\tilde{J}$ of A^+ with $p \in \tilde{J}^p$ such that $\varphi : A^+/\tilde{J} \to A^+/\tilde{J}^p$ is surjective. Hence (A, A^+) is a perfectoid pair. $\square$

Corollary 2.6.15. *For (R, R^+, I) corresponding to (A, A^+) as in Lemma 2.6.14, the construction of Lemma 2.6.12 defines a bijective map* $\mathrm{Spa}(A, A^+) \to \mathrm{Spa}(R, R^+)$.

Proof. For $v \in \mathrm{Spa}(R, R^+)$ corresponding to the pair (K, K^+), the triple $(K, K^+, IW(K^+))$ corresponds via Lemma 2.6.14 to a pair (L, L^+). By Corollary 2.6.8, L is an analytic field; this pair then corresponds to the unique valuation in $\mathrm{Spa}(A, A^+)$ mapping to v. $\square$

Corollary 2.6.16. *For (A, A^+) a uniform analytic Huber pair, the following conditions are equivalent.*

(a) *There exists a pseudouniformizer $\varpi \in A$ such that ϖ^p divides p in A^+ and $\varphi : A^+/(\varpi) \to A^+/(\varpi^p)$ is surjective.*
(b) *The ring A is Tate and perfectoid.*
(c) *The ring A is perfectoid and the ring $A^\flat$ is Tate.*
(d) *The ring A is perfectoid and there exists a uniformizer $\overline{\varpi} \in A^\flat$ such that $\sharp(\overline{\varpi})^p$ divides p in A^+ and $\varphi : A^+/(\sharp(\overline{\varpi})) \to A^+/(\sharp(\overline{\varpi}^p))$ is surjective.*

Proof. Since all four conditions imply that (A, A^+) is perfectoid (using Corollary 2.3.10), we may assume this from the outset. It is clear that (a) implies (b), (c) implies (a), and (d) implies (c); it thus remains to check that (b) implies (d).

Let $\varpi \in A$ be any pseudouniformizer. Lift ϖ to some $x_0 \in W(A^{\flat+})$, then form the sequence $x_0, x_1, \dots$ as in Lemma 2.6.9. Write $x_m = \sum_{n=0}^{\infty} p^n[\overline{x}_{m,n}]$ with $\overline{x}_{m,n} \in A^{\flat+}$. For each $v \in \mathrm{Spa}(A, A^+)$, corresponding to $w \in \mathrm{Spa}(A^\flat, A^{\flat+})$ via Corollary 2.6.15, we may apply Lemma 2.6.7 and Lemma 2.6.9 to see that for m sufficiently large, $w(\overline{x}_{m,0}) = v(\varpi) \neq 0$; in particular, there exists a neighborhood U of w in $\mathrm{Spa}(A^\flat, A^{\flat+})$ on which $\overline{x}_{m,0}$ does not vanish. Since $\mathrm{Spa}(A^\flat, A^{\flat+})$ is quasicompact, we may make a uniform choice of m for which $\overline{x}_{m,0}$ vanishes nowhere on $\mathrm{Spa}(A^\flat, A^{\flat+})$. By Corollary 1.5.21, $\overline{x}_{m,0}$ is a pseudouniformizer in $A^\flat$, and we may take $\overline{\varpi} = \overline{x}_{m,0}^{p^{-k}}$ for k sufficiently large to achieve the desired result. $\square$

Lemma 2.6.17. *For (R, R^+, I) corresponding to (A, A^+) as in Lemma 2.6.14, under the bijection* $\mathrm{Spa}(A, A^+) \to \mathrm{Spa}(R, R^+)$ *of Corollary 2.6.15, every rational subspace of* $\mathrm{Spa}(A, A^+)$ *arises from some rational subspace of* $\mathrm{Spa}(R, R^+)$.

Proof. Choose $f_1, \dots, f_n, g \in A$ generating the unit ideal. By Exercise 1.2.2, for $\epsilon > 0$ sufficiently small, perturbing $f_1, \dots, f_n, g$ by elements of norm at most ϵ does not change the resulting rational subspace. For $y = f_1, \dots, f_n, g$ in turn, choose $x_0 \in W^b(R)$ lifting y and define the sequence $x_0, x_1, \dots$ as in Lemma 2.6.9; then for sufficiently large m we have

$$\max\{(\alpha \circ \sharp)(y), \epsilon\} = \max\{\alpha(\overline{x}_m), \epsilon\} \qquad (\alpha \in \mathcal{M}(R)).$$

By replacing y with $\sharp(\overline{x}_m)$ in the list of parameters for our rational subspace, we achieve the desired result. $\square$

2.7. **Primitive ideals and tilting.** We now show that the construction of Lemma 2.6.14 accounts for all perfectoid pairs, completing the proof of Theorem 2.3.9.

Lemma 2.7.1. *For (A, A^+) a uniform Huber pair in which p is topologically nilpotent, topologize the set $A^\flat := \varprojlim_{x \mapsto x^p} A$ as in Definition 2.3.1.*

(a) *Let $(x_n)_n, (y_n)_n \in A^\flat$ be elements. The limit in the formula*

$$(x_n)_n + (y_n)_n = \left(\lim_{m \to \infty} (x_{m+n} + y_{m+n})^{p^m} \right)_n \tag{2.7.1.1}$$

exists and defines an element of $A^\flat$.

(b) *Using (2.7.1.1) to define addition, $A^\flat$ is a perfect uniform Huber ring of characteristic p.*

(c) *The subset $A^{\flat+} := \varprojlim_{x \mapsto x^p} A^+$ is a subring of $A^\flat$. Moreover, for any ideal of definition I of A^+ for which $p \in I^p$, the map*

$$A^{\flat+} = \varprojlim_{x \mapsto x^p} A^+ \to \varprojlim_{x \mapsto x^p} (A^+/I)$$

of topological rings, for the inverse limit of discrete topologies on the target, is an isomorphism.

Proof. In the ring $W(\mathbb{F}_p[x^{p^{-\infty}}, y^{p^{-\infty}}])$, we have

$$[x + y] = \lim_{m \to \infty} ([x^{p^{-m}}] + [y^{p^{-m}}])^{p^m}; \tag{2.7.1.2}$$

from this equality, we easily deduce (a).

In $A^\flat$, it is obvious that addition is commutative, multiplication distributes over addition, the p-power map is a bijection, and adding something to itself p times gives zero; using (2.7.1.2), we also see that addition is associative and continuous. It follows that $A^\flat$ is a perfect topological ring of characteristic p and that $x \mapsto |x^\sharp|$ is a norm defining the topology of $A^\flat$; in particular, $A^\flat$ is a uniform Huber ring. This proves (b), from which (c) follows easily using Lemma 2.2.4. $\square$

Hypothesis 2.7.2. For the remainder of §2.7, let (A, A^+) be a perfectoid pair. Keep in mind that we do not yet know that $A^\flat$ is analytic; this will be established in Lemma 2.7.8. Consequently, we need to be a bit wary about applying results from §2.6 to avoid creating a vicious circle (see Hypothesis 2.6.1).

Definition 2.7.3. By Lemma 2.2.7, there exists a unique homomorphism $\theta : W(A^{\flat+}) \to A^+$ satisfying $\theta([x]) = \sharp(x)$ for all $x \in A^{\flat+}$. Since $A^+/((p) + I) \to A^+/((p) + I^p)$ is surjective for an ideal of definition I as in Definition 2.1.1 (see Remark 2.1.5), θ is surjective.

Lemma 2.7.4. *For any ideal of definition I of A^+ with $p \in I^p$, there exist topologically nilpotent elements $\overline{x}_1, \dots, \overline{x}_n$ of $A^{\flat+}$ such that $\sharp(\overline{x}_1), \dots, \sharp(\overline{x}_n)$ generate I.*

Proof. Choose generators $x_1, \dots, x_n$ of I. The surjectivity of θ implies that $\overline{x}_1, \dots, \overline{x}_n$ can be chosen so that $x_i - \sharp(\overline{x}_i) \in I^p$ for $i = 1, \dots, n$; this yields the claim. □

Lemma 2.7.5. *The ideal* $\ker(\theta) \subseteq W(A^{\flat+})$ *is primitive.*

Proof. It suffices to exhibit a single primitive generator. Choose $\overline{x}_1, \dots, \overline{x}_n$ as in Lemma 2.7.4. Since $p \in I^p$, we can write p in the form $\sum_{i=1}^n y_i \sharp(\overline{x}_i)$ for some $y_i \in A^+$. Lift each y_i to $\tilde{y}_i \in W(A^{\flat+})$ and put

$$z := p - \sum_{i=1}^n \tilde{y}_i [\overline{x}_i] \in W(A^{\flat+});$$

then z is evidently primitive.

It remains to show that z generates $\ker(\theta)$. If on the contrary $y \in \ker(\theta)$ is not divisible by z, then we may apply Lemma 2.6.9 (which does not require $A^\flat$ to be analytic) to produce a prepared element $y' \in W(A^{\flat+})$ congruent to y modulo z. But now the same argument as in Lemma 2.6.6 yields a contradiction: for $\overline{y}' \in A^{\flat+}$ the reduction of y', the image of $y' - \theta(\overline{y}')$ in A^+ has norm strictly smaller than that of $\theta(\overline{y}')$, so we cannot have $\theta(y') = 0$. □

Lemma 2.7.6. *The map $v \mapsto v \circ \sharp$ defines an injective map* $\mathrm{Spa}(A, A^+) \to \mathrm{Spa}(A^\flat, A^{\flat+})$.

Proof. The map is well-defined by Lemma 2.7.5 and Lemma 2.6.12 (or simply imitating the proof of the latter). Since the image of $\sharp : A^{\flat+} \to A^+$ generates a dense $\mathbb{Z}$-subalgebra of A^+, the map $\mathrm{Spa}(A^+, A^+) \to \mathrm{Spa}(A^{\flat+}, A^{\flat+})$ is injective. Combining this observation with Corollary 1.1.4 yields the injectivity of $\mathrm{Spa}(A, A^+) \to \mathrm{Spa}(A^\flat, A^{\flat+})$. □

Lemma 2.7.7. *Suppose that A is Tate and admits a pseudouniformizer ϖ such that ϖ^p divides p in A^+ and $\varphi : A^+/(\varpi) \to A^+/(\varpi^p)$ is surjective. Then $A^\flat$ is Tate (hence perfectoid) and the map of Lemma 2.7.6 is bijective.*

Proof. By the proof of Lemma 2.7.4 in the case $n = 1$, we can find $\overline{\varpi} \in A^{\flat+}$ such that $\varpi - \sharp(\overline{\varpi}) \in \varpi^p A^+$ and hence $\varpi A^+ = \sharp(\overline{\varpi}) A^+$. It follows that $\overline{\varpi}$ is a pseudouniformizer of $A^\flat$, so $A^\flat$ is Tate. By Lemma 2.7.1, $(A^\flat, A^{\flat+})$ is a perfectoid pair; in fact, the triple $(A^\flat, A^{\flat+}, \ker(\theta))$ corresponds to (A, A^+) as in Lemma 2.6.14. By Corollary 2.6.15, the map of Lemma 2.7.6 is bijective. □

Lemma 2.7.8. *With notation as in Lemma 2.7.4, the elements $\overline{x}_1, \dots, \overline{x}_n$ generate the unit ideal in $A^\flat$. In particular, $A^\flat$ is analytic.*

Proof. Apply Lemma 2.7.5 to construct a primitive generator z of $\ker(\theta)$. Promote A to a uniform Banach ring as per Remark 1.5.4; pulling back along $\sharp$ then provides a norm promoting $A^\flat$. Since $\sharp(\overline{x}_1), \dots, \sharp(\overline{x}_n)$ generate the unit ideal in A, we may form the associated standard rational covering of $\mathrm{Spa}(A, A^+)$; namely, for $i = 1, \dots, n$, let $(A, A^+) \to (B_i, B_i^+)$ be the rational localization defined by the parameters $\sharp(\overline{x}_1), \dots, \sharp(\overline{x}_n), \sharp(\overline{x}_i)$. By Lemma 2.7.1, $B_i^\flat$ is a uniform Huber ring containing $\overline{x}_i$ as a pseudouniformizer (because $\sharp(\overline{x}_i)$ is invertible in B_i), and hence a Tate perfectoid ring of characteristic p. (This did not yet require Lemma 2.7.7 because we already had a choice of $\overline{\varpi}$ in mind.) The surjective map $W^b(A^\flat)\langle T_1, \dots, T_n \rangle \to A\langle T_1, \dots, T_n \rangle \to B$ factors through $W^b(B_i^\flat) \to B_i$ via the map taking

T_i to $[\overline{f}_i/\overline{g}]$, so $W^b(B_i^\flat) \to B_i$ is surjective. Let $(B_i', B_i'^+)$ be the untilt of $(B_i^\flat, B_i^{\flat+})$ corresponding to the ideal (z), which is perfectoid by Lemma 2.6.14; we now have a surjective map $B_i' \to B_i$. The map $\mathrm{Spa}(B_i, B_i^+) \to \mathrm{Spa}(B_i', B_i'^+)$ is thus a closed immersion, and hence a homeomorphism of $\mathrm{Spa}(B_i, B_i^+)$ onto a closed subset of $\mathrm{Spa}(B_i', B_i'^+)$. On the other hand, by Corollary 2.6.15, the image of $\mathrm{Spa}(B_i', B_i'^+) \to \mathrm{Spa}(A, A^+)$ consists entirely of points v for which $v(\sharp(\overline{x}_j)) \leq v(\sharp(\overline{x}_i))$ for $j = 1, \ldots, n$, and hence is contained in $\mathrm{Spa}(B_i, B_i^+)$. It follows that $B_i' \to B_i$ must in fact be an isomorphism, and so (B_i, B_i^+) is perfectoid.

For some suitably large m, $\varpi := \sharp(\overline{g}^{p^{-m}})$ is a pseudouniformizer of B_i such that ϖ^p divides p in B_i^+ and $\varphi : B_i^+/(\varpi) \to B_i^+/(\varpi^p)$ is bijective. We may thus apply Lemma 2.7.7 to deduce that $\mathrm{Spa}(B_i, B_i^+) \to \mathrm{Spa}(B_i^\flat, B_i^{\flat+})$ is bijective, and hence that $\mathcal{M}(B_i) \to \mathcal{M}(B_i^\flat)$ is bijective.

For $\alpha \in \mathcal{M}(A^\flat)$ with $\alpha(\overline{x}_1), \ldots, \alpha(\overline{x}_n)$ not all zero, we can find an index $i \in \{1, \ldots, n\}$ for which $\max\{\alpha(\overline{x}_1), \ldots, \alpha(\overline{x}_n)\} = \alpha(\overline{x}_i) \neq 0$, and then α belongs to the image of $\mathcal{M}(B_i^\flat)$. In particular, the joint zero locus Z of $\overline{x}_1, \ldots, \overline{x}_n$ in $\mathcal{M}(A^\flat)$, which is closed, has as complement the image of $\mathcal{M}(B_1^\flat \oplus \cdots \oplus B_n^\flat)$ in $\mathcal{M}(A^\flat)$, which is also closed (because the image of a continuous map from a quasicompact space to a Hausdorff space is closed). Consequently, Z is a closed-open subset of $\mathcal{M}(A^\flat)$; since $A^\flat$ is a Banach algebra over the trivially valued field $\mathbb{F}_p$, we may apply [13, Theorem 7.4.1] to realize Z as the zero locus of some idempotent element $\overline{e} \in A^\flat$. Put $e = \sharp(\overline{e})$; then e is an idempotent of A which vanishes nowhere, and so $e = 1$. It follows that $\overline{e} = 1$, $Z = \emptyset$, and $A^\flat$ is analytic. □

Lemma 2.7.9. *The formula*

$$(A, A^+) \mapsto (R := A^\flat, R^+ := A^{\flat+}, I := \ker(\theta : W(R^+) \to A^+))$$

defines a functor from perfectoid pairs (A, A^+) to triples (R, R^+, I), in which (R, R^+) is a perfectoid pair of characteristic p and I is a primitive ideal of $W(R^+)$. This functor and the functor from Lemma 2.6.14 are quasi-inverses of each other, so they are both equivalences of categories.

Proof. By Lemma 2.7.1 and Lemma 2.7.8, (R, R^+) is a perfectoid pair of characteristic p. By Lemma 2.7.5, I is a primitive ideal. It is evident that this functor and the functor from Lemma 2.6.14 are quasi-inverses of each other. □

We have now established Theorem 2.3.9, and thus are free to invoke it in subsequent proofs.

Lemma 2.7.10. *The map $\theta : W^b(R) \to A$ induces a surjection $W(R^\circ) \to A^\circ$ and an isomorphism $R^\circ/R^{\circ\circ} \to A^\circ/A^{\circ\circ}$. (Recall that $A^{\circ\circ}$ denotes the set of topologically nilpotent elements of A.)*

Proof. It is clear that $\theta(W(R^\circ)) \subseteq A^\circ$; conversely, by Lemma 2.6.9 every element of A° lifts to an element of $W(R^\circ)$. This produces the surjection $W(R^\circ) \to A^\circ$; since p is topologically nilpotent in A, the map $W(R^\circ) \to A^\circ \to A^\circ/A^{\circ\circ}$ factors through a surjection $R^\circ/R^{\circ\circ} \to A^\circ/A^{\circ\circ}$. Using Lemma 2.6.9 again, we see that this map is injective. □

2.8. **More proofs.** We continue to establish the basic properties of perfectoid rings.

Lemma 2.8.1. *Let $f : (A, A^+) \to (B, B^+)$ be a strict inclusion of perfectoid Huber pairs. Let $z \in W(A^{\flat+})$ be a generator of $\ker(\theta)$. Then within $W^b(B^\flat)$ we have the equalities*

$$zW(B^{\flat+}) \cap W(A^{\flat+}) = zW(A^{\flat+}), \qquad zW^b(B^\flat) \cap W^b(A^\flat) = zW(B^\flat)$$

Proof. We check the first assertion, the second being similar. If $x \in W(A^{\flat +})$ can be written in $W(B^{\flat +})$ as yz for some y, then from the shape of z we see that y is congruent modulo $[\overline{z}]$ to an element of $W(A^{\flat +})$. Writing $y = w_0 + zy_1$ with $w_0 \in W(A^{\flat +})$, we may repeat the argument to see that y_1 is congruent modulo $[\overline{z}]$ to an element of $W(A^{\flat +})$, and so on. Since $W(A^{\flat +})$ is complete for the $[\overline{z}]$-adic topology, we deduce the claim. $\square$

We have now established Theorem 2.4.3, and thus are free to use it in subsequent proofs.

Lemma 2.8.2. *Let (A, A^+) be a perfectoid Huber pair. Then for any positive integer m, there exists an ideal of definition I_m of A^+ such that $p \in I_m^{p^m}$.*

Proof. The case $m = 1$ is included in Definition 2.1.1. Given an ideal of definition I_m such that $p \in I^{p^m}$, choose generators $x_1, \dots, x_n$ of I_m. By Remark 2.1.5, there exist elements $y_1, \dots, y_n$ of A^+ such that $y_i^p \equiv x_i \pmod{I_m^p}$; it follows easily that $y_1^p, \dots, y_m^p$ are topologically nilpotent and generate I_m. Hence the ideal I_{m+1} of A^+ generated by $y_1, \dots, y_m$ is also an ideal of definition and satisfies $I_{m+1}^p = I_m$; hence $p \in I^{p^{m+1}}$ as desired. $\square$

The following metric criterion for the perfectoid property is adapted from [107, Proposition 3.6.2].

Lemma 2.8.3. *Let (A, A^+) be a uniform Huber pair and let I be an ideal of definition of A^+ such that $p \in I^p$. Using the ideal I, promote A to a uniform Banach ring as per Remark 1.5.4. Then A is perfectoid if and only if there exists some $c \in (0, 1)$ such that for every $x \in A$, there exists $y \in A$ with $|x - y^p| \leq c\,|x|$.*

Proof. If there exists some c as described, then for m sufficiently large, the ideal of definition I_m given by Lemma 2.8.2 has the property that $\varphi : A^\circ/I_m \to A^\circ/I_m^p$ is surjective, so A is perfectoid. Conversely, suppose that A is perfectoid. By Theorem 2.3.9, the map $\theta : W^b(A^\flat) \to A$ is surjective with kernel generated by some primitive element $z \in W(A^{\flat +})$. Put $c := |\overline{z}_0|$. By Lemma 2.6.9, for each $x \in A$ there exists $\overline{y} \in A^\flat$ such that $|x - \sharp(\overline{y})| \leq c\,|x|$; we may then take $y = \sharp(\overline{y}^{1/p})$. $\square$

Lemma 2.8.4. *Let $f : A \to B$ be a morphism of uniform Banach rings with dense image. If A is perfectoid, then B is perfectoid and $f^\flat$ has dense image.*

Proof. By arguing as in Lemma 2.7.1, we see that $B^\flat$ is a uniform Huber ring which is perfect of characteristic p; it is also analytic because it receives the continuous map $f^\flat$ from $A^\flat$. By Theorem 2.3.9, we have $A \cong W^b(A^\flat)/(z)$ for some primitive element $z \in W(A^{\flat +})$. Let $B_0^\flat$ be the closure of the image of $A^\flat$ in $B^\flat$, which is a perfectoid ring of characteristic p, and set $B_0 := W^b(B_0^\flat)/(z)$. Since the composition $W^b(A^\flat) \to A \to B$ has dense image, so does the induced map $B_0 \to B$. We may thus use B_0 to verify that B satisfies the condition of Lemma 2.8.3, so B is perfectoid. The map $B_0^\flat \to B^\flat$ is a strict inclusion, as then is $B_0 \to B$ by Theorem 2.4.3; since the latter map also has dense image, it is an isomorphism. Hence $f^\flat$ has dense image. $\square$

Lemma 2.8.5. *Let $f : A \to B$ be a surjective morphism of perfectoid Banach rings. Then the quotient norm induced by the spectral norm on A coincides with the spectral norm on B. (In other words, $B^+/f(A^+)$ is an almost zero B^+-module.)*

Proof. We adapt the argument from [107, Proposition 3.6.9(c)]. Since f is surjective, the induced map $\mathcal{M}(B) \to \mathcal{M}(A)$ is injective; by Lemma 1.5.22, it follows that f has operator norm at most 1 (i.e., it is *submetric*). By Theorem 1.1.9, the quotient norm on B is bounded by some constant $c > 1$ times the given norm. It will suffice to check that c can be replaced by $c^{1/p}$; in fact, it further suffices to check that for every $b \in B$, there exists $a \in A$ such that $|a| \leq c\,|b|$ and $|b - f(a)| \leq p^{-1/p}\,|b|$ (as we may then iterate the construction).

To begin, lift b^p to $a' \in A$ with $|a'| \leq c\,|b^p|$. Choose some lift x of a' to $W^b(A^\flat)$, then construct the sequence x_m as in Lemma 2.6.9 with respect to a primitive generator of $\ker(\theta : W^b(A^\flat) \to A)$ provided by Lemma 2.7.5. For m sufficiently large,

$$\alpha(a' - \sharp(\overline{x}_{m,0})) \leq p^{-1} \max\{\alpha(a'), |b^p|\} \qquad (\alpha \in \mathcal{M}(A)). \tag{2.8.5.1}$$

We claim that $a := \sharp(\overline{x}_{m,0}^{1/p})$ has the desired property. To see this, we may use Lemma 1.5.22 to reformulate the desired inequality as

$$\beta(b - f(a)) \leq p^{-1/p}\beta(b) \qquad (\beta \in \mathcal{M}(B)). \tag{2.8.5.2}$$

To check this, we first deduce from (2.8.5.1) that $\beta(b^p - f(a)^p) \leq p^{-1}\,|b^p|$. If $\beta(p) > 0$, we may deduce (2.8.5.2) from a simple analysis of the p-th power map in a mixed-characteristic nonarchimedean field [99, Lemma 10.2.2]. If instead $\beta(p) = 0$, we may instead deduce (2.8.5.2) more trivially. $\square$

Lemma 2.8.6. *Let $f : A \to B$ be a surjective morphism of perfectoid rings. Then $f^\flat$ is also surjective.*

Proof. We adapt the argument from [107, Proposition 3.6.9(d)]. Promote A and B to uniform Banach rings (equipped with their spectral norms) as per Remark 1.5.4. It will suffice to check that each $\overline{b} \in B^\flat$ can be lifted to some $\overline{a} \in A^\flat$ with $|\overline{a}| \leq p^{1/2}\,|\overline{b}|$; in fact, it further suffices to check that there exists $\overline{a}$ with $|\overline{a}| \leq p^{1/2}\,|\overline{b}|$ and $|\overline{b} - f^\flat(\overline{a})| \leq p^{-1/2}\,|\overline{b}|$ (as we may then iterate the construction).

Apply Lemma 2.8.5, we may lift $\sharp(\overline{b}) \in B$ to $a' \in A$ with $|a'| \leq p^{1/2}\,|\overline{b}|$. Using Lemma 2.6.9 as in the proof of Lemma 2.8.5, we can lift a' to $x \in W^b(A^\flat)$ so that

$$|a' - \sharp(\overline{x}_0)| \leq p^{-1}\max\{|a'|, |\overline{b}|\}.$$

We claim that $\overline{a} := \overline{x}_0$ has the desired property. To see this, apply Remark 2.2.8 to deduce that

$$\left|\sharp(\overline{b}) - \sharp(f^\flat(\overline{a})) - \sharp(\overline{b} - f^\flat(\overline{a}))\right| \leq p^{-1/2}\,|\overline{b}|.$$

From this, it follows that

$$\left|\overline{b} - f^\flat(\overline{a})\right| = \left|\sharp(\overline{b} - f^\flat(\overline{a}))\right| \leq p^{-1/2}\,|\overline{b}|$$

as desired. $\square$

Lemma 2.8.7. *Let (A, A^+) be a perfectoid pair. Let z be a generator of* $\ker(\theta : W(A^{\flat+}) \to A^+)$. *Let $(A, A^+) \to (B, B^+), (A, A^+) \to (C, C^+)$ be two morphisms of perfectoid pairs. Then $(B\widehat{\otimes}_A C, B^+\widehat{\otimes}_{A^+} C^+)$ is the untilt of $(B^\flat\widehat{\otimes}_{A^\flat} C^\flat, B^{\flat+}\widehat{\otimes}_{A^{\flat+}} C^{\flat+})$ corresponding to the ideal (z).*

Proof. This is clear in the case where C is equal to the perfectoid Tate algebra $A\langle T_s^{p^{-\infty}} : s \in S\rangle$ for some possibly infinite index set S (i.e., the completion of the perfect polynomial ring $A[T_s^{p^{-\infty}} : s \in S]$ for the Gauss norm) and $C^+ = A^+\langle T_s^{p^{-\infty}} : s \in S\rangle$, as then the completed tensor product is obtained by substituting B for A, and likewise on the tilt side. The general case follows from the right exactness of the tensor product (compare [108, Theorem 3.3.13]). $\square$

Lemma 2.8.8. *Suppose that $\overline{f}_1, \dots, \overline{f}_n, \overline{g} \in A^\flat$ are elements such that $\sharp(\overline{f}_1), \dots, \sharp(\overline{f}_n), \sharp(\overline{g})$ generate the unit ideal in A. Let $(A, A^+) \to (B, B^+)$ be the rational localization defined by these parameters. Then*

$$B \cong A\langle T_1^{p^{-\infty}}, \dots, T_n^{p^{-\infty}} \rangle / (\sharp(\overline{g}^{p^{-j}}) T_i^{p^{-j}} - \sharp(\overline{f}^{p^{-j}}) : i = 1, \dots, n; j = 0, 1, \dots)^\wedge.$$

In particular, (B, B^+) is an untilt of the localization of $(A^\flat, A^{\flat +})$ defined by $\overline{f}_1, \dots, \overline{f}_n, \overline{g}$.

Proof. We emulate [107, Remark 3.6.16]. Denote the quotient being compared to B as B'. Choose $h_1, \dots, h_n, k \in A$ such that $h_1 \sharp(\overline{f}_1) + \cdots + h_n \sharp(\overline{f}_n) + k \sharp(\overline{g}) = 1$. For $i_1, \dots, i_n \in \mathbb{Z}[p^{-1}]_{\geq 0}$, $T_1^{i_1} \cdots T_n^{i_n}$ represents the same class in B' as

$$(k + h_1 T_1 + \cdots + h_n T_n)^n \sharp(\overline{f}_1^{i_1 - \lfloor i_1 \rfloor}) \cdots \sharp(\overline{f}_n^{i_n - \lfloor i_n \rfloor}) \sharp(\overline{g}^{n - (i_1 - \lfloor i_1 \rfloor + \cdots + i_n - \lfloor i_n \rfloor)}) T_1^{\lfloor i_1 \rfloor} \cdots T_n^{\lfloor i_n \rfloor}.$$

We thus construct an inverse of the map $B \to B'$. ☐

Lemma 2.8.9. *Theorem 2.5.9 holds in the case where A is a perfectoid field. Moreover, the tilting operation preserves the degrees of field extensions.*

Proof. See [163, Lecture 2] or the references in Remark 2.5.12. ☐

Lemma 2.8.10. *Let $R \to S$ be a finite (resp. finite étale) morphism of perfectoid rings of characteristic p. Then any untilt of this morphism is finite (resp. finite étale).*

Proof. Let $A \to B$ be an untilt of $R \to S$. Choose $\overline{x}_1, \dots, \overline{x}_n$ which generate S as an R-module. By the open mapping theorem, the resulting map $R^n \to S$ is strict; it follows easily from this that $[\overline{x}_1], \dots, [\overline{x}_n]$ generate $W^b(S)$ over $W^b(R)$, and hence that $\sharp(\overline{x}_1), \dots, \sharp(\overline{x}_n)$ generate B over A.

Suppose now that $R \to S$ is finite étale. Since A is uniform, as in [107, Proposition 2.8.4] we may check that $A \to B$ is finite flat by checking that its rank is locally constant, which follows from Lemma 2.8.9. (Note that this argument uses essentially the fact that A is reduced; compare [48, Exercise 20.13].)

Now recall that $R \to S$ is finite étale if and only if both $R \to S$ and $S \otimes_R S \to S$ are finite flat. (The "only if" direction is obvious. The "if" direction holds because $S \otimes_R S \to S$ being flat implies that $R \to S$ is formally unramified [152, Tag 092M]. See also the discussion of *weakly étale morphisms* in [152, Tag 092A].) By the compatibility of untilting with tensor products (Lemma 2.8.7), we may repeat the previous argument to see that $B \otimes_A B \to B$ is finite flat, and then deduce that $A \to B$ is finite étale. ☐

Lemma 2.8.11. *Let (A, A^+) be a perfectoid pair. Let z be a generator of* $\ker(\theta : W(A^{\flat +}) \to A^+)$. *Then the functor $B^\flat \mapsto W^b(B^\flat)/(z)$ defines an equivalence of categories between finite étale $B^\flat$-algebras and finite étale B-algebras.*

Proof. The functor is well-defined by Lemma 2.8.10, and fully faithful by Theorem 2.3.9; it thus suffices to check essential surjectivity. In the case where A is a perfectoid field, essential surjectivity follows from Lemma 2.8.9. In the general case, given a finite étale morphism $A \to B$, we may combine the field case with the henselian property of local rings (Remark 1.2.5) to produce a rational covering $\{(A, A^+) \to (A_i, A_i^+)\}$ such that for each i, $B \otimes_A A_i$ is the untilt of some finite étale A_i-algebra. By full faithfulness, these modules collate to give a finite étale $\mathcal{O}$-module on $\mathrm{Spa}(A^\flat, A^{\flat +})$. Since $\mathrm{Spa}(A^\flat, A^{\flat +})$ is sheafy by Corollary 2.5.4, we may apply Theorem 1.4.2 (or Remark 1.4.3) to obtain a finite étale $A^\flat$-module $B^\flat$ which untilts to B. In particular, B is perfectoid. ☐

Exercise 2.8.12. Let A be a perfectoid ring. Note that in general, not every finite A-algebra is perfectoid, even if we restrict to characteristic p (trivially by adjoining nilpotents to destroy uniformity, or less trivially as in Exercise 2.5.8 where the result is still uniform). Nonetheless, show that $B \to B^\flat$ defines an equivalence of categories between *perfectoid* finite A-algebras and perfectoid finite $A^\flat$-algebras.

2.9. **Additional results about perfectoid rings.** We mention some additional results which we will not prove here.

Theorem 2.9.1 (Kedlaya)**.** *Any perfectoid ring whose underlying ring is a field is a perfectoid field. (That is, it is not necessary to assume in advance that the topology is given by a multiplicative norm.)*

Proof. Any such ring is Tate (not just analytic), so [104, Theorem 4.2] applies. □

Corollary 2.9.2. *Let A be a perfectoid ring. Let I be a maximal ideal of A (which is automatically closed; see Remark 1.1.1). If A/I is uniform, then it is a perfectoid field.*

Proof. If A/I is uniform, it is again a perfectoid ring, and so Theorem 2.9.1 applies. □

The following corollary is analogous to a standard fact about perfect rings.

Corollary 2.9.3. *Any noetherian perfectoid ring is a finite direct sum of perfectoid fields.*

Proof. Let A be a noetherian perfectoid ring. For $\overline{x} \in A^\flat$, the sequence of ideals $(\sharp(\overline{x}^{p^{-n}}))_n$ of A forms an ascending chain, and hence must stabilize. That is, there exists a positive integer n such that for $y = \sharp(\overline{x}^{p^{-n}})$, we have $y = wy^p$ for some $w \in A$. For such w, $y^{p-1}w$ is an idempotent in A, which defines a splitting $A \cong A_1 \oplus A_2$ of perfectoid rings by projecting onto A_1. Since $y(y^{p-1}w) = y$, y must project to a unit in A_1 and to zero in A_2; consequently, $\overline{x}$ must project to a unit in $A_1^\flat$ and to zero in $A_2^\flat$. In other words, every element of $A^\flat$ equals a unit times an idempotent. Since idempotent ideals in $A^\flat$ satisfy the ascending chain condition (as seen by applying $\sharp$), we deduce that $A^\flat$ is a finite direct sum of fields, each of which must be a perfectoid field by Theorem 2.9.1. □

Definition 2.9.4. For A a perfectoid ring, an A^+-module M is *almost zero* if it is annihilated by every topologically nilpotent element of A^+. Such modules form a thick Serre subcategory of the category of A^+-modules, so one may form the quotient category.

Theorem 2.9.5. *Let (A, A^+) be a perfectoid pair. Then for each $i > 0$, the A^+-modules $H^i(\mathrm{Spa}(A, A^+), \mathcal{O}^+)$ and $H^i(\mathrm{Spa}(A, A^+)_{\mathrm{et}}, \mathcal{O}^+)$ are almost zero.*

Proof. See [142, Lemma 6.3(iv)] in the case where A is an algebra over a perfectoid field, [107, Lemma 9.2.8] in the case where A is an algebra over $\mathbb{Q}_p$, or [108, Theorem 3.3.20] in the case where A is Tate. The analytic case is similar. □

In the Tate case, the following statement is [108, Theorem 3.7.4].

Exercise 2.9.6. A *seminormal* ring is a ring R in which the map

$$R \to \{(y, z) \in R \times R : y^3 = z^2\}, \qquad x \mapsto (x^2, x^3)$$

is an isomorphism. This definition is due to Swan [154]. Using Theorem 2.9.5, show that any perfectoid ring is seminormal. (Hint: work locally around a point $v \in \mathrm{Spa}(A, A^+)$, distinguishing between the cases where $v(y), v(z)$ are both zero or both nonzero.)

Theorem 2.9.7. *Let A be a uniform analytic Huber ring such that some faithfully finite étale (i.e., faithfully flat and finite étale) A-algebra is perfectoid. Then A is perfectoid.*

Proof. In the case where A is Tate, this is [108, Theorem 3.3.24]; the analytic case is similar. □

Problem 2.9.8. Does Theorem 2.9.7 remain true if "étale" is weakened to "flat"?

Theorem 2.9.9. *Let (A, A^+) be a (not necessarily sheafy) Huber pair in which A is Tate and p is topologically nilpotent, and put $X := \mathrm{Spa}(A, A^+)$. Then there exists a directed system $(A, A^+) \to (A_i, A_i^+)$ of faithfully finite étale morphisms such that the completion of $\varinjlim_i A_i$ for the seminorm induced by the spectral seminorm on each A_i is a perfectoid ring. (Note that the transition morphisms are isometric for the spectral seminorms.)*

Proof. For A a Huber ring over $\mathbb{Q}_p$, this follows from an argument of Colmez: for X affinoid, it suffices to repeatedly adjoin p-power roots of units. See [143, Proposition 4.8] (nominally in the locally noetherian case, but the argument does not depend on this) or [107, Lemma 3.6.26, Lemma 9.2.5]. In the Tate case, a modification of Colmez's argument by Scholze applies; see [108, Lemma 3.3.27]. (We have not checked whether this argument extends to the analytic case.) □

Remark 2.9.10. It is far from clear whether Theorem 2.9.9 remains true if we assume only that A is analytic, rather than Tate; there is no obvious mechanism to ensure in this case that A has "enough" finite étale extensions. Nonetheless, Theorem 2.9.9 implies that in any analytic adic space (or preadic space; see Definition 1.11.2) on which p is topologically nilpotent, in the pro-étale topology (to be introduced in Weinstein's third lecture [163, Lecture 3], but see also Definition 3.8.1) there exists a neighborhood basis consisting of perfectoid spaces. This fact underpins the use of perfectoid spaces in p-adic Hodge theory, as in the lectures of Bhatt [16] and Caraiani [26]; it also gives rise to the functor from analytic adic spaces in which p is topologically nilpotent to diamonds (Definition 4.3.1).

Remark 2.9.11. We postpone one more result until we have discussed fundamental groups: an amazing recent theorem of Achinger that asserts that adic affinoid spaces on which p is topologically nilpotent have no higher étale homotopy groups. See Theorem 4.1.26 and Corollary 4.1.27.

Remark 2.9.12. For discussion of various foundational problems concerning perfectoid rings and spaces in the spirit of the "Scottish Book" on functional analysis, see [106]. Another apt analogue in point-set topology is the book [153].

3. Sheaves on Fargues–Fontaine curves

We next pick up on a topic introduced in Weinstein's lectures [163]: the construction of Fargues–Fontaine which gives rise to a "moduli space of untilts" of a given perfectoid space. In this lecture, we study vector bundles and coherent sheaves on Fargues–Fontaine curves (associated to a perfectoid field) and relative Fargues–Fontaine curves (associated to a perfectoid ring or space), and a profound relationship between these sheaves and étale local systems. We will see in the final lecture how these results can be formally recast in a more suggestive manner that suggests how to put the analogy between mixed and equal characteristic on a firm footing.

Whereas in the first two lectures the notes constitute a fairly self-contained treatment of the material, some of the material in the last two lectures is far beyond the scope of what can be treated here. We thus revert to a more conventional order of presentation, in which we either prove statements on the spot or defer to external references.

3.1. Absolute and relative Fargues–Fontaine curves.

We begin by recalling the construction of Fargues–Fontaine [55, 56, 57] associated to a perfectoid field, and its generalization to perfectoid rings and spaces by Kedlaya–Liu [107, §8.7–8.8].

Hypothesis 3.1.1. Throughout §3.1, let (R, R^+) be a perfectoid pair of characteristic p and put $S = \mathrm{Spa}(R, R^+)$. Note that only the case where R is Tate is treated in [107].

Definition 3.1.2. Define the ring $\mathbf{A}_{\mathrm{inf}} := W(R^+)$. It is complete for the adic topology defined by the inverse image of some ideal of definition of R^+.

Lemma 3.1.3. *Choose topologically nilpotent elements $\overline{x}_1, \dots, \overline{x}_n \in R^+$ which generate the unit ideal in R.*

(a) *For the p-adic topology on $\mathbf{A}_{\mathrm{inf}}$, the ring $\mathbf{A}_{\mathrm{inf}}[p^{-1}]\langle \frac{[\overline{x}_1]}{p}, \dots, \frac{[\overline{x}_n]}{p} \rangle$ is stably uniform.*

(b) *For $i = 1, \dots, n$, for the $[\overline{x}_i]$-adic topology on $\mathbf{A}_{\mathrm{inf}}$, the ring $\mathbf{A}_{\mathrm{inf}}[[\overline{x}_i]^{-1}]\langle \frac{p}{[\overline{x}_i]}, \frac{[\overline{x}_1]}{[\overline{x}_i]}, \dots, \frac{[\overline{x}_n]}{[\overline{x}_i]} \rangle$ is stably uniform.*

Proof. We may check both claims using Corollary 2.5.5: in each case, taking the completed tensor product over $\mathbb{Z}_p$ with the p-adic completion of $\mathbb{Z}_p[p^{p^{-\infty}}]$ yields a perfectoid ring. Note that in case (b), we get an example of a perfectoid ring which is Tate but not a $\mathbb{Q}_p$-algebra. □

Remark 3.1.4. The ring appearing in Lemma 3.1.3(b) can be viewed as a subring of $W(R_i)$ where $(R, R^+) \to (R_i, R_i^+)$ is the rational localization with parameters $\overline{x}_1, \dots, \overline{x}_n, \overline{x}_i$. In particular, every element has a unique expansion $\sum_{n=0}^{\infty} p^n [\overline{y}_n]$ with $\overline{y}_n \in R_i$. By contrast, elements of the ring appearing in Lemma 3.1.3(a) do not necessarily admit expansions of the form $\sum_{n \in \mathbb{Z}} p^n [\overline{y}_n]$.

Definition 3.1.5. For the topology described in Definition 3.1.2, $\mathrm{Spa}(\mathbf{A}_{\mathrm{inf}}, \mathbf{A}_{\mathrm{inf}})$ is not analytic; the analytic locus $\mathrm{Spa}(\mathbf{A}_{\mathrm{inf}}, \mathbf{A}_{\mathrm{inf}})^{\mathrm{an}}$ consists of those v for which either $v(p) \neq 0$ *or* there exists a topologically nilpotent element $\overline{x}$ of R^+ for which $v([\overline{x}]) \neq 0$. For $\overline{x}_1, \dots, \overline{x}_n$ topologically nilpotent elements of R^+ which generate the unit ideal in R, $\mathrm{Spa}(\mathbf{A}_{\mathrm{inf}}, \mathbf{A}_{\mathrm{inf}})^{\mathrm{an}}$ can also be described as the set of v for which $v(p), v([\overline{x}_1]), \dots, v([\overline{x}_n])$ are not all zero. By Lemma 3.1.3, $\mathrm{Spa}(\mathbf{A}_{\mathrm{inf}}, \mathbf{A}_{\mathrm{inf}})^{\mathrm{an}}$ is a stably uniform adic space.

Let Y_S be the subspace of $\mathrm{Spa}(\mathbf{A}_{\mathrm{inf}}, \mathbf{A}_{\mathrm{inf}})^{\mathrm{an}}$ consisting of those v for which $v(p) \neq 0$ *and* there exists a topologically nilpotent element $\overline{x}$ of R^+ for which $v([\overline{x}]) \neq 0$. Again, the latter condition need only be tested for $\overline{x}$ running over a finite set of elements which generate the unit ideal in R.

The action of φ on Y_S is properly discontinuous. The quotient space $X_S := Y_S/\varphi^{\mathbb{Z}}$ in the category of locally v-ringed spaces is the *adic (relative) Fargues–Fontaine curve* over S, which we also denote by FF_S (especially in cases where we want to use X to mean another space).

Exercise 3.1.6. Prove that for any perfectoid space X over $\mathbb{Q}_p$, $X \times_{\mathbb{Q}_p} \mathrm{FF}_S$ is a perfectoid space.

Remark 3.1.7. With some effort, it can be shown that $\mathbf{A}_{\mathrm{inf}}$ is sheafy and hence $\mathrm{Spa}(\mathbf{A}_{\mathrm{inf}}, \mathbf{A}_{\mathrm{inf}})$ is itself a (nonanalytic) adic space (see Problem A.6.2). For the case where R is a nonarchimedean field, see Remark 3.1.10.

Remark 3.1.8. When developing the theory of relative Fargues–Fontaine curves, it is generally necessary to also consider the quotient of Y_S by $\varphi^{n\mathbb{Z}}$ for n a positive integer; this gives a finite étale covering of X_S with Galois group $\mathbb{Z}/n\mathbb{Z}$. To simplify the exposition, we (mostly) omit further mention of this construction.

Remark 3.1.9. Suppose that R is Tate, and let $\varpi \in R$ be a pseudouniformizer. We can then make the description of X_S somewhat more explicit. To begin with, Y_S is the subspace of $v \in \mathrm{Spa}(\mathbf{A}_{\mathrm{inf}}, \mathbf{A}_{\mathrm{inf}})$ for which $v(p[\varpi]) \neq 0$. This space can be covered by the subspaces

$$\begin{aligned} U_n &:= \{v \in Y_S : v(p)^{cp^n} \leq v(\varpi) \leq v(p)^{p^n}\}, \\ V_n &:= \{v \in Y_S : v(p)^{p^{n+1}} \leq v(\varpi) \leq v(p)^{cp^n}\} \qquad (n \in \mathbb{Z}), \end{aligned}$$

where $c \in (1,p) \cap \mathbb{Q}$ is arbitrary. The action of φ permutes the U_n (among themselves) and the V_n (among themselves), and hence is properly discontinuous. The spaces U_0 and V_0 map isomorphically to their images in X_S and cover the latter. In particular, X_S can be covered by two affinoid subspaces, so for every pseudocoherent sheaf $\mathcal{F}$ on X_S we have $H^i(X_S, \mathcal{F}) = 0$ for all $i > 1$.

Remark 3.1.10. Suppose that $R = F$ is a nonarchimedean field and $R^+ = \mathfrak{o}_F$. Then for any pseudouniformizer ϖ of F, the ring $\mathbf{A}_{\mathrm{inf}}[[\varpi]^{-1}]\langle \frac{p}{[\varpi]} \rangle$ admits euclidean division, and hence is a principal ideal domain; see [102, Corollary 2.10]. (This result appeared previously in [96, Lemma 2.6.3], but the proof contains several errors; see the online errata.) In addition, the ring $\mathbf{A}_{\mathrm{inf}}[[\varpi]^{-1}]\langle \frac{p}{[\varpi]} \rangle$ is strongly noetherian [102, Theorem 3.2]; consequently, in this case X_S is a noetherian adic space.

There is a useful illustration of the space $\mathrm{Spa}(\mathbf{A}_{\mathrm{inf}}, \mathbf{A}_{\mathrm{inf}})$ in the notes of Bhatt [16]. To summarize, as in Example 1.6.11 there is a unique point $v \in \mathrm{Spa}(\mathbf{A}_{\mathrm{inf}}, \mathbf{A}_{\mathrm{inf}})$ which is not analytic, namely the one with $v(p) = v([\varpi]) = 0$, and the only rational subspace containing v is the whole of $\mathrm{Spa}(\mathbf{A}_{\mathrm{inf}}, \mathbf{A}_{\mathrm{inf}})$. We may thus deduce that $\mathrm{Spa}(\mathbf{A}_{\mathrm{inf}}, \mathbf{A}_{\mathrm{inf}})$ is an adic space.

By contrast, if R is not a finite direct sum of perfectoid fields, then R itself cannot be noetherian (Corollary 2.9.3), and it is easily shown that X_S is not noetherian either. Also, one should not expect a particularly explicit description of X_S in this case, by analogy with the structure of Berkovich analytic spaces: these are reasonable to describe combinatorially in dimension 1 (or dimension 2 over a trivially valued base field) and unreasonable in higher dimensions.

Remark 3.1.11. The space Y_S is not affinoid, because it is not quasicompact. However, it is a *quasi-Stein space* in the category of adic spaces: it is a direct limit of affinoid subspaces where the transition maps induce dense inclusions of coordinate rings. For example, if R contains a pseudouniformizer ϖ, then the subspaces $\{v \in Y_S : v(p) \leq v(\varpi)^n, v(\varpi) \leq v(p)^n\}$ for $n = 0, 1, \dots$ form an ascending sequence of the desired form. Quasi-Stein spaces behave somewhat like affinoid spaces in that certain sheaves on them can be interpreted in terms of modules over coordinate rings. See [108, §2.6] for a detailed discussion.

The category of vector bundles on X_S can be interpreted as the category of φ-equivariant vector bundles on Y_S. In light of the previous paragraph, the latter can be interpreted as finite projective $\mathcal{O}(Y_S)$-modules equipped with φ-action.

Remark 3.1.12. Each point of X_S corresponds to a Huber pair (K, K^+) in which K is a perfectoid field; the tilting operation thus defines a map $X_S \to S$ which turns out to be a projection of topological spaces. This construction commutes with base change; in particular, for F a nonarchimedean field and $\mathrm{Spa}(F, \mathfrak{o}_F) \to S$ a morphism, the fiber of X_S over $\mathrm{Spa}(F, \mathfrak{o}_F)$ coincides with the adic Fargues–Fontaine curve over F. In this sense, X_S is a "family of curves" over S. However, this projection map is not a morphism of adic spaces. (It will acquire an interpretation in the language of diamonds; see Definition 4.3.7.)

Given an untilt (A, A^+) of (R, R^+), Theorem 2.3.9 produces a primitive element z of $W(R^+)$ such that $W(R^+)/(z) \cong A^+$ via the theta map. If A is a $\mathbb{Q}_p$-algebra, then the element z gives rise to a closed immersion of $\mathrm{Spa}(A^+, A^+)$ into $\mathrm{Spa}(\mathbf{A}_{\mathrm{inf}}, \mathbf{A}_{\mathrm{inf}})$, which restricts (using Lemma 1.6.5) to a closed immersion of $\mathrm{Spa}(A, A^+)$ into X_S. If we identify $\mathrm{Spa}(A, A^+)$ with $\mathrm{Spa}(R, R^+)$ via Theorem 2.5.1, then this closed immersion becomes a section of the projection $|X_S| \to |S|$ in the category of topological spaces.

Remark 3.1.13. Promote R to a uniform Banach ring with norm $|\bullet|$ as per Remark 1.5.4. The coordinate ring $\mathcal{O}(Y_S)$ can then be interpreted as the Fréchet completion of $\mathbf{A}_{\mathrm{inf}}[p^{-1}]$ for the family of Gauss norms (as in Definition 2.6.2) corresponding to the norms $|\bullet|^r$ for all $r > 0$. In [107], this ring appears under the notation $\tilde{\mathcal{R}}^{\infty}_R$ and is an example of an *extended Robba ring*, named by analogy with Remark 3.5.4.

Using the norm on R, one can construct a deformation retract of the maximal Hausdorff quotient of the space X_S (see Remark 1.5.15) onto a suitable section of $|X_S| \to |S|$; this implies that $|X_S| \to |S|$ has contractible fibers. See [100, Theorem 7.8].

Definition 3.1.14. Let $\mathcal{O}(1)$ be the line bundle on X_S corresponding to the trivial line bundle on Y_S on a generator $\mathbf{v}$, with the isomorphism $\varphi^*\mathcal{O}(1) \cong \mathcal{O}(1)$ given by $1 \otimes \mathbf{v} \mapsto p^{-1}\mathbf{v}$.

Via the following theorem, the sheaf $\mathcal{O}(1)$ may be viewed as an ample line bundle on X_S. This makes it possible to compare X_S to a schematic construction.

Theorem 3.1.15. *Suppose either that R is Tate and $\mathcal{F}$ is a vector bundle on X_S, or that $(R, R^+) = (F, \mathfrak{o}_F)$ for some nonarchimedean field F and $\mathcal{F}$ is a coherent sheaf on X_S. For $n \in \mathbb{Z}$, define the twisted sheaf $\mathcal{F}(n) := \mathcal{F} \otimes \mathcal{O}(1)^{\otimes n}$. Then for all sufficiently large n, the following statements hold.*

(a) *We have $H^1(X_S, \mathcal{F}(n)) = 0$.*
(b) *The sheaf $\mathcal{F}(n)$ is generated by finitely many global sections.*

Proof. See [107, Lemma 8.8.4, Proposition 8.8.6]. □

Definition 3.1.16. Define the graded ring

$$P_S := \bigoplus_{n=0}^{\infty} P_{S,n}, \qquad P_{S,n} := H^0(X_S, \mathcal{O}(n)).$$

The scheme $\mathrm{Proj}(P_S)$ is called the *schematic Fargues–Fontaine curve* over S. By construction, there is a morphism $X_S \to \mathrm{Proj}(P_S)$ of locally ringed spaces.

By analogy with Serre's GAGA theorem for complex algebraic varieties [150], we have the following result.

Theorem 3.1.17. *The morphism $X_S \to \mathrm{Proj}(P_S)$ has the following properties.*

(a) *Suppose that R is Tate. Then pullback of vector bundles from $\mathrm{Proj}(P_S)$ to X_S defines an equivalence of categories.*
(b) *Suppose that $(R, R^+) = (F, \mathfrak{o}_F)$ for some nonarchimedean field F. Then pullback of coherent sheaves from $\mathrm{Proj}(P_S)$ to X_S is an equivalence of categories.*
(c) *In both (a) and (b), the pullback functor preserves sheaf cohomology.*

Proof. As in the usual GAGA theorem, the strategy is to first prove preservation of H^1, then preservation of H^0, then full faithfulness of the pullback functor, then essential surjectivity of the pullback functor. At each stage, one uses Theorem 3.1.15 to reduce to considering the sheaves $\mathcal{O}(n)$ for $n \in \mathbb{Z}$, which one studies by comparing $\mathcal{O}(n)$ with $\mathcal{O}(n+1)$. For more details, see [107, Theorem 6.3.12] for (a) and [108, Theorem 4.7.4] for (b). □

Lemma 3.1.18. *Let $s \in H^0(X_S, \mathcal{O}(1))$ be a section which does not vanish on any fiber of X_S (that is, its pullback to $X_{\mathrm{Spa}(F,\mathfrak{o}_F)}$ does not vanish for any nonarchimedean field F). Let $\mathcal{I} \subset \mathcal{O}$ be the image of $s \otimes \mathcal{O}(-1)$ in $\mathcal{O}$. Then the zero locus Z of $\mathcal{I}$ is an untilt of S, and the projection $|X_S| \to |S|$ restricts to a homeomorphism $|Z(\mathcal{I})| \cong |S|$.*

Proof. We may work locally on S. For starters, we may assume S admits a pseudouniformizer $\overline{\varpi}$; as per Remark 3.1.9, we may further assume that Z is contained in the affinoid subspace $U = \{v \in Y_S : v(p)^c \leq v(\varpi) \leq v(p)\}$ for some $c \in (1,p) \cap \mathbb{Q}$. We may also assume that Z is cut out by a single element $f \in H^0(U, \mathcal{O})$.

In the case where S is a point, we can perform a Weierstrass factorization as in [96, Lemma 2.6.7] to write f as a multiple of a primitive element. In the general case, we may make the same argument in a suitably small neighborhood of any particular $x \in S$, and thus deduce the claim. (More details may be added here later.) □

Remark 3.1.19. It is unclear whether Theorem 3.1.15 and Theorem 3.1.17 extend to the case where R is analytic, not just Tate. One serious difficulty in that case is that by Corollary 2.6.16, if R is not Tate then it admits no untilts over $\mathbb{Q}_p$, so by Lemma 3.1.18, $H^0(X_S, \mathcal{O}(1))$ cannot contain an element which does not vanish identically on any fiber. As a result, even the nonvanishing of $H^0(X_S, \mathcal{O}(n))$ for n large is unclear.

Remark 3.1.20. Suppose that $R = F$ is a nonarchimedean field and $R^+ = \mathfrak{o}_F$. Let K be an untilt of F. As in Remark 3.1.12, the map $\theta : W(\mathfrak{o}_F) \to \mathfrak{o}_K$ is surjective with kernel generated by some primitive element z. If K is of characteristic p, then $K = F$ and we may take $z = p$.

Suppose hereafter that K is of characteristic 0. The zero locus of z in Y_S is a single point; projecting this point from Y_S to X_S amounts to forgetting the difference between F and its images under powers of Frobenius. From an algebraic point of view, this makes sense to do because Frobenius commutes with all automorphisms, and so this forgetting does not mess up any functoriality.

However, not all points of X_S arise in this fashion, even if F is algebraically closed. Consider by way of analogy the points on the adic projective line over F. There, the points of height 1 are conventionally divided into four types (following [13, Example 1.4.3]):

1. rigid-analytic points over a completed algebraic closure of F;
2. generic points of (virtual) closed discs of rational radius;
3. generic points of (virtual) closed discs of irrational radius;
4. points which witness the failure of F to be spherically complete (i.e., equivalence classes of descending chains of closed discs with empty rigid-analytic intersection).

The points of higher height are considered to be a fifth type; the type 5 points are specializations of type 2 points (see [142, Example 2.20] for an illustration).

The structure of X_S is quite analogous to this. For example, see [100, Theorem 8.17] for a classification of the height 1 points which reproduces many features of the Berkovich classification.

Remark 3.1.21. The previous construction globalizes to give an adic (relative) Fargues–Fontaine curve over any perfectoid space of characteristic p. Better yet, one may take the base space to be a suitably nice stack on the category of perfectoid spaces, such as a diamond; see Definition 4.3.7.

Remark 3.1.22. The original construction of Fargues–Fontaine dates back to 2006, although the manuscript [57] is still unfinished as of this writing. As this origin precedes the general promulgation of the theory of perfectoid fields, the original construction involved only algebraically closed nonarchimedean fields, and yielded only the schematic curves (Definition 3.1.16).

The relative Fargues–Fontaine curves, in both the schematic and adic versions over a Tate base ring, were introduced by Kedlaya–Liu in [107].

3.2. **An analogy: vector bundles on Riemann surfaces.** We continue with an analogy from classical algebraic geometry that will inform our work.

Definition 3.2.1. Let X be a smooth proper curve over a field k. Recall that the *degree* of a line bundle on X is defined as the degree of the divisor associated to any nonzero rational section (noting that any two such divisors differ by a principal divisor, whose degree is 0). Define the *degree* of a vector bundle V of rank n as the degree of $\wedge^n V$, denoted $\deg(V)$. Define the *slope* of a nonzero vector bundle V as the ratio

$$\mu(V) := \frac{\deg(V)}{\operatorname{rank}(V)};$$

we say V is *semistable* (resp. *stable*) if V contains no proper nonzero subbundle V' with $\mu(V') > \mu(V)$ (resp. $\mu(V') \geq \mu(V)$). Every semistable bundle is a successive extension of stable bundles of the same slope.

One can think of stable vector bundles as the building blocks out of which arbitrary vector bundles are built. One result in this direction is a theorem of Harder–Narasimhan [78], to the effect that every bundle admits a certain canonical filtration with semistable quotients. We will see a more general version of this result later (Theorem 3.4.11).

Remark 3.2.2. When $X \cong \mathbf{P}^1_k$, then $\mathrm{Pic}(X) \cong \mathbb{Z}$ with the inverse map being $n \mapsto \mathcal{O}(n)$, and a theorem of Grothendieck [71, 82] states that every vector bundle splits (nonuniquely) as a direct sum of line bundles. A nonzero vector bundle V is semistable if and only if it splits (noncanonically) as a direct sum $\mathcal{O}(n)^{\oplus m}$ for some m, n; in particular, every semistable bundle has integral slope. (These statements were extended by Harder [77] to G-bundles on $\mathbf{P}^1_k$, for G a split reductive algebraic group.)

This example is somewhat misleading in its simplicity. In general, not every semistable bundle has integral slope; in particular, not every bundle splits as a direct sum of line bundles. For example, if X is a curve of genus 1 and k is algebraically closed of characteristic 0, a theorem of Atiyah [9] implies that for any line bundle L on X of odd degree, there is a unique stable vector bundle V of rank 2 such that $\wedge^2 V \cong L$.

Remark 3.2.3. Because the definitions of stable and semistable vector bundles are nonexistence criteria, rather than existence criteria, they can be problematic to work with. For example, it is not apparent from the definition that the pullback of a (semi)stable bundle along a morphism of curves is again (semi)stable: the pullback bundle may have a subbundle that witnesses the failure of (semi)stability but is not itself the pullback of a subbundle on the original curve. For another example, for any two nonzero bundles V, V' on X we have

$$\mu(V \otimes V') = \mu(V) + \mu(V'),$$

but it is not apparent that the tensor product of two semistable bundles is again semistable.

The subtlety of this point is illustrated by the fact that the situation depends crucially on the characteristic of k. In characteristic 0, both pullback and tensor product preserve semistability; this can be proved either algebraically (see [89, Chapter 3]) or by using the Lefschetz principle to reduce to the case $k = \mathbb{C}$, then appealing to a "positive" but transcendental characterization of semistability (see Theorem 3.2.4). By contrast, in characteristic p, semistability is preserved by pullback along separable morphisms [89, Lemma 3.2.2] but not along Frobenius (see [136] for further discussion of this phenomenon); semistability is also not preserved by tensor products, as first shown by Gieseker [67].

In characteristic 0, the issues in the previous remark are resolved by the following theorem of Narasimhan–Seshadri [131], later reproved by Donaldson [42].

Theorem 3.2.4. *Assume that $k = \mathbb{C}$, and choose a closed point $x_0 \in X$. Then a vector bundle V of rank n on X is stable of slope 0 if and only if it admits a connection $\nabla : V \to V \otimes_{\mathcal{O}_X} \Omega_{X/k}$ whose holonomy representation $\rho : \pi_1(X^{\mathrm{an}}, x_0) \to \mathrm{GL}_n(\mathbb{C})$ (i.e., the one from the Riemann–Hilbert correspondence, obtained by analytic continuation of local sections in the kernel of ∇) is irreducible and unitary. In the latter case, ∇ and ρ are uniquely determined by V.*

Remark 3.2.5. The use of the terms *stable* and *semistable* in this manner stems from the original context in which these notions were studied, via geometric invariant theory. Assume (for simplicity) that k is of characteristic 0. For $G \subseteq \mathrm{GL}(n)_k$ a reductive k-algebraic group, a point $x \in \mathbf{A}^n_k$ is *stable* for the action of G if its stabilizer in G is finite and its G-orbit is closed in $\mathbf{A}^n_k$ (the finite-stabilizer condition is comparable to the definition of a *stable curve* as one with a finite automorphism group) Given a vector bundle V of rank n of a particular rank and degree on X, one can tensor with a suitable ample line bundle to obtain a bundle generated by global sections; this gives a point x in a certain affine space carrying a linear action of $G = \mathrm{GL}(n)_k$, which is stable for the action if and only if V is stable as a bundle. This makes it possible to construct and study moduli spaces of stable bundles by quotienting a certain orbit space for the action of G.

3.3. **The formalism of slopes.** We next describe a more general framework in which slopes and (semi)stability can be considered. The presentation is based on [137]; see André [4] for an alternate point of view based on tannakian categories.

Definition 3.3.1. A *slope category* consists of the following data.

- An exact faithful functor $F : \mathcal{C} \to \mathcal{D}$ for some exact category $\mathcal{C}$ and some abelian category $\mathcal{D}$ such that for every $V \in \mathcal{C}$, the category of admissible monomorphisms into V (i.e., monomorphisms which occur as kernels of epimorphisms) is equivalent via F to the category of monomorphisms into $F(V)$.
- An assignment $\mathrm{rank} : \mathcal{D} \to \mathbb{Z}_{\geq 0}$ which is constant on isomorphism classes, additive on short exact sequences, and takes only the zero object to 0.
- An assignment $\deg : \mathcal{C} \to \Gamma$ (to some totally ordered abelian group Γ) which is constant on isomorphism classes, additive on short exact sequences, and with the property that for every morphism $f : V_1 \to V_2$ in $\mathcal{C}$ for which $F(f)$ is an isomorphism, we have $\deg(V_1) \leq \deg(V_2)$ with equality if and only if f is an isomorphism.

In order to parse this definition, we first translate the motivating example of vector bundles on curves into this framework.

Example 3.3.2. Let X be a smooth proper algebraic curve over a field k with generic point η. Let $\mathcal{C}$ be the exact category of vector bundles on X; a monomorphism $V' \to V$ is admissible if and only if V' is isomorphic to a *saturated* subbundle of V, i.e., one for which the quotient V/V' is torsion-free. Let $\mathcal{D}$ be the category of finite-dimensional $\kappa(\eta)$-vector spaces; there is an obvious exact faithful functor $F : \mathcal{C} \to \mathcal{D}$ taking a bundle V to its stalk V_η. For $V \in \mathcal{C}$ and $K \to F(V)$ a monomorphism, the subsheaf of V given by $U \mapsto \ker(V(U) \to F(V)/K)$ is an admissible subobject of V because $\mathcal{O}(U)$ is a Dedekind domain. Let $\mathrm{rank} : \mathcal{D} \to \mathbb{Z}_{\geq 0}$ be the dimension function.

Take $\Gamma = \mathbb{Z}$ and let $\deg : \mathcal{C} \to \Gamma$ be the usual degree function: for $\mathcal{F} \in \mathcal{C}$ of rank $n > 0$, $\deg(\mathcal{F})$ is the degree of the divisor defined by any nonzero rational section of the line bundle $\wedge^n \mathcal{F}$. (The unambiguity of this definition relies on the fact that any principal divisor on X has degree 0.) The fact that this is additive in short exact sequences comes down to the fact that if $0 \to V' \to V \to V'' \to 0$ is exact, then there is a natural isomorphism

$$\wedge^{\mathrm{rank}(V)} V \cong \wedge^{\mathrm{rank}(V')} V' \otimes \wedge^{\mathrm{rank}(V'')} V''. \tag{3.3.2.1}$$

If $f : V \to V'$ is a morphism in $\mathcal{C}$ for which $F(f)$ is an isomorphism, then $\mathrm{rank}(V)$ and $\mathrm{rank}(V')$ are equal to a common value n, $\wedge^n f : \wedge^n V \to \wedge^n V'$ is injective with cokernel supported on some finite set S of closed points, and $\deg(V) - \deg(V')$ is a nonnegative linear combination of the degrees of the points in S. (The difference $\deg(V) - \deg(V')$ can also be interpreted as $\dim_k H^0(X, \mathrm{coker}(\wedge^n f))$, but this interpretation will not persist for abstract curves as in Definition 3.3.4.)

Definition 3.3.3. For $k = \mathbb{C}$, let X^{an} denote the *analytification* of X, which is a compact Riemann surface. There is a canonical morphism $X^{\mathrm{an}} \to X$ in the category of locally ringed spaces; by (a very special case of) Serre's GAGA theorem [150], pullback along this morphism equates the categories of vector bundles (and coherent sheaves) on X and X^{an} and preserves sheaf cohomology. We can thus formally restate Example 3.3.2 in terms of vector bundles on X^{an}, or even in terms of Γ-equivariant vector bundles on Y where $Y \to X^{\mathrm{an}}$ is a Galois covering space map with deck transformation group Γ. For example, if X is of genus at least 2, then the universal covering space is an open unit disc with deck transformations by the fundamental group of X^{an}.

Our subsequent discussion will involve a generalization of Example 3.3.2.

Definition 3.3.4. An *abstract curve* is a connected, separated, noetherian scheme X which is regular of dimension 1; any such scheme has a unique generic point η which is also the unique nonclosed point. An *abstract complete curve* is an abstract curve X equipped with a nonzero map $\deg : \mathrm{Div}(X) \to \mathbb{Z}$ which is nonnegative on effective divisors and zero on principal divisors. For X an abstract complete curve, we may emulate Example 3.3.2 to obtain a slope category with $\mathcal{C}$ being the category of vector bundles on X.

This generalization is sufficient to discuss Fargues–Fontaine curves. However, we will also introduce some additional examples of the slope formalism in §3.5.

3.4. **Harder–Narasimhan filtrations.** Fix now a formalism of slopes. We now define and construct the Harder–Narasimhan filtrations associated to objects of $\mathcal{C}$.

Definition 3.4.1. For $f : V \to V'$ a monomorphism in $\mathcal{C}$, $F(f)$ lifts to an admissible monomorphism $\tilde{f} : \tilde{V} \to V'$ in $\mathcal{C}$ through which f factors. We call $\tilde{f}$ the *saturation* of f, and call $\tilde{V}$ the *saturation* of V in V'; we have $\deg(V) \leq \deg(\tilde{V})$ with equality if and only if $V = \tilde{V}$.

Remark 3.4.2. For $f : V \to V'$ an arbitrary morphism in $\mathcal{C}$, the kernel of $F(f)$ corresponds to an admissible subobject of V which is a kernel of f. By the same token, the cokernel of this admissible monomorphism is an image of f.

The poset of subobjects of a given object in $\mathcal{C}$ is a lattice. For $V_1 \to V', V_2 \to V'$ two monomorphisms in $\mathcal{C}$, we write $V_1 \cap V_2$ and $V_1 + V_2$ for the meet and join, respectively (mimicking the notation for vector bundles); these fit into an exact sequence

$$0 \to V_1 \cap V_2 \to V_1 \oplus V_2 \to V_1 + V_2 \to 0.$$

Beware that the join of two admissible subobjects need not be admissible.

Remark 3.4.3. A consequence of the previous discussion is that for any admissible subobject V' of V, if we form the associated exact sequence

$$0 \to V' \to V \to V'' \to 0$$

and take W to be another (not necessarily admissible) subobject of V, we have another short exact sequence

$$0 \to W' \to W \to W'' \to 0$$

where $W' = V' \cap W$ is a subobject of V' (and an admissible subobject of W) and W'' is a subobject of V''.

Definition 3.4.4. Given a slope category, define the *slope* of a nonzero object $V \in \mathcal{C}$ as the ratio

$$\mu(V) := \frac{\deg(V)}{\mathrm{rank}(F(V))} \in \Gamma \otimes_{\mathbb{Z}} \mathbb{Q}.$$

If $0 \to V' \to V \to V'' \to 0$ is an exact sequence in $\mathcal{C}$ with $V', V'' \neq 0$, then

$$\min\{\mu(V'), \mu(V'')\} \leq \mu(V) \leq \max\{\mu(V'), \mu(V'')\}$$

with equality if and only if $\mu(V') = \mu(V'')$.

A nonzero object $V \in \mathcal{C}$ is *semistable* (resp. *stable*) if V contains no proper nonzero subobject V' with $\mu(V') > \mu(V)$ (resp. $\mu(V') \geq \mu(V)$); this implies that V admits no proper quotient V'' with $\mu(V'') < \mu(V)$ (resp. $\mu(V'') \leq \mu(V)$). Note that our hypotheses ensure that any rank 1 object is semistable. (It would be reasonable to treat the zero object as being semistable of *every* slope, but we won't do this.)

Lemma 3.4.5. *For $V, V' \in \mathcal{C}$ which are semistable with $\mu(V) > \mu(V')$, we have* $\mathrm{Hom}_{\mathcal{C}}(V, V') = 0$*.*

Proof. Suppose by way of contradiction that $f : V \to V'$ is a nonzero morphism. Let W be the image of V in V'; then $\mu(V) \leq \mu(W) \leq \mu(V')$, a contradiction. □

Lemma 3.4.6. *Let*

$$0 \to V' \to V \to V'' \to 0$$

be a short exact sequence of nonzero objects in $\mathcal{C}$. If V', V'' are semistable of the same slope μ, then so is V.

Proof. For any nonzero subobject W of V, with notation as in Remark 3.4.3 we have

$$\deg(W) = \deg(W') + \deg(W'') \leq \mu\,\mathrm{rank}(W') + \mu\,\mathrm{rank}(W'') \leq \mu\,\mathrm{rank}(W),$$

so $\mu(W) \leq \mu$ (even in the corner cases where $W' = 0$ or $W'' = 0$). □

Corollary 3.4.7. *For any $\mu \in \Gamma \otimes_{\mathbb{Z}} \mathbb{Q}$, the objects of $\mathcal{C}$ which are semistable of slope μ (plus the zero object) form an exact abelian subcategory of $\mathcal{C}$ which is closed under extensions.*

Proof. Augment the previous lemma with the observation that if $V \in \mathcal{C}$ is semistable of slope μ, any subobject W of V which is semistable of slope μ is admissible: otherwise, the saturation of W would witness the failure of semistability of V. □

Definition 3.4.8. For $V \in \mathcal{C}$, a *Harder–Narasimhan filtration* (or *HN filtration*) of V is a filtration

$$0 = V_0 \subset \cdots \subset V_l = V \tag{3.4.8.1}$$

such that each inclusion $V_{i-1} \to V_i$ is admissible with cokernel being semistable of some slope μ_i, and $\mu_1 > \cdots > \mu_l$. By convention, the trivial filtration of the zero object is an HN filtration. If $V \neq 0$, then the sequence

$$0 = V_1/V_1 \subset \cdots \subset V_l/V_1 = V/V_1$$

constitutes an HN filtration of V/V_1; this provides the basis for various inductive arguments.

In order to better digest this definition, we give an alternate characterization of the first step of the HN filtration.

Lemma 3.4.9. *Suppose that $V \in \mathcal{C}$ is nonzero and admits an HN filtration labeled as in* (3.4.8.1)*. Then μ_1 is the maximum slope of any nonzero subobject of V, and V_1 is the maximal subobject of V of slope μ_1.*

Proof. We proceed by induction on $\mathrm{rank}(V)$. There is nothing to check if V is semistable. Otherwise, for any nonzero subobject W of V, set notation as in Remark 3.4.3 with $V' = V_1$. Using the semistability of V_1 and applying the induction hypothesis to V/V_1, we see that

$$\deg(W) \leq \deg(W') + \deg(W'') \leq \mu_1 \,\mathrm{rank}(W') + \mu(V_2/V_1)\,\mathrm{rank}(W'') \leq \mu_1 \,\mathrm{rank}(W),$$

with strict inequality whenever $W'' \neq 0$. This proves the claim. $\square$

We now turn around and construct the object with the properties of V_1 identified in Lemma 3.4.9. It is relatively easy to see that the possible slopes of subobjects of V are bounded above, but this would only imply that the maximum is achieved if Γ is discrete (because the slopes of subobjects of an object of rank n belong to $\frac{1}{1}\Gamma \cup \cdots \cup \frac{1}{n}\Gamma$, which would then itself be a discrete set). However, it is easy to give an alternate argument that works even when Γ is not discrete, and which even in the discrete case gives additional crucial information.

Lemma 3.4.10. *Suppose that $V \in \mathcal{C}$ is nonzero. Then V admits a nonzero subobject V_1 of some slope $\mu_1 \geq \mu(V)$ such that μ_1 is the maximum slope of any nonzero subobject of V, and V_1 is the maximal subobject of V of slope μ_1.*

Proof. We proceed by induction on $\mathrm{rank}(V)$, with the case of V semistable serving as a trivial base case. If V is not semistable, then the set of nonzero proper subobjects W with $\mu(W) > \mu(V)$ is nonempty; by saturating, we may find an admissible subobject W of this form of maximal rank. (Note that we do not attempt to maximize the *slope* of W, just its rank; hence this is *a priori* a maximization over a finite set.) By the induction hypothesis, W admits a subobject V_1 of the claimed form; we will show that this subobject also has the desired effect for V. This amounts to showing that any subobject X of V satisfying $\mu(X) \geq \mu_1$ must be contained in W; to see this, write the exact sequence

$$0 \to W \cap X \to W \oplus X \to W + X \to 0,$$

note that $\mu(W \cap X) \leq \mu_1$ if $W \cap X \neq 0$, and then compute that

$$\begin{aligned}
\deg(W + X) &= \deg(W) + \deg(X) - \deg(W \cap X) \\
&= \mathrm{rank}(W)\mu(W) + \mathrm{rank}(X)\mu(X) - \mathrm{rank}(W \cap X)\mu(W \cap X) \\
&\geq \mathrm{rank}(W)\mu(W) + (\mathrm{rank}(X) - \mathrm{rank}(W \cap X))\mu_1 \\
&= \mathrm{rank}(W + X)\mu_1.
\end{aligned}$$

Since $\mu_1 \geq \mu(W) > \mu(V)$, $W + X$ is a subobject of V of slope strictly greater than $\mu(V)$; its saturation is an admissible subobject with the same property. By the maximality of $\mathrm{rank}(W)$, this is only possible if $\mathrm{rank}(W + X) = \mathrm{rank}(W)$, and hence if $W + X = W$ because W is admissible. $\square$

Putting the two preceding lemmas together gives us HN filtrations in general.

Theorem 3.4.11 (after Harder–Narasimhan)**.** *Every object $V \in \mathcal{C}$ admits a unique HN filtration.*

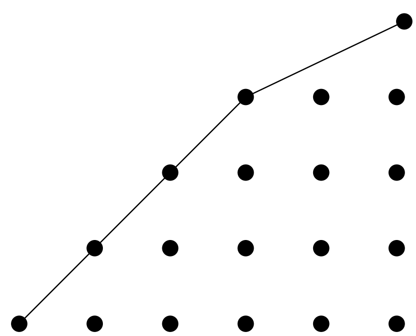
Proof. We check both existence and uniqueness by induction on $\mathrm{rank}(V)$, the case $V = 0$ serving as a trivial base case. To establish uniqueness, note that Lemma 3.4.9 implies that the choice of V_1 is uniquely determined, and then applying the induction hypothesis to V/V_1 forces the rest of the filtration. To establish existence, take V_1 as in Lemma 3.4.10; the maximal slope condition ensures that V_1 is semistable. For any subobject W'' of V/V_1, the inverse image W of W'' in V is strictly larger than V_1, so $\mu(W) < \mu(V_1)$ and so $\mu(W'') < \mu(V_1)$. Consequently, we obtain an HN filtration of V by starting with V_1, then lifting the terms of an HN filtration of V/V_1 produced by the induction hypothesis. $\square$

Definition 3.4.12. Suppose now that $\Gamma \subseteq \mathbb{R}$. For $V \in \mathcal{C}$, with notation as in (3.4.8.1), the *slope multiset* of V is the multisubset of $\Gamma \otimes_{\mathbb{Z}} \mathbb{Q}$ of cardinality $\operatorname{rank}(V)$ consisting of $\mu(V_i/V_{i-1})$ with multiplicity $\operatorname{rank}(V_i/V_{i-1})$ for $i = 1, \dots, l$. As is typical when studying nonarchimedean fields, it is convenient and customary to repackage these values as the slopes of a piecewise affine function. We define the *HN polygon* of V, denoted $\operatorname{HN}(V)$ to be the graph of the continuous, concave-down function from $[0, \operatorname{rank}(V)]$ to $\mathbb{R}$ given by the formula

$$x \mapsto \deg(V_{i-1}) + (x - \operatorname{rank}(V_{i-1}))\mu(V_i) \qquad (i = 1, \dots, l; \operatorname{rank}(V_{i-1}) \leq x \leq \operatorname{rank}(V_i)).$$

That is, start at $(0,0)$ and draw n segments of width 1 whose slopes are the elements of the HN multiset in *decreasing* order, counting multiplicities. See Figure 1 for an illustration.

FIGURE 1. The HN polygon associated to an object V admitting a filtration $0 = V_0 \subset V_1 \subset V_2 = V$ with $\operatorname{rank}(V_1) = 3$, $\deg(V_1) = 3$, $\operatorname{rank}(V_2/V_1) = 2$, $\deg(V_2/V_1) = 1$.



Lemma 3.4.13. *For $V, V' \in \mathcal{C}$, the slope multiset of $V \oplus V'$ is the multiset union of the slope multisets of V and V'. We may characterize this by writing* $\operatorname{HN}(V \oplus V') = \operatorname{HN}(V) \oplus \operatorname{HN}(V')$.

Proof. We proceed by induction on $\operatorname{rank}(V) + \operatorname{rank}(V')$, with all cases where either summand is zero serving as base cases. If $V, V' \neq 0$, let V_1, V_1' be the first step in the respective HN filtrations, and let μ_1, μ_1' be the respective slopes. Without loss of generality suppose that $\mu_1 \geq \mu_1'$. By Lemma 3.4.9, the largest element of the slope multiset of $V_1 \oplus V_2$ is μ_1, with the corresponding subobject being V_1 if $\mu_1 > \mu_1'$ or $V_1 \oplus V_1'$ if $\mu_1 = \mu_1'$. In either case, we may then conclude using the induction hypothesis. $\square$

Remark 3.4.14. At this point, it is necessary to comment on two different sign conventions that we have implicitly adopted at this point. The first is the direction of the inequality in the definition of semistability (or equivalently, the choice of sign in the definition of the degree function): we are using the sign convention compatible with the literature on geometric invariant theory (Remark 3.2.5), which is incompatible with the literature on Dieudonné modules. The second is the choice of concavity (up or down) in the definition of the HN polygon (which can be interpreted as the choice to label filtrations in ascending order); we are using the sign convention compatible with the literature on algebraic groups, which is incompatible with the usual definition of Newton polygons. When comparing results between sources, it is important to keep track of both possible sign discrepancies.

We may characterize the HN polygon directly (without overt reference to the HN filtration) as follows.

Lemma 3.4.15. *For $V \in \mathcal{C}$, the HN polygon is the boundary of the upper convex hull of the set of points $(\mathrm{rank}(W), \deg(W)) \in \mathbb{R}^2$ as W runs over all subobjects of V.*

Proof. On one hand, the steps of the HN filtration show that the boundary of the upper convex hull lies on or above the HN. We establish the reverse inequality by induction on $\mathrm{rank}(V)$. Given a subobject W of V, set notation as in Remark 3.4.3 with $V' = V_1$. By the definition of semistability, the point $(\mathrm{rank}(W'), \deg(W''))$ lies under the line $y = \mu(V_1)x$; by the induction hypothesis, the point $(\mathrm{rank}(W'), \deg(W''))$ lies on or below the HN polygon of V/V_1. This yields the claim. □

In terms of slope multisets, we may formally promote Lemma 3.4.5 as follows.

Corollary 3.4.16. *For $V, V' \in \mathcal{C}$, if the least element of the slope multiset of V is greater than the greatest element of the slope multiset of V', then $\mathrm{Hom}_{\mathcal{C}}(V, V') = 0$.*

Proof. If V, V' are both semistable, then this is exactly the assertion of Lemma 3.4.5. If V is general and V' is semistable, then the first step of the slope filtration of V cannot map nontrivially to V'; we may thus deduce this case by induction on $\mathrm{rank}(V)$. If V, V' are both general, then V maps trivially to the final quotient of the slope filtration of V'; we may thus deduce this case by induction on $\mathrm{rank}(V')$. □

Corollary 3.4.16 has various consequences about the possibilities for the set of slope multisets for the three terms in a short exact sequence. A full analysis of this in the case of the Fargues–Fontaine curves is part of the student project; we limit ourselves here to one simple observation.

Lemma 3.4.17. *If $0 \to V' \to V \to V'' \to 0$ is a short exact sequence, then $\mathrm{HN}(V) \leq \mathrm{HN}(V' \oplus V'')$ with the same endpoint.*

Proof. For every subobject W of V, Remark 3.4.3 gives rise to a subobject $W' \oplus W''$ of $V' \oplus V''$ of the same degree and rank; moreover, this correspondence is compatible with inclusion of subobjects. Using the criterion from Lemma 3.4.15, we deduce the claim. □

Corollary 3.4.18. *Suppose that $0 = V_0 \subset \cdots \subset V_m = V$ is a filtration of V by admissible subobjects such that each quotient V_i/V_{i-1} is semistable of some slope μ_i. Then $\mathrm{HN}(V) \leq \mathrm{HN}(V_1/V_0 \oplus \cdots \oplus V_m/V_{m-1})$ with the same endpoint.*

Remark 3.4.19. So far, we have said nothing about tensor products; in fact, we did not even include a symmetric monoidal structure on the category $\mathcal{C}$ in the definition of a slope formalism. In practice, all of the examples we will consider in this lecture admit such a structure which satisfies the property

(3.4.19.1)
$$\mathrm{rank}(V \otimes V') = \mathrm{rank}(V)\,\mathrm{rank}(V'), \quad \deg(V \otimes V') = \deg(V)\,\mathrm{rank}(V') + \deg(V')\,\mathrm{rank}(V).$$

For V, V' nonzero, we again have

$$\mu(V \otimes V') = \mu(V) + \mu(V'),$$

but it is again not apparent that the tensor product of two semistable bundles is again semistable. When this occurs, the HN filtrations are *determinantal* in the sense that slopes of objects behave like the determinants of linear transformations (or more precisely, their images under some valuation). This is the same arrangement that one encounters initially in the study of the Weil conjectures, as in [35]: before one can define the *weights* of a coefficient object, one must work with the ultimately equivalent concept of *determinantal weights*, whose definition is based on the fact that there is no ambiguity about weights for objects of rank 1.

Remark 3.4.20. A thoroughly modern twist on slope formalisms comes from the work of Bridgeland [24], which gives rise to slope formalisms for *triangulated* categories; the motivating example is the bounded derived category of coherent sheaves on an algebraic variety. In this setting, one assigns to each object a complex number in the upper half-plane (called a *central charge* for presently irrelevant physical reasons), with the argument of this value playing the role of the slope; under fairly mild hypothesis, every object has a Harder–Narasimhan filtration.

3.5. **Additional examples of the slope formalism.** To provide some indication of the power of this formalism, we describe some other classes of examples. In each case, the key question is whether or not the tensor product of semistable objects is semistable; the example of vector bundles on an algebraic curve in positive characteristic shows that this is not guaranteed by the slope formalism (see Remark 3.2.3).

Example 3.5.1. Let R be an integral domain in which every finitely generated ideal is projective (a *Bézout domain*); note that every finitely presented torsion-free R-module is principal. Let Φ be a monoid acting on R via ring homomorphisms. Let $\mathcal{C}$ be the category of finite projective R-modules with Φ-actions: that is, one must specify an underlying module M together with isomorphisms $\varphi^* M \cong M$ for each $\varphi \in \Phi$ compatible with composition (with the identity element of Φ acting via the identity map). Let $\mathcal{D}$ be the category of finite-dimensional $\mathrm{Frac}(R)$-vector spaces with Φ-actions. Let $\mathrm{rank} : \mathcal{D} \to \mathbb{Z}_{\geq 0}$ be the dimension function.

Let $v : H^1(\Phi, R^\times) \to \Gamma$ be a homomorphism with the following property: if $x \in R$ is nonzero and satisfies $\varphi(x)/x \in R^\times$ for all $\varphi \in \Phi$, then the cocycle c taking φ to $\varphi(x)/x$ satisfies $v(c) \geq 0$ with equality if and only if $x \in R^\times$. Define $\deg : \mathcal{C} \to \Gamma$ as follows: for $V \in \mathcal{C}$ of rank n, choose a generator $\mathbf{v}$ of $\wedge^n V$; let $c : \Phi \to R^\times$ be the cocycle taking φ to the element r for which the specified isomorphism $\varphi^* \wedge^n V = (\wedge^n V) \otimes_{R,\varphi} R \to \wedge^n V$ takes $\mathbf{v} \otimes 1$ to $r\mathbf{v}$; and put $\deg(V) = v(c)$.

It is obvious that deg is constant on isomorphism classes and (from (3.3.2.1)) additive in short exact sequences. If $f : V_1 \to V_2$ is a morphism in $\mathcal{C}$ for which $F(f)$ is an isomorphism, then $\mathrm{rank}(V_1)$ and $\mathrm{rank}(V_2)$ are equal to a common value n, $\wedge^n f : \wedge^n V_1 \to \wedge^n V_2$ is injective with cokernel isomorphic to R/xR for some $x \in R$ with $\varphi(x)/x \in R^\times$, and $\deg(V_2) - \deg(V_1) = v(x) \geq 0$ with equality only if $V_1 = V_2$.

Remark 3.5.2. Example 3.5.1 is formulated so as to bring to mind the following case. Take R to be the holomorphic functions on the open unit disc in $\mathbb{C}$; this is a Bézout ring which is not noetherian (consider an infinite sequence accumulating at the boundary, and form the ideal of functions which vanish at all but finitely many of these points). Take Φ to be the deck transformation group for a Teichmüller uniformization of a Riemann surface X of genus at least 2; then the objects of $\mathcal{C}$ are precisely the vector bundles on X, and it is straightforward to reverse-engineer the map v so as to recover the usual degree function.

Remark 3.5.3. Another possibly familiar context for Example 3.5.1 is where R is the fraction field of the ring of Witt vectors over a perfect field k of characteristic p, and Φ is the monoid generated by the Witt vector Frobenius map; in this case, $\mathcal{C}$ is the category of *isocrystals* over k. While this example is important in the theory of Dieudonné modules, from the point of view of the slope formalism it is misleadingly simple: if $f : V_1 \to V_2$ is a morphism in $\mathcal{C}$ for which $F(f)$ is an isomorphism, then f is itself an isomorphism. In any case, one can use the standard Dieudonné–Manin classification theorem in the case where k is algebraically closed (see Theorem 3.6.19) to show that for arbitrary k, the tensor product of semistable objects is semistable.

Somewhere between the two previous examples, we find the following example of great interest in p-adic Hodge theory.

Remark 3.5.4. Let F be a complete discretely valued field with valuation $v_0 : F^\times \to \mathbb{Z}$. Let R be the ring consisting of all formal Laurent series $\sum_{n\in\mathbb{Z}} c_n t^n$ over F which converge in some region of the form $* < |t| < 1$ (where $*$ depends on the series). By analogy with the complex-analytic case, results of [122] (on the theory of divisors in rigid-analytic discs) show that R is a Bézout domain; note that the units in R consist precisely of the nonzero series with bounded coefficients, so v_0 induces a valuation map $v : R^\times \to \mathbb{Z}$. This construction occurs commonly in the theory of p-adic differential equations, where it is commonly known as the *Robba ring* over F (in the variable t).

Let Φ be the monoid generated by a single endomorphism $\varphi : R \to R$ given as a substitution $\sum_{n\in\mathbb{Z}} c_n t^n \mapsto \sum_{n\in\mathbb{Z}} c_n \varphi(t)^n$, where $\varphi(t) = t^m u$ for some integer $m > 1$ and some $u \in R^\times$ with $v(u) = 0$. Let $v : H^1(\Phi, R^\times) \to \mathbb{Z}$ be the homomorphism taking the cocycle c to $v(c(\varphi))$; again using the results of [122], one sees that this satisfies the condition of Example 3.5.1.

When F is of residue characteristic p and $m = p$, it is known that the tensor product of semistable objects is semistable, by a classification theorem similar to the one we will give for vector bundles on the Fargues–Fontaine curve. See [97].

A closely related example from p-adic Hodge theory is the following.

Example 3.5.5. Let k be a perfect field of characteristic p. Let K be the fraction field of the ring of Witt vectors over k. Let L be a finite totally ramified extension of K. Let $\mathcal{C}$ be the category of *filtered φ-modules* over L; such an object is a finite-dimensional K-vector space V equipped with a φ-action, for φ the Witt vector Frobenius, plus an exhaustive separated $\mathbb{Z}$-indexed filtration on $V \otimes_K L$ by L-subspaces. Morphisms in $\mathcal{C}$ are maps which are φ-equivariant and respect the filtration. Note that there are monomorphisms which are not admissible, because the inverse image of the target filtration is the "wrong" filtration on the source.

We define deg by using exterior powers to reduce to the case of one-dimensional spaces. For V of dimension 1, the filtration jumps at a unique integer i, the action of Frobenius on a generator is multplication by some $r \in K^\times$, and we set $\deg(V) = i - v_p(r)$. This yields a slope formalism in which preservation of semistability by tensor product was shown by Faltings [50] using the relationship to Galois representations which are *crystalline* in Fontaine's sense, and more directly by Totaro [158]; another approach can be obtained by going through Remark 3.5.4 using a construction of Berger [12]. (Here the semistable objects are commonly called *weakly admissible* objects.)

A related construction is that of *filtered* (φ, N)*-modules*, in which one adds to the data a (necessarily nilpotent) K-linear endomorphism N of V satisfying $N\varphi = p\varphi N$. Again, Berger's method can be used to show that the tensor product of semistable objects is semistable, ultimately using the relationship to Galois representations which are *log-crystalline*[8] in Fontaine's sense.

Remark 3.5.6. Another closely related example is that of *Banach-Colmez spaces*. Roughly speaking, for F a complete algebraically closed nonarchimedean field of mixed characteristics, a Banach-Colmez space is a Banach space over $\mathbb{Q}_p$ which is obtained by forming an extension of a finite-dimensional F-vector space by a finite-dimensional $\mathbb{Q}_p$-vector space, then quotienting by another finite-dimensional $\mathbb{Q}_p$-vector spaces. In this setup, the rank is given by the dimension of the F-vector space and the degree by the difference between the dimensions of the two $\mathbb{Q}_p$-vector spaces.

In order to make this definition precise, we need to formulate the construction in such a way that fixes the F-dimension without fixing the F-linear structure (which we do not want morphisms to respect). This is most naturally done using the *pro-étale topology* on the category of perfectoid spaces of characteristic p; see [163, Lecture 4].

In all of the preceding examples, the group Γ is discrete. Let us end with a few examples where Γ is not discrete.

Example 3.5.7. Let k be a perfect field of characteristic p. Let X be the scheme obtained by glueing together the rings $\operatorname{Spec} k[T^{1/p^\infty}]$ and $\operatorname{Spec} k[T^{-1/p^\infty}]$ together along their common open subscheme $\operatorname{Spec} k[T^{\pm 1/p^\infty}]$. By emulating the argument for the usual projective line, one can exhibit a homomorphism $\mathbb{Z}[p^{-1}] \to \operatorname{Pic}(X)$ taking $n \in \mathbb{Z}[p^{-1}]$ to a line bundle $\mathcal{O}(n)$ whose global sections are homogeneous polynomials of degree n (when $n \geq 0$), and show that this is an isomorphism (see for example [29]). We may emulate Example 3.3.2 to obtain a slope formalism on vector bundles whose degree function takes values in $\mathbb{Z}[p^{-1}]$.

If we view X as the inverse limit of $\mathbf{P}^1_k$ along Frobenius, then every vector bundle on X is the pullback of a vector bundle on some copy of $\mathbf{P}^1_k$, and so by Grothendieck's theorem splits as a direct sum of line bundles. Beware however that one cannot derive this splitting by directly imitating the proof for $\mathbf{P}^1_k$: in that argument, it is crucial that every exact sequence of the form

$$0 \to \mathcal{O} \to V \to \mathcal{O}(1) \to 0$$

splits, but that fails here. The corresponding Ext group is spanned by homogeneous monomials in x, y of total degree -1 in which each variable occurs with degree strictly less than 0, and hence is nonzero.

In any case, we see that the tensor product of semistable bundles is semistable.

Example 3.5.8. Let K be a nonarchimedean field of residue characteristic p. Let X be the adic space obtained by glueing $\operatorname{Spa}(K\langle T^{p^{-\infty}}\rangle, K\langle T^{p^{-\infty}}\rangle^\circ)$ and $\operatorname{Spa}(K\langle T^{-p^{-\infty}}\rangle, K\langle T^{-p^{-\infty}}\rangle^\circ)$ together along $\operatorname{Spa}(K\langle T^{\pm p^{-\infty}}\rangle, K\langle T^{\pm p^{-\infty}}\rangle^\circ)$. Using Exercise 1.5.27, we see that every line

[8] The term *semistable* is more common here, and its etymology in this usage is entirely defensible, but the ensuing terminological conflict renders the term *log-crystalline* a preferred alternative.

bundle on either $\mathrm{Spa}(K\langle T^{p^{-\infty}}\rangle, K\langle T^{p^{-\infty}}\rangle^\circ)$ or $\mathrm{Spa}(K\langle T^{-p^{-\infty}}\rangle, K\langle T^{-p^{-\infty}}\rangle^\circ)$ is trivial; using this, one can imitate the argument in [29] to see that $\mathrm{Pic}(X) \cong \mathbb{Z}[p^{-1}]$.

By contrast with the previous example, it is not the case that every vector bundle is a direct sum of line bundles! We illustrate this by showing that for $p > 2$, there is a vector bundle V of rank 2 with $\wedge^2 V \cong \mathcal{O}(1)$ which cannot be written as the direct sum of two line bundles. (To cover $p = 2$, one should be able to construct a vector bundle V of rank 3 with $\wedge^3 V \cong \mathcal{O}(1)$ which does not split as a direct sum of a line bundle and another bundle.)

To construct V, identify $\mathrm{Ext}^1_{\mathcal{C}}(\mathcal{O}(1), \mathcal{O})$ with the completion for the supremum norm of the K-vector space on the monomials $x^{-i}y^{-1+i}$ for $i \in \mathbb{Z}[p^{-1}] \cap (0,1)$, and take V to be an extension corresponding to an element of this space of the form $s = \sum_{n=0}^\infty c_n x^{-i_n} y^{-1+i_n}$ where c_n is a null sequence in K and i_n is an increasing sequence in $\mathbb{Z}[p^{-1}] \cap (0, \frac{1}{2})$ with limit $\frac{1}{2}$. If we had an isomorphism $V \cong \mathcal{O}(j) \oplus \mathcal{O}(1-j)$ for some $j \in \mathbb{Z}[p^{-1}]$, we would have to have $j \in (0,1)$ (otherwise s would be forced to split), and without loss of generality $j \in (0, \frac{1}{2})$ (since $p > 2$ we have $\frac{1}{2} \notin \mathbb{Z}[p^{-1}]$). Moreover, we would have $\mathrm{Hom}_{\mathcal{C}}(\mathcal{O}, V(j-1)) = K$. However, from the exact sequence

$$0 \to \mathcal{O}(j-1) \to V(j-1) \to \mathcal{O}(j) \to 0$$

we obtain an exact sequence

$$0 \to \mathrm{Hom}_{\mathcal{C}}(\mathcal{O}, V(j-1)) \to \mathrm{Hom}_{\mathcal{C}}(\mathcal{O}, \mathcal{O}(j)) \to \mathrm{Ext}_{\mathcal{C}}(\mathcal{O}, \mathcal{O}(j-1))$$

where the last arrow is a connecting homomorphism. If we represent the source and target of this map as homogeneous sums of degree j and $j-1$, then the map between them is given by multiplication by s. More precisely, the source is (topologically) spanned by monomials $x^i y^{j-i}$ for $i \in \mathbb{Z}[p^{-1}] \cap [0, j]$, while the target is obtained by quotienting out by monomials in which either x or y occurs with nonnegative degree. For $t = \sum_{0<i<j} d_i x^i y^{j-i}$ in the source, the corresponding element of the target is $\sum_{i,n: i<i_n} c_n d_i x^{i-i_n} y^{j-1-i+i_n}$ (the exponent of y is always negative because $i_n < 1/2 < 1-j$); if t represents an element of the kernel of the connecting homomorphism, in $K\langle T^{p^{-\infty}}\rangle$ we must have

$$\left(\sum_{0<i<j} d_i T^i\right)\left(\sum_{n=0}^\infty c_n T^{1-i_n}\right) \equiv 0 \pmod{T}.$$

However, by considering Newton polygons, we see that this congruence cannot even hold modulo $T^{1/2+j+\epsilon}$ for any $\epsilon > 0$ unless $t = 0$. This yields the desired contradiction.

Unfortunately, this example does not give rise to a slope formalism for general K, because the underlying rings are not Bézout domains unless K is discretely valued. In that case, we may emulate Example 3.3.2 to obtain a slope category whose degree function takes values in $\mathbb{Z}[p^{-1}]$. For $p > 2$, the above example is a semistable object of rank 2 and degree $\frac{1}{2}$.

We expect that if V is semistable of rank r and degree d, then $V(-n)$ is spanned by horizontal sections whenever $d > rn$. If so, this would imply (using the fact that $\mathbb{Z}[p^{-1}]$ is not discrete in $\mathbb{R}$) that the tensor product of semistable objects is semistable.

Remark 3.5.9. When K is perfectoid of characteristic p, the two preceding examples are related via an analytification morphism from the adic space to the scheme. However, the discrepancies between the two cases make it clear that there is no version of the GAGA theorem applicable to this morphism.

In connection with the analogy between archimedean and p-adic Hodge theory (see §A.4), Sean Howe has suggested the following example.

Exercise 3.5.10. Consider the category $\mathcal{C}$ of $\mathbf{G}_m$-equivariant vector bundles on $\mathbf{P}^1$, equipped with the fiber functor

$$V \mapsto \{\mathbf{G}_m\text{-invariant sections of } V \text{ over } \mathbf{P}^1 \setminus \{0, \infty\}\}.$$

Equip $\mathbb{Z}^2$ with the lexicographic ordering. Consider the degree function $\mathcal{C} \to \mathbb{Z}^2$ induced by the function $L \mapsto (n, p)$ on $\mathbf{G}_m$-equivariant line bundles, in which n is the usual degree of L and p is the order of vanishing at 0 of an invariant section over $\mathbf{P}^1 \setminus \{0, \infty\}$.

(a) Show that this gives a slope formalism.
(b) What are the stable and semistable bundles?
(c) Classify the $\mathbf{G}_m$-equivariant vector bundles on $\mathbf{P}^1$.
(d) Give an equivalence between $\mathbf{G}_m$-vector bundles on $\mathbf{P}^1$ and a linear-algebraic category, and describe the slope formalism in these terms. Then give a linear-algebraic description of the subcategory of objects of slope $(n, *)$ for a fixed n.

We conclude with an exotic example coming from Arakelov theory (compare [4, §3.2.1]).

Example 3.5.11. Let $\mathcal{C}$ be the category of Euclidean lattices, i.e., finite free $\mathbb{Z}$-modules equipped with positive-definite inner products, in which morphisms are homomorphisms of lattices which have operator norm at most 1 with respect to the inner products. Let deg be the function assigning to a lattice L the quantity $-\log \det L$. This gives an example of the slope formalism; the Harder–Narasimhan filtrations in this context were previously known as *Grayson–Stuhler filtrations* before the analogy between them was observed. The preservation of semistability by tensor products has been conjectured by Bost and known in some cases.

One may similarly replace $\mathbb{Z}$ with $\mathfrak{o}_K$ for K a number field, considering finite projective $\mathfrak{o}_K$-modules equipped with Hermitian norms with respect to all real and complex embeddings. This again gives a slope filtration in which the preservation of semistability by tensor products is conjectured by Bost and known in some cases; see [4, §3.2.1] for further discussion.

3.6. **Slopes over a point.** We now consider the slope formalism associated to vector bundles on Fargues–Fontaine curves, as treated in [57]. This provides an improved perspective on some of my previous work on φ-modules over the Robba ring [94, 96].

Hypothesis 3.6.1. Throughout §3.6, let F be a perfectoid field of characteristic p and take $S = \mathrm{Spa}(F, \mathfrak{o}_F)$.

Theorem 3.6.2 (Fargues–Fontaine)**.** *The scheme* $\mathrm{Proj}(P_S)$ *is an abstract curve. Every closed point x has residue field which is an untilt of a finite extension of F; setting* $\deg(x)$ *to be the degree of that finite extension gives* $\mathrm{Proj}(P_S)$ *the structure of an abstract complete curve.*

Proof. See [57, §10]. More details may be added here later. □

Remark 3.6.3. For any pseudouniformizer $\overline{\varpi} \in F$, the series

$$\sum_{n \in \mathbb{Z}} p^{-n} [\overline{\varpi}^{p^n}]$$

converges to an element $\mathbf{v}$ of $H^0(Y_S, \mathcal{O})$ satisfying $\varphi(\mathbf{v}) = p\mathbf{v}$, and hence to an element of $H^0(X_S, \mathcal{O}(1))$. This section vanishes at a single closed point whose residue field is an untilt of F itself. It follows that $\deg(\mathcal{O}(1)) = 1$.

In light of Theorem 3.6.2, we may apply Definition 3.3.4 to obtain a slope formalism on the vector bundles on $\mathrm{Proj}(P_S)$, or equivalently (by Theorem 3.1.17) on the vector bundles on FF_S. These obey an analogue of Grothendieck's theorem, although with slightly more basic objects than just the powers of $\mathcal{O}(1)$.

Definition 3.6.4. For $d = \frac{r}{s}$ a rational number in lowest terms (which is to say $r, s \in \mathbb{Z}$, $s > 0$, and $\gcd(r, s) = 1$), let $\mathcal{O}(d)$ be the vector bundle on X_S corresponding to the trivial vector bundle on Y_S of rank s on the basis $\mathbf{v}_1, \dots, \mathbf{v}_s$ with the isomorphism $\varphi^*\mathcal{O}(d) \to \mathcal{O}(d)$ sending $1 \otimes \mathbf{v}_1, \dots, 1 \otimes \mathbf{v}_s$ to $\mathbf{v}_2, \dots, \mathbf{v}_s, p^{-r}\mathbf{v}_1$. This bundle is the pushforward of the line bundle $\mathcal{O}(r)$ on the s-fold cover of FF_S described in Remark 3.1.8.

The following is an easy variant of Remark 3.6.3.

Exercise 3.6.5. For $d > 0$, we have $H^0(\mathrm{FF}_S, \mathcal{O}(d)) \neq 0$ whenever $d > 0$.

Exercise 3.6.6. Suppose that F is algebraically closed. For $d, d' \in \mathbb{Q}$, $\mathcal{O}(d) \otimes \mathcal{O}(d')$ is isomorphic to a direct sum of copies of $\mathcal{O}(d + d')$. If you don't see how to check this "by hand", use the Dieudonné–Manin classification theorem (see Theorem 3.6.19).

Corollary 3.6.7. *For $d = \frac{r}{s}$ in lowest terms, $\mathcal{O}(d)$ is stable of slope d and degree r.*

Proof. All of the claims reduce to the case where F is algebraically closed. We start with the degree statement. For $d \in \mathbb{Z}$, the claim follows from Remark 3.6.3. For $d \notin \mathbb{Z}$, using (3.4.19.1) we reduce to checking that $\deg(\mathcal{O}(d)^{\otimes s}) = ds^{s+1}$; this follows from the previous case using Exercise 3.6.6.

We next check semistability. Suppose that $\mathcal{F}$ is a nonzero subobject of $\mathcal{O}(d)$ of slope greater than d. Then $\mathcal{F}^{\otimes s}$ is a subobject of $\mathcal{O}(d)^{\otimes s}$ of slope greater than $ds = r$; the first step $\mathcal{G}$ in the HN filtration of $\mathcal{F}^{\otimes s}$ is a semistable subobject of slope greater than r. By Exercise 3.6.6, $\mathcal{O}(d)^{\otimes s} \cong \mathcal{O}(r)^{\oplus s^s}$, so the existence of a nonzero map $\mathcal{G} \to \mathcal{O}(d)^{\otimes s}$ implies the existence of a nonzero map $\mathcal{G}(-r) \to \mathcal{O}$. Transposing yields a nonzero map $\mathcal{O} \to \mathcal{G}^\vee(r)$ whose target is semistable of negative degree, a contradiction.

We finally note that since $\gcd(r, s) = 1$, $\mathcal{O}(d)$ cannot admit any nonzero proper submodule of slope exactly d. It follows that $\mathcal{O}(d)$ is stable. $\square$

Remark 3.6.8. The stability of $\mathcal{O}(d)$ is not preserved by base extension from $\mathbb{Q}_p$ to a larger field. See Remark 4.3.11.

Corollary 3.6.9. *Suppose that F is algebraically closed. For $d, d' \in \mathbb{Q}$, the following statements hold.*

(a) *If $d \leq d'$, then* $\mathrm{Hom}(\mathcal{O}(d), \mathcal{O}(d')) \neq 0$.
(b) *If $d > d'$, then* $\mathrm{Hom}(\mathcal{O}(d), \mathcal{O}(d')) = 0$.

Proof. Using Exercise 3.6.6 and the identification

$$\mathrm{Hom}(\mathcal{F}, \mathcal{F}') \cong H^0(\mathrm{FF}_S, \mathcal{F}^\vee \otimes \mathcal{F}'),$$

this reduces to checking that $H^0(\mathrm{FF}_S, \mathcal{O}(d)) \neq 0$ whenever $d \geq 0$, which is Exercise 3.6.5; and that $H^0(\mathrm{FF}_S, \mathcal{O}(d)) = 0$ whenever $d < 0$, which follows from Corollary 3.6.7 (again, there are no nonzero maps from $\mathcal{O}$ to a semistable bundle of negative degree). $\square$

Exercise 3.6.10. For $\mathcal{F}, \mathcal{F}'$ vector bundles on FF_S, produce a canonical isomorphism

$$\mathrm{Ext}^1(\mathcal{F}, \mathcal{F}') \cong H^1(\mathrm{FF}_S, \mathcal{F}^\vee \otimes \mathcal{F}').$$

Deduce that if F is algebraically closed, then for $d, d' \in \mathbb{Q}$ with $d \geq d'$, we have $\mathrm{Ext}^1(\mathcal{O}(d), \mathcal{O}(d')) = 0$. (For general F, this remains true when $d > d'$.)

Exercise 3.6.11. For $d \in \mathbb{Q}$, show that as a (noncommutative) $\mathbb{Q}$-algebra, $\mathrm{End}(\mathcal{O}(d))$ is isomorphic to the division algebra over $\mathbb{Q}_p$ of invariant d. Remember (from local class field theory) that this algebra is split by *every* degree-d extension of $\mathbb{Q}_p$.

Exercise 3.6.12. Suppose that F is algebraically closed. Prove that $\mathrm{Pic}(\mathrm{FF}_S) \cong \mathbb{Z}$, i.e., every line bundle on FF_S is isomorphic to $\mathcal{O}(d)$ for some $d \in \mathbb{Z}$ (namely its degree). This is not true for general F; see Corollary 3.6.17 for the reason why.

Theorem 3.6.13 (Kedlaya, Fargues–Fontaine)**.** *Suppose that F is algebraically closed. Then every vector bundle on FF_S splits as a direct sum of vector bundles of the form $\mathcal{O}(d_i)$ for some $d_i \in \mathbb{Q}$.*

Proof. Using the alternate formulation in terms of finite projective modules over an extended Robba ring equipped with a semilinear φ-action (as in Remark 3.1.13), this result first appears in [96, Theorem 4.5.7]; the case where F is the completed algebraic closure of a power series field was previously treated in [94, Theorem 4.16] using similar methods. Using the formulation in terms of vector bundles on $\mathrm{Proj}(P_S)$, a different proof of this result was obtained by Fargues–Fontaine [57, Théorème 13.7].

The general strategy of both arguments can be characterized as follows. (This is further axiomatized in [57] into a theory of *generalized Riemann spheres*, but we give only a summary here.) We may proceed by induction on $\mathrm{rank}(\mathcal{F})$, the rank 1 case being Exercise 3.6.12. Given a bundle $\mathcal{F}$, one knows from Theorem 3.1.15 that it admits a filtration in which each successive quotient splits as a direct sum of copies of $\mathcal{O}(d)$ for a single value of d. By Corollary 3.4.18, the HN polygon of the associated graded module is an upper bound on $\mathrm{HN}(\mathcal{F})$. Since the degree function is discretely valued in this setting, we may choose a filtration whose associated graded module has minimal HN polygon (the minimal polygon need not *a priori* be unique but this doesn't matter); the crux of the argument is to show that any such filtration must split. By the induction hypothesis, this reduces to the case of a two-step filtration

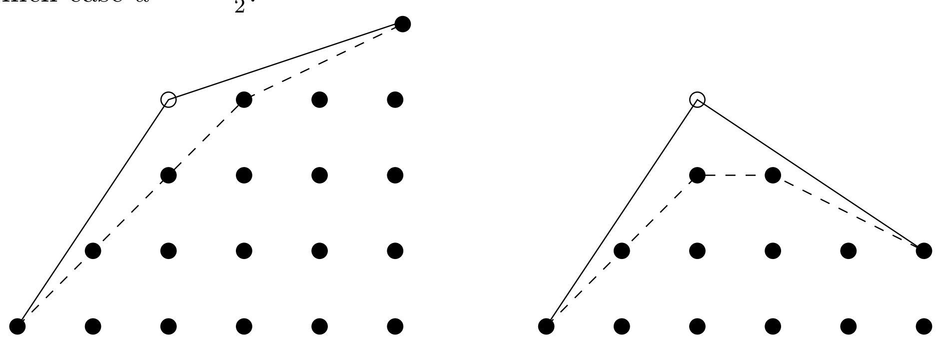
$$0 \to \mathcal{O}(d) \to \mathcal{F} \to \mathcal{O}(d') \to 0.$$

By Exercise 3.6.10, there is nothing to check unless $d < d'$; in this case, one must show that either the sequence splits or the filtration is not minimal. The former condition is equivalent to the HN polygon of $\mathcal{F}$ having slopes d and d'; see Remark 3.6.14.

Before discussing the proof of this further, we make a motivating observation. Write $d = \frac{r}{s}, d' = \frac{r'}{s'}$ in lowest terms, so that $\mu(\mathcal{F}) = \frac{r+r'}{s+s'}$. Now consider the subset of $\mathbb{Z}^2$ obtained by taking all of the points under $\mathrm{HN}(\mathcal{O}(d) \oplus \mathcal{O}(d'))$ *except* for the interior vertex (s', r'); the upper convex envelope is an upper bound for $\mathrm{HN}(\mathcal{F})$ as long as the latter is not equal to $\mathrm{HN}(\mathcal{O}(d) \oplus \mathcal{O}(d'))$. Let d'' be the least slope of this envelope. (See Figure 2 for two illustrated examples of this definition.)

In particular, if the filtration is not minimal, then $\mathrm{Hom}(\mathcal{O}(d''), \mathcal{F}) \neq 0$; in fact, this also holds if the filtration splits because $\mathrm{Hom}(\mathcal{O}(d''), \mathcal{O}(d')) \neq 0$ by Corollary 3.6.9. This suggests that our next step should be to prove that $\mathrm{Hom}(\mathcal{O}(d''), \mathcal{F}) \neq 0$ in all cases; using the

FIGURE 2. The effect of removing the interior vertex of the HN polygons of $\mathcal{O}(d) \oplus \mathcal{O}(d')$ with $(d, d') = \left(\frac{1}{3}, \frac{3}{2}\right)$, in which case $d'' = \frac{1}{2}$; and $(d, d') = \left(-\frac{2}{3}, \frac{3}{2}\right)$, in which case $d'' = -\frac{1}{2}$.



induction hypothesis, it is not hard to show that this is in fact sufficient to complete the proof (by showing that either $\mathcal{F}$ splits or the original filtration was not minimal).

Note that d, d'' are the slopes of two sides of a triangle with vertices at lattice points and containing no lattice points in its interior; if we write $d'' = \frac{r''}{s''}$ in lowest terms, this implies that $rs'' - r''s = 1$. By considering $\mathcal{F} \otimes \mathcal{O}(-d'')$ and invoking Exercise 3.6.6, we reduce to checking the following special case: for any short exact sequence

$$0 \to \mathcal{O}\left(-\frac{1}{n}\right) \to \mathcal{F} \to \mathcal{O}(1) \to 0 \tag{3.6.13.1}$$

one has $H^0(\mathrm{FF}_S, \mathcal{F}) \neq 0$, or equivalently the connecting homomorphism $H^0(\mathrm{FF}_S, \mathcal{O}(1)) \to H^1(\mathrm{FF}_S, \mathcal{O}\left(-\frac{1}{n}\right))$ is not injective. This can be done in several ways.

- One option is to check this directly using an *ad hoc* calculation, as in [94, Proposition 4.15].
- A cleaner option is to use the dimension theory for Banach–Colmez spaces as in [57, Théorème 12.9]. This ultimately depends on some calculations of Colmez [27, §7]. (Conversely, Theorem 3.6.13 can be used to establish the dimension theory for Banach–Colmez spaces; we will not work this out here.)
- It may also be possible to check this by identifying the moduli space of nonsplit extension as in (3.6.13.1) with a certain moduli space of p-divisible groups considered in [148]. However, to avoid a vicious circle, the relevant arguments from [148] would need to be reworked to avoid dependence on either [94] or [57]; for instance, this can be done using results of Hartl [80] and Faltings [54].

Using any of these approaches, the proof is completed. □

Remark 3.6.14. In Theorem 3.6.13, the multiset consisting of each d_i with multiplicity $\mathrm{rank}(\mathcal{O}(d_i))$ equals the slope multiset of the bundle, and hence is independent of the choice of the decomposition. Moreover, for each μ, the sum of all summands $\mathcal{O}(d_i)$ with $d_i \geq \mu$ is a step of the HN filtration, and hence independent of all choices. We will exploit these observations in the following corollaries.

Corollary 3.6.15. *For any inclusion $F \to F'$ of perfectoid fields, base extension from $X_{\mathrm{Spa}(F,\mathfrak{o}_F)}$ to $X_{\mathrm{Spa}(F',\mathfrak{o}_{F'})}$ preserves semistability of vector bundles.*

Corollary 3.6.16. *For F arbitrary, the tensor product of semistable vector bundles is semistable.*

Proof. Immediate from Exercise 3.6.6 and Theorem 3.6.13. □

By Theorem 3.6.13, if F is algebraically closed, then a vector bundle is semistable of degree 0 if and only if it is trivial. This formally promotes to a statement directly analogous to the Narasimhan–Seshadri theorem.

Corollary 3.6.17. *For F arbitrary, the category of vector bundles on FF_S which are semistable of degree 0 is equivalent to the category of continuous representations of G_F on finite-dimensional $\mathbb{Q}_p$-vector spaces via the functor $\mathcal{F} \mapsto H^0(X_{\mathrm{Spa}(\mathbb{C}_F, \mathfrak{o}_{\mathbb{C}_F})}, \mathcal{F})$ for $\mathbb{C}_F$ a completed algebraic closure of F.*

Remark 3.6.18. Corollary 3.6.17 is closely related to the theory of (φ, Γ)-*modules*, an important tool in p-adic Hodge theory which gives a useful description of the category of continuous representations of G_F, for F a finite extension of $\mathbb{Q}_p$, on finite-dimensional $\mathbb{Q}_p$-vector spaces. See [110] for a detailed discussion of how this older theory fits into the framework of perfectoid fields and spaces.

It is worth comparing Theorem 3.6.13 with the Dieudonné–Manin classification theorem.

Theorem 3.6.19. *In the notation of Remark 3.5.3, suppose that k is algebraically closed. Then every object of $\mathcal{C}$ splits as a direct sum of objects of the form $\mathcal{O}(d_i)$ for various $d_i \in \mathbb{Q}$, where for $d_i = \frac{r}{s}$ the object $\mathcal{O}(d_i)$ is a vector space on the generators $\mathbf{v}_1, \dots, \mathbf{v}_s$ equipped with the φ-action sending $\mathbf{v}_1, \dots, \mathbf{v}_s$ to $\mathbf{v}_2, \dots, \mathbf{v}_s, p^{-r}\mathbf{v}_1$.*

Proof. The original references are [41, 124]. Alternatively, see [37, §4.4] or [99, Theorem 14.6.3]. □

Remark 3.6.20. The main distinction between Theorem 3.6.13 and Theorem 3.6.19 is that when $d > d'$, the group $\mathrm{Hom}(\mathcal{O}(d), \mathcal{O}(d'))$ vanishes in the category of isocrystals but not in the category of vector bundles on FF_S. This implies that in the category of isocrystals over some perfect field k, the HN filtration splits uniquely; whereas in the category of vector bundles over FF_S, the HN filtration splits (by Exercise 3.6.10) but not uniquely.

Remark 3.6.21. In [57], the construction of FF_S is generalized from what we consider here by allowing the role of the field $\mathbb{Q}_p$ to be played by an arbitrary local field E of residue characteristic p. When E is of mixed characteristic, this does not give any essentially new results compared to the case we have considered. When E is of positive characteristic, one gets a distinct but closely analogous situation originally considered by Hartl–Pink [81], who proved the analogue of Theorem 3.6.13. See Remark 3.7.7 for more discussion.

3.7. **Slopes in families.** We next indicate how the slope formalism behaves in families, i.e., for vector bundles on relative Fargues–Fontaine curves.

Definition 3.7.1. Let **Pfd** be the category of perfectoid spaces of characteristic p. For $S \in$ **Pfd** and $\mathcal{F}$ a vector bundle on FF_S, for any morphism $\mathrm{Spa}(F, \mathfrak{o}_F) \to S$ with F a perfectoid field, we may pull back $\mathcal{F}$ to $X_{\mathrm{Spa}(F, \mathfrak{o}_F)}$ and compute its HN polygon. By Corollary 3.6.15, this depends only on the underlying point in S. We thus get a well-defined function $\mathrm{HN}(\mathcal{F}, \bullet)$ on S which is constant under specialization (i.e., it factors through the maximal Hausdorff quotient of S).

Theorem 3.7.2 (Kedlaya–Liu)**.** *For $S \in \mathbf{Pfd}$, let $\mathcal{F}$ be a vector bundle on FF_S.*

(a) *The function $\mathrm{HN}(\mathcal{F}, \bullet)$ is upper semicontinuous. That is, for any given polygon P, the set of $x \in S$ for which $\mathrm{HN}(\mathcal{F}, x) \leq P$ is open; moreover, this set is partially proper (i.e., stable under generization).*

(b) *If $\mathrm{HN}(\mathcal{F}, \bullet)$ is constant on S, then $\mathcal{F}$ admits a filtration which pulls back to the HN filtration at any point.*

Proof. For (a), see [107, Theorem 7.4.5]; note that what is asserted in [107] is *lower* semicontinuity because of the sign convention therein that HN polygons are concave up, not concave down (Remark 3.4.14). For (b), see [107, Corollary 7.4.10]. □

Corollary 3.7.3. *For $S \in \mathbf{Pfd}$ and $\mathcal{F} \in \mathbf{Vec}_{\mathrm{FF}_S}$, the set of points in S at which $\mathrm{HN}(\mathcal{F}, \bullet)$ has all slopes equal to zero is an open (and partially proper) subset of S, called the* étale locus *of $\mathcal{F}$.*

Definition 3.7.4. For $S \in \mathbf{Pfd}$, let S_{proet} denote the *pro-étale site* as defined in [163, Lecture 3]. To do later: add a quick summary of the definition.

For $S^\sharp$ an untilt of S, Corollary 2.5.10 induces a functorial homeomorphism $\sharp^* : S^\sharp_{\mathrm{proet}} \cong S_{\mathrm{proet}}$; if $S^\sharp$ is a space over $\mathbb{Q}_p$, this map factors through a functorial map $\sharp^* : X_{S,\mathrm{proet}} \to S_{\mathrm{proet}}$. We emphasize that this is *not* the pullback along a genuine map of spaces $X_S \to S$, although there is such a map in the category of diamonds (see Definition 4.3.7).

For $S \in \mathbf{Pfd}$, by an *étale $\mathbb{Q}_p$-local system* on S, we will mean a sheaf of $\underline{\mathbb{Q}_p}$-modules on S_{proet} which is locally finite free. For $S = \mathrm{Spa}(F, \mathfrak{o}_F)$, this is equivalent to a continuous representation of G_F on a finite-dimensional $\mathbb{Q}_p$-vector space; for S connected, there is a similar interpretation in terms of continuous representations of the étale fundamental group of S (see Remark 4.1.6).

Theorem 3.7.5 (Kedlaya–Liu)**.** *For $S \in \mathbf{Pfd}$, the functor $V \mapsto \sharp^{-1}(V) \otimes_{\underline{\mathbb{Q}_p}} \mathcal{O}_{\mathrm{FF}_S}$ defines an equivalence of categories between étale $\mathbb{Q}_p$-local systems on S and vector bundles on FF_S which at every point of S are semistable of degree 0; more precisely, there is a quasi-inverse functor taking $\mathcal{F}$ to the sheaf $\sharp_*\mathcal{F}$ given by $U \mapsto H^0(\mathrm{FF}_U, \mathcal{F})$. Moreover, this equivalence of categories equates sheaf cohomology groups on both sides.*

Proof. In light of Definition 3.8.1, this follows from [107, Theorem 9.3.13] (for the equivalence of categories) and [107, Theorem 8.7.13, Theorem 9.4.5] (for the comparison of cohomology). □

Remark 3.7.6. For S a point, every étale $\mathbb{Q}_p$-local system can be realized as the base extension of an étale $\mathbb{Z}_p$-local system, by using the compactness of G_F to obtain a stable lattice in the associated Galois representation. By contrast, for general (or even affinoid) S, an étale $\mathbb{Q}_p$-local systems only *locally* admits a stable lattice; this is unsurprising if one thinks of examples of étale coverings of rigid analytic spaces with noncompact groups of deck transformations, such as the Tate uniformization of an elliptic curve or the Lubin-Tate period mapping.

Remark 3.7.7. Suppose one were to construct a "moduli space" of vector bundles on Fargues–Fontaine curves with a certain property (which really just means a particular vector bundle on the curve over a particular base space). Then Theorem 3.7.2 would give rise to a locally closed stratification of the moduli space by HN polygons, and Theorem 3.7.5 would give rise to an étale $\mathbb{Q}_p$-local system over the (possibly empty) open stratum corresponding to the zero polygon.

In the next lecture, we will be interested precisely in moduli spaces of this type, parametrizing vector bundles with certain additional structures (reductions of the structure group, modifications along certain sections of the structure morphism). The category of diamonds provides a substantive (i.e., not meaninglessly formal) context in which such moduli spaces can be constructed, providing an approach to emulating certain constructions in positive characteristic which provide a geometric approach to the Langlands correspondence.

These developments are largely motivated by developments in the analogous setting in equal positive characteristic (see Remark 3.6.21), particularly the work of Hartl [79, 80] and Genestier–Lafforgue [66].

Theorem 3.7.5 has various applications beyond the scope of these notes. For example, it is an ingredient (together with the methods of [143], the properties of pseudocoherent sheaves such as Theorem 1.4.17, and a number of additional ideas) into the following theorem.

Theorem 3.7.8 (Kedlaya–Liu). *Let $f : X \to S$ be a smooth proper morphism of relative dimension n of rigid analytic spaces over a nonarchimedean field of mixed characteristics $(0, p)$. Let V be an étale $\mathbb{Q}_p$-local system on X.*

(a) *Let $f_{\mathrm{proet}} : X_{\mathrm{proet}} \to S_{\mathrm{proet}}$ be the morphism induced by f. Then for $i \geq 0$, $R^i f_{\mathrm{proet}\,*} V$ is an étale $\mathbb{Q}_p$-local system on S, which vanishes for $i > 2n$.*

(b) *The construction in (a) is compatible with the correspondence between étale $\mathbb{Q}_p$-local systems and vector bundles on Fargues–Fontaine curves described in Theorem 3.7.5. Formally, put $\mathcal{F} := \sharp^{-1}(V) \otimes_{\underline{\mathbb{Q}_p}} \mathcal{O}_{\mathrm{FF}_X}$, and let $g : \mathrm{FF}_X \to \mathrm{FF}_S$ be the projection induced by $f : X \to S$; then for $i \geq 0$, the natural morphism*
$$\sharp^{-1}(R^i f_{\mathrm{proet}\,*} V) \otimes_{\mathbb{Q}_p} \mathcal{O}_{\mathrm{FF}_S} \to R^i g_{\mathrm{proet}\,*} \mathcal{F}$$
is an isomorphism. (Note that X and S are not perfectoid here, so FF_X and FF_S must be interpreted in the context of diamonds, as in §4.3.)

(c) *Suppose that $S = \mathrm{Spa}(F, \mathfrak{o}_F)$ for F algebraically closed. Then for $i \geq 0$, $H^i(X_{\mathrm{proet}}, V)$ is a finite-dimensional $\mathbb{Q}_p$-vector space, which vanishes for $i > 2n$.*

(d) *Suppose that $S = \mathrm{Spa}(F, \mathfrak{o}_F)$ for F a finite extension of $\mathbb{Q}_p$. Then for $i \geq 0$, $H^i(X_{\mathrm{proet}}, V)$ is a finite-dimensional $\mathbb{Q}_p$-vector space, which vanishes for $i > 2n + 2$.*

Proof. See [109], but also Definition 3.8.1 below. □

3.8. **More on exotic topologies.** We have just seen and used one example of a topology on adic spaces finer than the étale topology, the *pro-étale topology* on the category of perfectoid spaces of characteristic p. In fact, there are quite a few exotic topologies at work in the theory of perfectoid spaces; we focus on a couple that will occur in the last lecture.

Definition 3.8.1. Recall the definition of the *pro-étale topology* from Definition 3.7.4. This definition was formulated for perfectoid spaces of characteristic p, but may also be used for any adic space (or even any preadic space; see §1.11).

This construction is the natural "pro" analogue[9] of the étale topology according to the general discussion of pro-categories in SGA 4 [8, Exposé I]. However, this is not the *pro-étale*

[9]This is an argument for spelling *pro-étale* with a hyphen: even if you share my preference for suppressing hyphens on prefixes, the *pro-étale topology* is really the *pro-(étale topology).*

topology as originally introduced in [143, §3] for locally noetherian spaces, then generalized to arbitrary adic spaces in [107, §9.1]. In that definition, one only considers inverse systems in which eventually all of the morphisms are finite and surjective.[10] This definition has the advantage of retaining certain features of the étale topology, such as the fact that a pro-étale morphism in this sense induces an open map of underlying topological spaces [143, Lemma 3.10(iv)], [107, Lemma 9.1.6(b)]. More seriously, under some conditions, the ring morphism associated to a pro-étale morphism of adic affinoid spaces is either flat, or at least preserves the category of complete pseudocoherent modules; for example, this holds when the base ring is perfectoid [108, Theorem 3.4.6], or when the morphism of spaces is a perfectoid subdomain of a seminormal (Exercise 2.9.6) affinoid space over a mixed-characteristic nonarchimedean field (see [143, Lemma 8.7(ii)] for the case where the base affinoid is smooth, and [108, Lemma 8.3.3] in the general case).

For this last reason, we propose to retronymically refer to the older version of the pro-étale topology as the *flattening pro-étale topology*.

Theorem 3.8.2. *For (A, A^+) a perfectoid pair, the structure presheaf on $\mathrm{Spa}(A, A^+)_{\mathrm{proet}}$ is an acyclic sheaf. The same is also true if we replace the pro-étale topology with the v-topology (Definition 3.8.5).*

Proof. For A Tate, this (in both cases) is a consequence of [108, Theorem 3.5.5]. The general case follows from this statement plus Theorem 1.3.4. □

Remark 3.8.3. By Theorem 3.8.2, the pro-étale topology on the category of perfectoid spaces is subcanonical (i.e., representable functors are sheaves; see Definition 1.11.1). By contrast, the pro-étale topology on other types of adic spaces is often not subcanonical. For instance, for K a nonarchimedean field of mixed characteristics, the largest subcategory of the category of rigid analytic spaces over K on which the pro-étale topology is subcanonical is the category of *seminormal* spaces (Exercise 2.9.6); see [108, Theorem 8.2.3].

Remark 3.8.4. In the category of schemes, it is useful to refine the étale topology to the *fpqc*[11] topology, in which any faithfully flat quasicompact morphism is treated as a covering. It would be useful to do something similar for affine spaces, but flatness is a tricky concept to deal with in the presence of topological completions (compare Remark 1.3.5).

However, there is an even finer topology for schemes which does admit a suitable adic analogue: the *h-topology* introduced by Voevodsky for use in $\mathbf{A}^1$-homotopy theory [160], in which the coverings are the universal topological epimorphisms (e.g., blowups). This topology is so fine that it is not even subcanonical on the full category of schemes; analogously to Remark 3.8.3, for excellent schemes over $\mathbb{Q}$, the maximal subcategory on which the h-topology is subcanonical is the category of seminormal schemes [160, Proposition 3.2.10], [88, Proposition 4.5], [108, Proposition 1.4.21]. For nonnoetherian schemes, a more useful variant of this topology has been introduced by Rydh [140].

Definition 3.8.5. By analogy with the h-topology, in [145] one finds consideration of the *faithful topology* on **Pfd**, in which a morphism $f : Y \to X$ is considered as a covering if f is surjective and every quasicompact open subset of X is contained in the image of a

[10]It should be possible to replace this condition with a Mittag-Leffler condition without changing the resulting topos.

[11]Acronym for *fidèlement plat quasi-compact*.

quasicompact open subset of Y. Based on the usage in [19] and the advance knowledge that the terminology would be changed in [147], this was renamed the *v-topology* in [108, §3.5], and we retain that terminology here. The v-topology on **Pfd** is subcanonical by Theorem 3.8.2.

Definition 3.8.6. By a *vector bundle* on an adic space with respect to the pro-étale topology or the v-topology, we mean a sheaf of $\mathcal{O}$-modules which is locally finite free. It is not reasonable to try to work with pseudocoherent sheaves at this level of generality, due to the use of blatantly nonflat covers; however, in [108] one does find such a notion for the flattening pro-étale topology.

Theorem 3.8.7. *For (A, A^+) a perfectoid pair, the pullback functor $\mathbf{FPMod}_A \to \mathbf{Vec}_{\mathrm{Spa}(A,A^+)_{\mathrm{proet}}}$ is an equivalence of categories. The same is also true if we replace the pro-étale topology with the v-topology (Definition 3.8.5).*

Proof. For A Tate, this (in both cases) is a consequence of [108, Theorem 3.5.8]. The general case follows from this statement plus Theorem 1.4.2. $\square$

Remark 3.8.8. Theorem 3.8.2 and Theorem 3.8.7, in the case of the v-topology, are analogous to certain results of Gabber about the h-topology on perfect schemes. See [20, §3], [19, Theorem 1.2].

4. Shtukas

The Langlands correspondence describes a relationship between Galois representations and automorphic forms extending class field theory, appropriately formulated as a statement about the algebraic group $\mathbf{G}_m$, to more general algebraic groups. In the setting where the Galois group in question is that of a function field over a finite field, there is a geometric approach pioneered by Drinfeld [43] (for the group GL_2) and subsequently extended by L. Lafforgue [118] (for the group GL_n) and V. Lafforgue [120] (for more general groups).

In this final lecture, we give some hints as to how the preceding discussion can be reformulated, using the language of *diamonds* introduced in [163, Lecture 3], in a manner that is consistent with geometric Langlands. This amounts to a segue into Scholze's Berkeley lecture notes [145].

4.1. Fundamental groups.

4.1. Fundamental groups. We first review basic facts about the profinite fundamental groups of schemes. A standard introduction to this topic is the book of Murre [130]; our presentation draws heavily on a course of de Jong [33].

Definition 4.1.1. For X a scheme, let $\mathbf{FEt}(X)$ denote the category of finite étale coverings of X; for A a ring, we write $\mathbf{FEt}(A)$ as shorthand for $\mathbf{FEt}(\mathrm{Spec}(A))$ and confuse an object of this category with its coordinate ring. The following observations about $\mathbf{FEt}$ will be useful.

(a) If $A = \varinjlim_i A_i$ in the category of rings, then the base extension functor from the 2-direct limit $\varinjlim_i \mathbf{FEt}(A_i)$ to the category $\mathbf{FEt}(A)$ is an equivalence of categories. (By [152, Tag 01ZC], the functor is fully faithful. By [152, Tag 00U2], any $B \in \mathbf{FEt}(A)$ is the base extension of some étale A_i-algebra B_i for some i. By [152, Tags 01ZO, 07RR], we may increase i to ensure that B_i is also finite and faithfully flat over A_i; hence the functor is essentially surjective.)

(b) If $f : Y \to X$ is a proper surjective morphism of schemes, then the functor from $\mathbf{FEt}(X)$ to descent data with respect to f (i.e., objects of $\mathbf{FEt}(Y)$ equipped with isomorphisms of their two pullbacks to $Y \times_X Y$ satisfying the cocycle condition on $Y \times_X Y \times_X Y$) is an equivalence of categories. For a similar statement with a much weaker hypothesis on f, see [140, Theorem 5.17].

The concept of a *Galois category* was originally introduced in SGA1 [73, Exposé V]. We instead take the approach of [152, Tag 0BMQ], starting with the definition from [152, Tag 0BMY].

Definition 4.1.2. Let $\mathcal{C}$ be a category and let $F : \mathcal{C} \to \mathbf{Set}$ be a covariant functor. We say that $(\mathcal{C}, F)$ is a *Galois category* if the following conditions hold.

(a) The category $\mathcal{C}$ admits finite limits and finite colimits.

(b) Every object of $\mathcal{C}$ is a (possibly empty) finite coproduct of connected objects. (Here $X \in \mathcal{C}$ is *connected* if it is not initial and every monomorphism $Y \to X$ is either a monomorphism or a morphism out of an initial object.)

(c) For every $X \in \mathcal{C}$, $F(X)$ is finite.

(d) The functor F is exact and reflects isomorphisms.

We often refer to F in this context as a *fiber*[12] *functor* by analogy with the primary example (Definition 4.1.3).

A key property of this definition is its relationship with profinite groups. Let G be the automorphism group of the functor F; then G is a finite group and the action of G on F induces an equivalence of categories between $\mathcal{C}$ and the category of finite G-sets [152, Tag 0BN4]. This abstracts the usual construction of the absolute Galois group of a field; see Remark 4.1.4.

Definition 4.1.3. For X a connected scheme, the category $\mathbf{FEt}(X)$ is a Galois category [152, Tag 0BNB]. For $\overline{x}$ a geometric point of X (i.e., a scheme over X of the form $\mathrm{Spec}(k)$ for k some algebraically closed field), the *profinite fundamental group* $\pi_1^{\mathrm{prof}}(X, \overline{x})$ is the automorphism group of the functor $\mathbf{FEt}(X) \to \mathbf{Set}$ taking Y to $|Y \times_X \overline{x}|$ (noting that $Y \times_X \overline{x}$ is a disjoint union of copies of $\overline{x}$); the point $\overline{x}$ is called the *basepoint* in this definition.

From the construction, we obtain a natural functor from $\mathbf{FEt}(X)$ to the category of finite sets equipped with $\pi_1^{\mathrm{prof}}(X, \overline{x})$-actions. Using properties of Galois categories, we see that $\pi_1^{\mathrm{prof}}(X, \overline{x})$ is profinite with a neighborhood basis of open subgroups given by the point stabilizers in $|Y \times_X \overline{x}|$ for each $Y \in \mathbf{FEt}(X)$. Moreover, the previous functor defines an equivalence between $\mathbf{FEt}(X)$ and the category of *finite* $\pi_1^{\mathrm{prof}}(X, \overline{x})$*-sets* for the profinite topology on $\pi_1^{\mathrm{prof}}(X, \overline{x})$ (i.e., finite sets with the discrete topology carrying continuous group actions).

Remark 4.1.4. For $X = \mathrm{Spec}(K)$ with K a field, a geometric point $\overline{x}$ of X amounts to a field embedding $K \to L$ with L algebraically closed, and $\pi_1^{\mathrm{prof}}(X, \overline{x})$ is the absolute Galois group of K acting on the separable closure of K in L. Similarly, for general X, if $\overline{x} = \mathrm{Spec}(L)$ is a geometric point lying over $x = \mathrm{Spec}(K) \in X$, then $\pi_1^{\mathrm{prof}}(X, \overline{x})$ remains (naturally) unchanged if we replace $\overline{x}$ by the spectrum of the separable or algebraic closure of K in L.

Remark 4.1.5. In Definition 4.1.3, the definition of $\pi_1^{\mathrm{prof}}(X, \overline{x})$ is independent of the choice of $\overline{x}$, but only in a weak sense: any two choices of basepoints gives a pair of groups and an isomorphism between them, but the latter is only specified up to composition with an inner automorphism. This includes the familiar fact that "the" absolute Galois group of a field F is only functorial up to inner automorphism, as its definition depends on the choice of an algebraic closure of F. It also corresponds to an analogous ambiguity for topological fundamental groups, arising from the fact that changing the basepoint of a loop requires choosing a particular isotopy class of paths from one point to the other. For this reason, the choice of an isomorphism $\pi_1^{\mathrm{prof}}(X, \overline{x}_1) \cong \pi_1^{\mathrm{prof}}(X, \overline{x}_2)$ is sometimes referred to as a *path* (in French, *chemin*) between the two basepoints $\overline{x}_1$ and $\overline{x}_2$.

Remark 4.1.6. The profinite fundamental group of a scheme is often called the *étale fundamental group* and denoted $\pi_1^{\mathrm{et}}(X, \overline{x})$. We avoid this terminology here for the following reasons.

For an ordinary topological space X (which is connected, locally path-connected, and locally simply connected) and a point $x \in X$, the fundamental group $\pi_1(X, x)$ (or retronymically, the *topological fundamental group*) can be interpreted in terms of deck transformations of covering space maps which need not be finite. If one uses only the finite covering space

[12] Also frequently spelled *fibre*, but this is due more to the influence of Francophones on the early development of this topic than to the standard discrepancies between US-style and UK-style spelling.

maps as in Definition 4.1.3, one instead obtains the profinite completion of $\pi_1(X, x)$, which we call the *profinite fundamental group* of X (with basepoint x) and denote by $\pi_1^{\mathrm{prof}}(X, x)$.

For a rigid analytic space X (or Berkovich space or adic space) and a geometric point $\overline{x}$ of X, one can again define a profinite fundamental group $\pi_1^{\mathrm{prof}}(X, x)$ using finite étale coverings as in Definition 4.1.3. However, there are interesting étale coverings which are not finite, such as the Tate uniformizations of elliptic curves of bad reduction. To account for this, de Jong [31] defines the étale fundamental group $\pi_1^{\mathrm{et}}(X, \overline{x})$ in terms of coverings which locally-on-the-target[13] split as disjoint unions of finite étale coverings. Again, the profinite completion of this group yields the profinite fundamental group. Despite this, though, the profinite fundamental group fails to detect many interesting examples; for instance, the Hodge–Tate period map discussed in [26] (which reinterprets the Gross-Hopkins period map [70], as discussed in [148]) gives rise to a connected étale covering of $\mathbf{P}^{1,\mathrm{an}}_{\mathbb{C}_p}$ with deck transformations by $\mathrm{PSL}_2(\mathbb{Q}_p)$, a group with no nontrivial finite quotients (consistent with the triviality of the profinite fundamental group of $\mathbf{P}^{1,\mathrm{an}}_{\mathbb{C}_p}$).

Let us now return to the case of schemes. Motivated by the previous examples, let us define the étale fundamental group in terms of deck transformations of coverings which are locally-on-the-target the disjoint unions of finite étale coverings. For X a normal connected scheme, X is irreducible and we may thus choose the base point $\overline{x}$ to lie over the generic point η of X; to compute fundamental groups, there is no harm in replacing X with its reduced closed subscheme, which has the same finite étale covers. We may then argue (see [152, Tag 0BQM]) that $\pi_1^{\mathrm{et}}(X, \overline{x})$ is a quotient of the absolute Galois group of η (i.e., the automorphism group of the integral closure of $\kappa(\eta)$ in $\kappa(\overline{x})$), hence is profinite, hence coincides with $\pi_1^{\mathrm{prof}}(X, \overline{x})$.

By contrast, if X is a scheme which is not normal, then its étale fundamental group need not be profinite. For example, let X be a nodal cubic curve in $\mathbf{P}^2_{\mathbb{C}}$. Let Y be the normalization of X, and let y_1, y_2 be the two distinct points in Y mapping to the node in X. Then for any basepoint $\overline{x}$, $\pi_1^{\mathrm{et}}(X, \overline{x})$ is isomorphic to $\mathbb{Z}$, with the corresponding cover being the "helical" covering of X obtained from the disjoint union $\coprod_{n \in \mathbb{Z}} Y_n$ of $\mathbb{Z}$-many copies of Y by identifying $y_2 \in Y_n$ with $y_1 \in Y_{n+1}$ for each $n \in \mathbb{Z}$. (Similar considerations apply when X is the scheme obtained by glueing two copies of $\mathbf{P}^1_{\mathbb{C}}$ along two distinct closed points.)

Remark 4.1.7. In order to construct the non-profinite fundamental groups described in Remark 4.1.6 using the formalism of Galois categories, one must modify the definition of a Galois category by relaxing some of the finiteness hypotheses. One candidate for a replacement definition is the concept of an *infinite Galois theory* given in [18, Definition 7.2.1]; this generalizes a construction of Noohi [132].

Remark 4.1.8. Another possible name for the profinite algebraic group is the *algebraic fundamental group*, but this terminology has at least two defects of its own. One is that in the context of complex manifolds, it may be interpreted as referring to the pro-algebraic completion with respect to the images of finite-dimensional linear representations; see for example [49]. The other is that it may be confused with Nori's *fundamental group scheme* of a variety over a field [133, 134].

Remark 4.1.9. Remark 4.1.6 is consistent with the behavior of étale $\mathbb{Q}_p$-local systems, which for analytic spaces correspond to representations of the étale fundamental group rather than

[13]This is not the same as defining this condition locally on the source. However, in the context of topological covering spaces the two would be equivalent.

the profinite fundamental group. This is also true for schemes for any natural definition of étale $\mathbb{Q}_p$-local systems, e.g., as locally finite free modules over the locally constant sheaf $\underline{\mathbb{Q}_p}$ on the pro-étale topology of X in the sense of Bhatt–Scholze [18].

Remark 4.1.10. Let $Y \to X$ be a morphism of connected schemes. Suppose that for every connected $Z \in \mathbf{FEt}(X)$, the scheme $Y \times_X Z$ is connected. Then for any geometric point $\overline{y}$ of Y, the map $\pi_1^{\mathrm{prof}}(Y, \overline{y}) \to \pi_1^{\mathrm{prof}}(X, \overline{y})$ is surjective.

Lemma 4.1.11. *Let $k \to k'$ be an extension of algebraically closed fields. Let X be a connected scheme over k. Then $X_{k'}$ is also connected.*

Proof. See [72, EGA IV.2, Théorème 4.4.4]. $\square$

Definition 4.1.12. We would like to think of the profinite fundamental group of a scheme as a "topological invariant", but this goal is hampered by a fundamental defect: it is not stable under base change. More precisely, if $k \to k'$ is an extension of algebraically closed fields and X is a connected scheme over k, then $X_{k'}$ is again connected by Lemma 4.1.11; for any geometric point $\overline{x}$ of $X_{k'}$, the morphism $\pi_1^{\mathrm{prof}}(X_{k'}, \overline{x}) \to \pi_1^{\mathrm{prof}}(X, \overline{x})$ is surjective. However, it is easy to exhibit examples where this map fails to be injective; see Example 4.1.13. If $\pi_1^{\mathrm{prof}}(X_{k'}, \overline{x}) \to \pi_1^{\mathrm{prof}}(X, \overline{x})$ is an isomorphism for any $k', \overline{x}$, we say that the morphism $X \to k$ is *π_1-proper*; this (highly nonstandard!) terminology is motivated by the fact that proper morphisms with connected total space have this property (Corollary 4.1.19).

Example 4.1.13. Let $k \to k'$ be an extension of algebraically closed fields of characteristic $p > 0$ and put $X := \mathrm{Spec}(k[T])$. For any geometric point $\overline{x}$ of X, the Artin–Schreier construction provides an identification

$$\mathrm{Hom}_{\mathbf{TopGp}}(\pi_1^{\mathrm{prof}}(X, \overline{x}), \mathbb{Z}/p\mathbb{Z}) \cong \bigoplus_{n>0, n \neq 0 \pmod{p}} kT^i$$

(for **TopGp** the category of topological groups). This group is not invariant under enlarging k.

Example 4.1.14. Let $k \to k'$ be an extension of algebraically closed fields of characteristic $p > 0$. Let X be a smooth, projective, connected curve of genus g over k. Then for any geometric point $\overline{x}$ of $X_{k'}$, $\mathrm{Hom}(\pi_1^{\mathrm{prof}}(X, \overline{x}), \mathbb{Z}/p\mathbb{Z})$ is a finite free $\mathbb{Z}/p\mathbb{Z}$-module of rank equal to the p-rank of X. This rank can be computed in terms of the geometric points of the p-torsion subscheme of the Jacobian, and thus is invariant under base change from k to k'. Thus the argument of Example 4.1.13 does not apply in this case, and indeed Corollary 4.1.19 below will imply that $\pi_1^{\mathrm{prof}}(X_{k'}, \overline{x}) \to \pi_1^{\mathrm{prof}}(X, \overline{x})$ is an isomorphism.

It turns out that the essential feature of Example 4.1.14 which separates it from Example 4.1.13 is properness. We show this through a series of arguments

Lemma 4.1.15. *Let $f : Y \to X$ be a morphism of schemes which are qcqs (quasicompact and quasiseparated). Suppose that the base change functor $\mathbf{FEt}(X) \to \mathbf{FEt}(Y)$ is an equivalence of categories.*

(a) *The map $\pi_0(X) \to \pi_0(Y)$ is a homeomorphism.*
(b) *Suppose that one of X or Y is connected. Then so is the other, and for any geometric point $\overline{y}$ of Y the map $\pi_1^{\mathrm{prof}}(Y, \overline{y}) \to \pi_1^{\mathrm{prof}}(X, \overline{y})$ is a homeomorphism.*

Proof. See [152, Tag 0BQA]. $\square$

Lemma 4.1.16. *Let $k \to k'$ be an extension of algebraically closed fields of characteristic* 0. *Let X be a k-scheme.*

(a) *The base change functor $\mathbf{FEt}(X) \to \mathbf{FEt}(X_{k'})$ is an equivalence of categories.*
(b) *If X is connected (as then is $X_{k'}$ by Lemma 4.1.11), then for any geometric point $\overline{x}$ of $X_{k'}$, the map $\pi_1^{\mathrm{prof}}(X_{k'}, \overline{x}) \to \pi_1^{\mathrm{prof}}(X, \overline{x})$ is a homeomorphism. That is, the morphism $X \to k$ is π_1-proper.*

Proof. We start with some initial reductions. We need only prove (a), as then (b) follows from Lemma 4.1.15. We may assume that X is affine. By writing the coordinate ring A of X as a direct limit of finitely generated k-subalgebras A_i and applying Definition 4.1.1(a) to both A and to $A \otimes_k k' = \varinjlim_i (A_i \otimes_k k')$, we may further reduce to the case where X is of finite type over k. By forming a hypercovering of X by smooth varieties using resolution of singularities and applying Definition 4.1.1(b), we may also assume that X is smooth. Using the Lefschetz principle, we may also assume that k and k' are contained in $\mathbb{C}$; we may then assume without loss of generality that $k' = \mathbb{C}$.

If X is connected, then so is $X_{\mathbb{C}}$ by Lemma 4.1.11, as then is $X_{\mathbb{C}}^{\mathrm{an}}$ by [73, SGA1, Exposé X, Proposition 2.4]; from this, it follows that $\mathbf{FEt}(X) \to \mathbf{FEt}(X_{\mathbb{C}})$ is fully faithful. To prove essential surjectivity, apply resolution of singularities to construct a compactification $\overline{X}$ of X whose boundary is a divisor Z of simple normal crossings. Given a finite étale cover of $X_{\mathbb{C}}$, we obtain a corresponding $\mathbb{Z}$-local system on $X_{\mathbb{C}}^{\mathrm{an}}$ with finite global monodromy; by the Riemann–Hilbert correspondence plus GAGA, this gives rise to a vector bundle on $X_{\mathbb{C}}$ equipped with an integrable connection having regular logarithmic singularities along $Z_{\mathbb{C}}$. The moduli stack of such objects is the base extension from k to $\mathbb{C}$ of a corresponding stack of finite type over k; since the base extension must consist of discrete points, these points coincide with the connected components of the stack, which remain invariant under base extension (Lemma 4.1.11 again). We thus obtain a vector bundle with integrable meromorphic connection on X itself; the sheaf of sections of this bundle is the underlying $\mathcal{O}_X$-module of a finite étale $\mathcal{O}_X$-algebra descending the original cover of $X_{\mathbb{C}}$. $\square$

Remark 4.1.17. From the proof of Lemma 4.1.16, we see that if X is a smooth scheme over an algebraically closed field k of characteristic 0, $\pi_1^{\mathrm{prof}}(X, \overline{x})$ can be computed as the profinite completion of $\pi_1(X_{\mathbb{C}}^{\mathrm{an}}, \overline{x})$ for any embedding $k \to \mathbb{C}$ (and any closed point $\overline{x}$ of $X_{\mathbb{C}}$). However, even if X is projective, the group $\pi_1(X_{\mathbb{C}}^{\mathrm{an}}, \overline{x})$ is not in general independent of the choice of the embedding $k \to \mathbb{C}$, as first observed by Serre [151].

Lemma 4.1.18. *Let A be a henselian local ring with residue field κ. Let $f : X \to S :=$ $\mathrm{Spec}(A)$ be a proper morphism of schemes. Then the base change functor $\mathbf{FEt}(X) \to \mathbf{FEt}(X \times_S \mathrm{Spec}(\kappa))$ is an equivalence of categories.*

Proof. This is a relatively easy argument in terms of relatively difficult theorems (on algebraization and approximation). See [152, Tag 0A48]. $\square$

Corollary 4.1.19. *Let $k \to k'$ be an extension of algebraically closed fields (of any characteristic). Let X be a proper k-scheme.*

(a) *The base change functor $\mathbf{FEt}(X) \to \mathbf{FEt}(X_{k'})$ is an equivalence of categories.*

(b) *If X is connected (as then is $X_{k'}$ by Lemma 4.1.11), then for any geometric point $\overline{x}$ of $X_{k'}$, the map $\pi_1^{\mathrm{prof}}(X_{k'}, \overline{x}) \to \pi_1^{\mathrm{prof}}(X, \overline{x})$ is a homeomorphism. That is, the morphism $X \to k$ is π_1-proper.*

Proof. Part (a) is obtained from Lemma 4.1.18 by writing k' as a direct limit of finitely generated k-algebras; see [152, Tag 0A49]. Given (a), (b) follows from Lemma 4.1.15. ☐

Remark 4.1.20. If $k = \mathbb{C}$ and X is proper over k, then the GAGA theorem, as extended to the proper case in SGA1 [73, Exposé XII], implies that any finite covering space map of the analytification X^{an} of X is in fact the analytification of a finite étale cover of X. Hence if $\overline{x}$ is a geometric point lying over a closed point x of X, then $\pi_1^{\mathrm{prof}}(X, \overline{x})$ can be interpreted as the profinite completion of $\pi_1(X^{\mathrm{an}}, x)$.

We now turn to analogues of the homotopy exact sequence of a fiber bundle of topological spaces. The following result, similar in spirit to Stein factorization, is a refinement of [73, SGA1, Exposé X, Corollaire 1.3] adapted from a similar result for diamonds [145, Proposition 17.3.10].

Lemma 4.1.21. *Let $X \to S$ be a qcqs morphism of schemes with connected, π_1-proper geometric fibers. Assume in addition that for every geometric point $\overline{s}$ of S, every connected finite étale covering of $X \times_S \overline{s}$ extends to a finite étale covering of $X \times_S U$ with connected geometric fibers over some étale neighborhood U of $\overline{s}$ in S. Then for any finite étale morphism $X' \to X$, there exists a commutative diagram*

$$\begin{array}{ccc} X' & \longrightarrow & X \\ \downarrow & & \downarrow \\ S' & \longrightarrow & S \end{array}$$

such that $S' \to S$ is finite étale and $X' \to S'$ has geometrically connected fibers. Additionally, this diagram is initial among diagrams

$$\begin{array}{ccc} X' & \longrightarrow & X \\ \downarrow & & \downarrow \\ T & \longrightarrow & S \end{array}$$

where $T \to S$ is finite étale; in particular, it is unique up to unique isomorphism.

Proof. In light of the uniqueness statement, the claim is fpqc-local on S; by Lemma 4.1.11, the hypothesis is also fpqc-local on S. We may thus assume first that S is affine and reduced (since replacing S by its reduced closed subscheme does not change its étale site), and second that S is *strictly w-local* in the sense of [18]; in particular, every finite étale covering of a closed-open subspace of S splits. In this case, the uniqueness property is vacuously true, and we need only check existence; this amounts to showing that X' splits as a finite disjoint union of closed-open subspaces, each of which maps to some closed-open subspace of S with geometrically connected fibers.

It suffices to work étale-locally around some geometric point $\overline{s} \in S$. By the qcqs hypothesis, the functor

$$2\text{-}\varinjlim_{U \ni \overline{y}} \mathbf{FEt}(X \times_S U) \to \mathbf{FEt}(X \times_S \overline{s}), \tag{4.1.21.1}$$

where U runs over étale neighborhoods of $\overline{s}$ in S, is an equivalence of categories. We may thus reduce to the case where $X \times_S \overline{s}$ is connected, in which case we must produce U so that $X' \times_S U$ has connected geometric fibers over U. By shrinking U, we may first ensure that $X' \times_S \overline{s}$ lifts to some finite étale cover of $X \times_S U$ with connected geometric fibers over U (by hypothesis), and second that this cover is isomorphic to X' (again because (4.1.21.1) is an equivalence). $\square$

This then yields a variant of [73, SGA1, Exposé X, Corollaire 1.4] adapted from [145, Proposition 17.3.13].

Corollary 4.1.22. *With notation and hypotheses as in Lemma 4.1.21, suppose in addition that S is connected. Then X is connected, and for any geometric point $\overline{x}$ of X mapping to the geometric point $\overline{s}$ of S, the sequence*

$$\pi_1^{\mathrm{prof}}(X \times_S \overline{s}, \overline{x}) \to \pi_1^{\mathrm{prof}}(X, \overline{x}) \to \pi_1^{\mathrm{prof}}(S, \overline{s}) \to 1$$

is exact.

Proof. We first check that X is connected. It is apparent that $X \neq \emptyset$. Suppose by way of contradiction that X disconnects as $X_1 \sqcup X_2$. For any geometric point $\overline{s} \in S$, $X \times_S \overline{s}$ is connected by hypothesis, so one of $X_1 \times_S \overline{s}, X_2 \times_S \overline{s}$ must be empty. Suppose that $X_1 \times_S \overline{s}$ is empty; this space can be rewritten as the inverse limit $\varprojlim_U X_1 \times_S U$ for U running over étale neighborhoods of $\overline{s}$ in S. At the level of topological spaces, we have an inverse limit of spectral spaces and spectral morphisms, which can only be empty if it is empty at some term. (For the constructible topologies, this is an inverse limit of compact Hausdorff spaces, which by Tikhonov's theorem cannot be empty if none of the terms is empty.) It follows that $\{s \in S : X_{1,s} = \emptyset\}$ is open, as then is $\{s \in S : X_{2,s} = \emptyset\}$. Since these sets cannot overlap, they form a disconnect of S, a contradiction.

By the previous paragraph, if $S' \to S$ is finite étale and S' is connected, then so is $X \times_S S'$. By Remark 4.1.10, $\pi_1^{\mathrm{prof}}(X, \overline{x}) \to \pi_1^{\mathrm{prof}}(S, \overline{s})$ is surjective.

Let G be a finite quotient of $\pi_1^{\mathrm{prof}}(X, \overline{x})$ corresponding to $X' \in \mathbf{FEt}(X)$. Let $G \to H$ be the quotient corresponding to a Galois cover $S' \to S$ as produced by Lemma 4.1.21 (the uniqueness property of that result implies the Galois property of the cover). Since $X' \to S'$ has geometrically connected fibers, the map $\pi_1^{\mathrm{prof}}(X \times_S \overline{s}, \overline{x}) \to \ker(G \to H)$ must be surjective. This completes the proof of exactness. $\square$

This in turn yields a variant of [73, Exposé X, Corollaire 1.7], giving a Künneth formula for fundamental groups of products.

Corollary 4.1.23. *Let k be an algebraically closed field and put $S := \mathrm{Spec}(k)$. Let $X \to S, Y \to S$ be morphisms such that Y is connected and $X \to S$ is qcqs and π_1-proper. (The π_1-proper condition holds if k is of characteristic 0, by Lemma 4.1.16, or if $X \to S$ is proper, by Corollary 4.1.19.) Then $Z := X \times_S Y$ is connected, and for any geometric point $\overline{z}$ of Z the map*

$$\pi_1^{\mathrm{prof}}(Z, \overline{z}) \to \pi_1^{\mathrm{prof}}(X, \overline{z}) \times \pi_1^{\mathrm{prof}}(Y, \overline{z})$$

is an isomorphism of topological groups.

Proof. Apply Corollary 4.1.22 to the morphism $Z \to Y$; both hypotheses of Lemma 4.1.21 are satisfied because $X \to S$ is π_1-proper. We then have a commutative diagram of groups

$$\begin{tikzcd}
\pi_1^{\mathrm{prof}}(Z_{\overline{s}}, \overline{z}) \arrow[r] \arrow[dr, equal] & \pi_1^{\mathrm{prof}}(Z, \overline{z}) \arrow[r] \arrow[d] & \pi_1^{\mathrm{prof}}(Y, \overline{z}) \arrow[r] & 1 \\
& \pi_1^{\mathrm{prof}}(X, \overline{z}) & &
\end{tikzcd}$$

in which the top row is exact. This proves the claim. □

Although we do not use it here, we wish to point out the following recent result of Achinger [3, Theorem 1.1.1].

Definition 4.1.24. For X a connected scheme, we say that X is a $K(\pi, 1)$ *scheme* if for some (hence any) geometric point $\overline{x}$ of X, for every locally constant sheaf of finite abelian groups $\mathcal{F}$ on X_{et}, the natural maps

$$H^*(\pi_1^{\mathrm{prof}}(X, \overline{x}), \mathcal{F}_{\overline{x}}) \to H^*(X_{\mathrm{et}}, \mathcal{F}) \tag{4.1.24.1}$$

are isomorphisms. This is analogous to the corresponding definition in topology, which can be formulated as the assertion that the higher homotopy groups of X all vanish. We may similarly define the concept of a $K(\pi, 1)$ *adic space*.

Remark 4.1.25. The usual definition of a $K(\pi, 1)$ scheme imposes the condition on (4.1.24.1) only for torsion sheaves whose order is invertible on X (see for example [135, Definition 5.3], [1, Definition 9.20]). We need the stronger restriction here in order to pass the condition through the tilting equivalence.

Theorem 4.1.26 (Achinger)**.** *Let X be a connected affine scheme over $\mathbb{F}_p$. Then X is a $K(\pi, 1)$ scheme.*

As in [3, Theorem 6.4.2], this yields the following corollary.

Corollary 4.1.27. *Let $X := \mathrm{Spa}(A, A^+)$ be a connected Tate adic affinoid space on which p is topologically nilpotent. Then X is a $K(\pi, 1)$ adic space.*

Proof. In case X is affinoid perfectoid, the statements follow by applying Corollary 2.5.10 to reduce to the case of an affinoid perfectoid space in characteristic p, then reducing to Theorem 4.1.26 via an algebraization argument (see [3, Proposition 6.4.1]). This implies the general case using Theorem 2.9.9. □

Remark 4.1.28. In Theorem 4.1.26, the isomorphism in (4.1.24.1) is easy to verify for p-torsion coefficients using the Artin–Schreier construction. The subtle part is to extend this argument to all coefficients; this makes use of certain very strong results on the presentation of schemes of finite type over a positive-characteristic field as finite étale covers of affine spaces, in the spirit of [93, 95]. (The one-dimensional cases of such results may be viewed as positive-characteristic analogues of Belyi's theorem on covers of $\mathbf{P}^1$ ramified over three points, as in [68, §4].)

Remark 4.1.29. In Corollary 4.1.27, the condition that p be topologically nilpotent is essential: there exist affinoid spaces over $\mathbb{C}((t))$ which are not $K(\pi, 1)$ spaces. An explicit example is the closed subspace of the unit 3-ball in x, y, z cut out by the equation $xy = z^2 - t$; see [2, §7] for a closely related example.

4.2. **Drinfeld's lemma.** We next introduce a fundamental result of Drinfeld[14] which gives a replacement for the Künneth formula for fundamental groups (Corollary 4.1.23) for products of schemes in characteristic p. More precisely, the original result of Drinfeld [45, Theorem 2.1], [46, Proposition 6.1] gives a key special case (see Remark 4.2.13); the general case is due to E. Lau [121, Theorem 8.1.4], except for a superfluous restriction to schemes of finite type. See also [145, Theorem 17.2.4].

Definition 4.2.1. For any scheme X over $\mathbb{F}_p$, let $\varphi_X : X \to X$ be the *absolute Frobenius* morphism, induced by the p-th power map on rings. For $f : Y \to X$ a morphism of schemes, define the *relative Frobenius* $\varphi_{Y/X} : Y \to \varphi_X^* Y$ to be the unique morphism making the diagram

$$\begin{tikzcd}
Y \arrow[dr, "\varphi_{Y/X}"] \arrow[drr, bend left, "\varphi_Y"] \arrow[ddr, bend right, "f"'] & & \\
& \varphi_X^* Y \arrow[r, "f^*\varphi_X"] \arrow[d, "\varphi_X^* f"] & Y \arrow[d, "f"] \\
& X \arrow[r, "\varphi_X"] & X
\end{tikzcd}$$

commute.

The following argument is similar in style to the proof of Serre's GAGA theorem [150].

Lemma 4.2.2. *Let X be a projective scheme over $\mathbb{F}_p$. Let k be a separably closed field of characteristic p. Then pullback along $X_k \to X$ defines an equivalence of categories between coherent sheaves on X and coherent sheaves on X_k equipped with isomorphisms with their φ_k-pullbacks. Moreover, for $\mathcal{F}$ a coherent sheaf on X, the induced maps*

$$H^i(X, \mathcal{F}) \otimes_{\mathbb{F}_p} k \to H^i(X_k, \mathcal{F})$$

are φ-equivariant isomorphisms.

Proof. The assertion about comparison of cohomology is a consequence of flat base change (this step is trivial compared to the analogous step in GAGA), and immediately implies that the pullback functor is fully faithful (by forming internal Homs and comparing H^0 groups).

It thus remains to prove essential surjectivity. In the case $R = \mathbb{F}_p$, this is a result of Lang, as reported by Katz in SGA 7 [36, Exposé XXII, Proposition 1.1]. We summarize the argument in the style of [107, Lemma 3.2.6]: if V is a vector space with basis $\mathbf{e}_1, \dots, \mathbf{e}_n$ over k equipped with the action of φ_k taking $\mathbf{e}_j$ to $\sum_i A_{ij}\mathbf{e}_i$, then the closed subscheme X of $\operatorname{Spec} k[U_{ij} : i, j = 1, \dots, n]$ cut out by the matrix equation $\varphi(U) = A^{-1}U$ is finite (evidently) and étale (by the Jacobian criterion) over $\operatorname{Spec}(k)$, and so splits as a disjoint union of k-rational points (because k is separably closed). Projecting to a component of this disjoint union, we obtain elements $\mathbf{v}_1, \dots, \mathbf{v}_n$ of V defined by $\mathbf{v}_j = \sum_i U_{ij}\mathbf{e}_i$ which are fixed by φ; for a suitable choice of component, these elements form a basis of V.

To treat the general case, fix an ample line bundle $\mathcal{O}(1)$ on X; we can then identify X with the Proj of the graded ring $\bigoplus_{n=0}^\infty \Gamma(X, \mathcal{O}(n))$, X_k with the Proj of the graded ring $\bigoplus_{n=0}^\infty \Gamma(X_k, \mathcal{O}(n))$, and $\mathcal{F}$ with the sheaf associated to the graded module $\bigoplus_{n=0}^\infty \Gamma(X_k, \mathcal{F}(n))$. Each graded piece of this module is a finite-dimensional k-vector space, so we may apply the

[14]A more accurate transliteration of Дринфельд would be *Drinfel'd*, but this would lead to the typographical monstrosity of *Drinfel'd's lemma*.

previous paragraph to write it as $S_n \otimes_{\mathbb{F}_p} k$ for $S_n = \Gamma(X_k, \mathcal{F}(n))^{\varphi_k}$. The sheaf $\mathcal{F}$ then arises as the pullback of the sheaf on X associated to the graded module $\bigoplus_{n=0}^\infty S_n$. (Compare [46, Proposition 1.1], [117, I.3, Lemme 3], [121, Lemma 8.1.1], [145, Lemma 17.2.6].) $\square$

Remark 4.2.3. As with the GAGA theorem (see SGA 1 [73, Expose XII]), using Chow's lemma one can immediately promote Lemma 4.2.2 to the case where X is proper over $\mathbb{F}_p$. However, it does not hold if we only require X to be of finite type over $\mathbb{F}_p$. For example, take $X = \operatorname{Spec}(k[T^\pm])$ and $\mathcal{F} = \tilde{M}$ for M the free module on the single generator $\mathbf{v}$ equipped with the φ_k-action taking $\mathbf{v}$ to $T\mathbf{v}$; then M cannot have a φ_k-invariant element.

Using the previous argument, we may show that "quotienting by relative Frobenius" can be used to mitigate failures of π_1-properness.

Definition 4.2.4. For X a scheme and Γ a group of automorphisms of X, let $\mathbf{FEt}(X/\Gamma)$ denote the category of finite étale coverings Y equipped with an action of Γ. That is, we must specify isomorphisms $Y \to \gamma^* Y$ for each $\gamma \in \Gamma$, subject to the condition that for $\gamma_1, \gamma_2 \in Y$, composing the γ_1-pullback of $Y \to \gamma_2^* Y$ with $Y \to \gamma_1^* Y$ yields the chosen map $Y \to (\gamma_1\gamma_2)^* Y$.

We say that X is Γ-*connected* if X is nonempty and its only Γ-stable closed-open subset are itself and the empty set. If X is Γ-connected, then for any geometric point $\overline{x}$ of X, the category $\mathbf{FEt}(X/\Gamma)$ equipped with the fiber functor $Y \mapsto |Y \times_X \overline{x}|$ is a Galois category in the sense of Definition 4.1.2; the argument is the same as in [152, Tag 0BNB] except for condition (b), in which the Γ-connected hypothesis is used. We then write $\pi_1^{\mathrm{prof}}(X/\Gamma, \overline{x})$ for the automorphism group of this fiber functor.

In these notations, when Γ is generated a single element γ, we will typically write X/γ in place of X/Γ.

We need the following variant of Definition 4.1.1(a).

Lemma 4.2.5. *Let $X = \operatorname{Spec}(A)$ be an affine scheme over $\mathbb{F}_p$. Let k be a field of characteristic p. Write A as a filtered direct limit of finitely generated $\mathbb{F}_p$-subalgebras A_i. Then the base extension functor*

$$2\text{-}\varinjlim \mathbf{FEt}((A_i \otimes_{\mathbb{F}_p} k)/\varphi_k) \to \mathbf{FEt}((A \otimes_{\mathbb{F}_p} k)/\varphi_k)$$

is an equivalence of categories.

Proof. By the same argument as in Definition 4.1.1(a) , the functor

$$2\text{-}\varinjlim \mathbf{FEt}(A_i \otimes_{\mathbb{F}_p} k) \to \mathbf{FEt}(A \otimes_{\mathbb{F}_p} k)$$

is an equivalence of categories. This implies immediately that the given functor is fully faithful. To establish essential surjectivity, note that for $B \in \mathbf{FEt}((A \otimes_{\mathbb{F}_p} k)/\varphi_k)$, we know that for some index i, B descends to $B_i \in \mathbf{FEt}(A_i \otimes_{\mathbb{F}_p} k)$ while $\varphi_k^* B$ descends to $\varphi_k^* B_i'$ for some $B_i' \in \mathbf{FEt}(A_i \otimes_{\mathbb{F}_p} k)$. In addition, the isomorphisms

$$B_i \otimes_{A_i} A \cong B_i' \otimes_{A_i} A, \qquad \varphi_k^*(B_i' \otimes_{A_i} A) \cong B_i \otimes_{A_i} A$$

both descend to $\mathbf{FEt}(A_j \otimes_{\mathbb{F}_p} k)$ for some j. This proves the claim. $\square$

Lemma 4.2.6. *Let X be a scheme over $\mathbb{F}_p$. Let k be an algebraically closed field of characteristic p. Then the base extension functor*

$$\mathbf{FEt}(X) \to \mathbf{FEt}(X_k/\varphi_k)$$

is an equivalence of categories, with the quasi-inverse functor being given by taking φ_k-invariants.

Proof. We first reduce to the case where X is affine. Using Lemma 4.2.5, we further reduce to the case where X is of finite type over $\mathbb{F}_p$. Applying Definition 4.1.1(b) to a suitable covering, we further reduce[15] to the case where X is normal and connected. Choose an open immersion $X \to X'$ with X' normal and projective over $\mathbb{F}_p$. Now note that the following categories are equivalent (using Lemma 4.2.2 between (b) and (c)):

(a) finite étale morphisms $Y \to X_k$ with isomorphisms $\varphi_k^* Y \cong Y$;
(b) finite morphisms $Y \to X'_k$ with Y normal and étale over X_k with isomorphisms $\varphi_k^* Y \cong Y$;
(c) finite morphisms $Y \to X'$ with Y normal and étale over X;
(d) finite étale morphisms $Y \to X$.

This proves the claim. (Compare [117, IV.2, Théorème 4], [121, Lemma 8.1.2], [145, Lemma 17.2.6].) $\square$

Example 4.2.7. Let k be an algebraic closure of $\mathbb{F}_p$ and put $X = \mathrm{Spec}(k)$. Then X_k is highly disconnected: there is a natural homeomorphism $\pi_0(X_k) \cong \mathrm{Gal}(k/\mathbb{F}_p) \cong \widehat{\mathbb{Z}}$. However, the action of φ_k on $\pi_0(X)$ is via translations by the dense subgroup $\mathbb{Z}$ of $\widehat{\mathbb{Z}}$; consequently, there is no φ_k-stable disconnection of X, as predicted by Lemma 4.2.6.

Corollary 4.2.8. *Let X be a connected scheme over $\mathbb{F}_p$. Let k be an algebraically closed field of characteristic p.*

(a) *The scheme X_k is φ_k-connected.*
(b) *For any geometric point $\overline{x}$ of X, the map*

$$\pi_1^{\mathrm{prof}}(X, \overline{x}) \to \pi_1^{\mathrm{prof}}(X_k/\varphi_k, \overline{x})$$

is a homeomorphism of profinite groups.

Proof. Let k_0 be the integral closure of $\mathbb{F}_p$ in k; by Lemma 4.1.11, we have $\pi_0(X_{k_0}) = \pi_0(X_k)$. We may thus argue as in Example 4.2.7, i.e., by identifying $\pi_0(X_{k_0})$ with a quotient of $\widehat{\mathbb{Z}}$ on which φ_k acts via translation by the dense subgroup $\mathbb{Z}$. This proves (a). Given (a), (b) follows immediately from Lemma 4.2.6. $\square$

This then leads to a corresponding mitigation for products of varieties.

Remark 4.2.9. For X a connected scheme over $\mathbb{F}_p$, if we view $\mathbf{FEt}(X/\varphi)$ as the category of pairs (Y, σ) where $Y \in \mathbf{FEt}(X)$ and $\sigma : X \to \varphi_X^* Y$ is a single isomorphism, then the forgetful functor $\mathbf{FEt}(X/\varphi) \to \mathbf{FEt}(X)$ admits a distinguished section taking Y to $(Y, \varphi_{Y/X})$. However, this section is not an isomorphism: whereas every *connected* finite étale cover Y of X admits only the action by $\varphi_{Y/X}$ (which commutes with all automorphisms of Y over X), for a disconnected cover this action may be twisted by an automorphism of Y over X that permutes connected components. From this, one deduces that for $\overline{x}$ a geometric point of X, there is a canonical isomorphism

$$\pi_1^{\mathrm{prof}}(X/\varphi, \overline{x}) \cong \pi_1^{\mathrm{prof}}(X, \overline{x}) \times \widehat{\mathbb{Z}} \cong \pi_1^{\mathrm{prof}}(X, \overline{x}) \times G_{\mathbb{F}_p}.$$

[15] Had it been helpful to do so, we could have added de Jong's alterations theorem [32] into this argument to further reduce to the case where X is smooth and admits a compactification with good boundary.

Definition 4.2.10. Let $X_1, \dots, X_n$ be schemes over $\mathbb{F}_p$ and put $X := X_1 \times_{\mathbb{F}_p} \cdots \times_{\mathbb{F}_p} X_n$. Write φ_i as shorthand for φ_{X_i}. Define the category

$$\mathbf{FEt}(X/\Phi) := \mathbf{FEt}(X/\langle \varphi_1, \dots, \varphi_n \rangle) \times_{\mathbf{FEt}(X/\varphi_X)} \mathbf{FEt}(X)$$

via the functor $\mathbf{FEt}(X) \to \mathbf{FEt}(X/\varphi_X)$ described in Remark 4.2.9. In other words, an object of $\mathbf{FEt}(X/\varphi)$ is a finite étale covering $Y \to X$ equipped with commuting isomorphisms $\beta_i : Y \cong \varphi_i^* Y$ whose composition is $\varphi_{Y/X}$. (Here "composition" and "commuting" must be interpreted suitably: by the "composition" $\beta_i \circ \beta_j$, we really mean $(\beta_j^* \beta_i) \circ \beta_j$.) Note that for any $i \in \{1, \dots, n\}$, there is a canonical equivalence of categories

$$\mathbf{FEt}(X/\Phi) \cong \mathbf{FEt}(X/\langle \varphi_1, \dots, \widehat{\varphi}_i, \dots, \varphi_n \rangle).$$

In case $X_1, \dots, X_n$ are connected, by Lemma 4.2.11 we may obtain a Galois category in the sense of Definition 4.1.2 by considering the usual fiber functor defined by any geometric point $\overline{x}$ of X; we denote the corresponding group by $\pi_1^{\mathrm{prof}}(X/\Phi, \overline{x})$.

Lemma 4.2.11. *With notation as in Definition 4.2.10, if $X_1, \dots, X_n$ are connected, then X is $\langle \varphi_1, \dots, \widehat{\varphi}_i, \dots, \varphi_n \rangle$-connected for any $i \in \{1, \dots, n\}$. We say for short that X is Φ-connected.*

Proof. Using Corollary 4.2.8, this follows as in the proof of Corollary 4.1.22. □

Theorem 4.2.12 ("Drinfeld's lemma"). *Let $X_1, \dots, X_n$ be connected qcqs schemes over $\mathbb{F}_p$ and put $X := X_1 \times_{\mathbb{F}_p} \cdots \times_{\mathbb{F}_p} X_n$. Then for any geometric point $\overline{x}$ of X, the map*

$$\pi_1^{\mathrm{prof}}(X/\Phi, \overline{x}) \to \prod_{i=1}^n \pi_1^{\mathrm{prof}}(X_i, \overline{x})$$

is an isomorphism of topological groups.

Proof. In light of Definition 4.2.10, we may rewrite the group on the left as

$$\pi_1^{\mathrm{prof}}(X_1 \times_{\mathbb{F}_p} (X_2/\varphi) \times_{\mathbb{F}_p} \cdots \times_{\mathbb{F}_p} (X_n/\varphi), \overline{x}).$$

We may then proceed by induction on n, with the base case $n = 1$ being trivial. The induction step follows from Lemma 4.2.6 as in the proof of Corollary 4.1.23; more details may be added later. □

Remark 4.2.13. The original result of Drinfeld [46, Proposition 6.1] is somewhat more restrictive than Theorem 4.2.12; it treats the case where $n = 2$ and $X_1 = X_2 = \mathrm{Spec}(F)$ where F is the function field of a curve over a finite field. See [120, Lemme 8.2] for further discussion of this case, including additional references.

In the spirit of the theory of diamonds, one may reinterpret Drinfeld's lemma as follows.

Remark 4.2.14. Let **Perf** denote the category of perfect schemes over $\mathbb{F}_p$. Identify each $X \in \mathbf{Perf}$ with the representable functor $h_X : \mathbf{Perf} \to \mathbf{Set}$ taking Y to $\mathrm{Hom}_{\mathbf{Perf}}(Y, X)$, which is a sheaf for the Zariski, étale, and fpqc topologies. Let $X/\varphi : \mathbf{Pfd} \to \mathbf{Set}$ be the functor taking $Y \in \mathbf{Perf}$ to the set of pairs (f, g) where $f : Y \to X$ is a morphism and $g : Y \to \varphi_X^* Y$ is an isomorphism (using f to define $\varphi_X^* Y$). That is, $Y(X)$ consists of the *torsors* over X with respect to the group $\varphi^{\mathbb{Z}}$ (by analogy with the formation of group quotients in algebraic topology). Beware that X/φ is no longer a sheaf for any of the topologies in question; it is only a *stack* over **Perf** in the sense of [152, Tag 026F]. (See Problem A.6.3.)

For a suitable definition of the étale topology on stacks (as in Definition 1.11.2), Lemma 4.2.6 asserts an equivalence

$$\mathbf{FEt}(X) \to \mathbf{FEt}(X \times_{\mathrm{Spec}(\mathbb{F}_p)} (\mathrm{Spec}(k)/\varphi)) \qquad (X \in \mathbf{Perf}),$$

which we may think of as formally defining an isomorphism

$$\pi_1^{\mathrm{prof}}(X, \overline{x}) \to \pi_1^{\mathrm{prof}}(X \times_{\mathrm{Spec}(\mathbb{F}_p)} (\mathrm{Spec}(k)/\varphi), \overline{x}).$$

Similarly, let $X_1, \dots, X_n$ be connected schemes over $\mathbb{F}_p$ and put $X := X_1 \times_{\mathbb{F}_p} \cdots \times_{\mathbb{F}_p} X_n$. Let $X/\Phi : \mathbf{Perf} \to \mathbf{Set}$ be the functor taking $Y \in \mathbf{Perf}$ to the set of tuples $(f, \beta_1, \dots, \beta_n)$ where $f : Y \to X$ is a morphism and $\beta_i : Y \to \varphi_{X_i}^* Y$ are commuting isomorphisms which compose to $\varphi_{Y/X}$. We may then think of Theorem 4.2.12 as formally defining an isomorphism

$$\pi_1^{\mathrm{prof}}\left(\left(\prod_{i=1}^n X_i\right)/\Phi, \overline{x}\right) \to \prod_{i=1}^n \pi_1^{\mathrm{prof}}(X_i, \overline{x}).$$

Remark 4.2.15. Following up on Remark 4.2.14, using Remark 4.2.9 we obtain an isomorphism of

$$\pi_1^{\mathrm{prof}}((X_1/\varphi) \times_{\mathrm{Spec}(\mathbb{F}_p)/\varphi} \cdots \times_{\mathrm{Spec}(\mathbb{F}_p)/\varphi} (X_n/\varphi), \overline{x})$$

with the limit (i.e., fiber product) of the diagram

$$\begin{tikzcd}
\pi_1^{\mathrm{prof}}(X_1/\varphi, \overline{x}) \arrow[dr] & \cdots \arrow[d] & \pi_1^{\mathrm{prof}}(X_n/\varphi, \overline{x}) \arrow[dl] \\
 & \pi_1^{\mathrm{prof}}(\mathrm{Spec}(\mathbb{F}_p)/\varphi, \overline{x}). &
\end{tikzcd}$$

This statement admits a highly suggestive topological analogue. Namely, let $X_1 \to S, \dots, X_n \to S$ be Serre fibrations of topological spaces, and let x be a basepoint of $X := X_1 \times_S \cdots \times_S X_n$ mapping to $x_1, \dots, x_n, s$ in $X_1, \dots, X_n, S$. Suppose further that S is a $K(\pi, 1)$ (this being analogous to the algebro-geometric situation, e.g., in light of Theorem 4.1.26). Since $\pi_2(S) = 0$, we may combine the long exact sequence of homotopy groups associated to a fibration with the formula for the fundamental group of an ordinary product to deduce that $\pi_1(X, x)$ is the limit of the diagram

$$\begin{tikzcd}
\pi_1(X_1, x_1) \arrow[dr] & \cdots \arrow[d] & \pi_1(X_n, x_n) \arrow[dl] \\
 & \pi_1(S, s). &
\end{tikzcd}$$

4.3. **Drinfeld's lemma for diamonds.** We now establish an analogue of Drinfeld's lemma for diamonds (and somewhat more general sheaves). This involves a reinterpretation of relative Fargues–Fontaine curves in the language of diamonds (already discussed in [163, Lecture 4]), which can be taken as a retroactive justification for their construction.

Beware that [145] represents only a first attempt to lay down foundations for the theory of diamonds, and that this process is still ongoing; a more definitive treatment will eventually appear in [147]. While we have attempted to align our definitions with the expected final forms, due caution is nonetheless advised.

Definition 4.3.1. Let **Pfd** again denote the category of perfectoid spaces of characteristic p. Identify each $S \in \mathbf{Pfd}$ with the representable functor $h_X : \mathbf{Pfd} \to \mathbf{Set}$; the latter is a pro-étale sheaf (see Remark 3.8.3). A *diamond* is a pro-étale sheaf of sets on **Pfd** which is a quotient of an object of **Pfd** by a pro-étale equivalence relation. These form a category via natural transformations of functors. (The definition of a diamond in [145] is slightly different; this is the definition that should appear in [147].)

For X a perfectoid space (not necessarily of characteristic p), let $X^\diamond$ be the representable functor $h_{X^\flat}$. Using Remark 2.9.10, we may extend this construction to a functor $X \mapsto X^\diamond$ from analytic adic spaces on which p is topologically nilpotent to diamonds: explicitly, for $Y \in \mathbf{Pfd}$, $X^\diamond(Y)$ consists of isomorphism classes of pairs $(Y^\sharp, f)$ in which $Y^\sharp$ is an untilt of Y (i.e., a perfectoid space equipped with an isomorphism $(Y^\sharp)^\flat \cong Y$) and $f : Y^\sharp \to X$ is a morphism of adic spaces. Beware that this functor is not fully faithful (see Remark 3.8.3).

For (A, A^+) a Huber pair in which p is topologically nilpotent (with A analytic as usual), we write $\mathrm{Spd}(A, A^+)$ (the "diamond spectrum") as shorthand for $\mathrm{Spa}(A, A^+)^\diamond$. Furthermore, if $A = F$ is a nonarchimedean field and $A^+ = \mathfrak{o}_F$, we usually just write $\mathrm{Spd}(F)$.

We will also need a more permissive construction.

Definition 4.3.2. Recall that the pro-étale topology is refined by the *v-topology* (see Definition 3.8.5), which is still subcanonical on **Pfd**. A *small v-sheaf* is a sheaf on **Pfd** which admits a surjective morphism from some perfectoid space; any diamond is a small v-sheaf.

Using small v-sheaves, we may extend the functor $(A, A^+) \to \mathrm{Spd}(A, A^+)$ to some non-analytic Huber pairs. For example, $\mathrm{Spd}(\mathbb{F}_p)$ is a terminal object in the category of small v-sheaves. For another example, by analogy with the interpretation of $\mathrm{Spd}(\mathbb{Q}_p)$ as the functor taking $S \in \mathbf{Pfd}$ to the set of isomorphism classes of untilts of S over $\mathbb{Q}_p$ (see [163, Lecture 3]), one can interpret $\mathrm{Spd}(\mathbb{Z}_p)$ as the functor taking $S \in \mathbf{Pfd}$ to the set of isomorphism classes of untilts of S, or more precisely of pairs $(S^\sharp, \iota)$ in which $S^\sharp$ is a perfectoid space and $\iota : (S^\sharp)^\flat \cong S$ is an isomorphism. (For example, $\mathrm{Spa}(\mathbb{Z}_p((T)), \mathbb{Z}_p[\![T]\!]) \to \mathrm{Spa}(\mathbb{Z}_p, \mathbb{Z}_p)$ is an admissible covering for the v-topology.)

Remark 4.3.3. Note that the definition of a small v-sheaf does not include any properties on an equivalence relation (or any relative representability condition). Somewhat surprisingly, such conditions are superfluous! Namely, if $Y \to X$ is a surjective morphism from a diamond (e.g., a perfectoid space) to a small v-sheaf, then $Y \times_X Y$ is also a diamond. See [147] for further discussion.

Remark 4.3.4. For X a perfectoid space not necessarily of characteristic p, the functor $X \mapsto X^\diamond$ depends only on $X^\flat$, and thus loses information. However, X also determines a morphism $X^\diamond \to \mathrm{Spd}(\mathbb{Z}_p)$ of small v-sheaves, and the resulting functor from X to small v-sheaves over $\mathrm{Spd}(\mathbb{Z}_p)$ is fully faithful.

As in Remark 4.2.14, we consider quotients by Frobenius.

Definition 4.3.5. For X a small v-sheaf, let $X/\varphi : \mathbf{Pfd} \to \mathbf{Set}$ denote the functor taking $Y \in \mathbf{Pfd}$ to the set of pairs (f, g) where $f : Y \to X$ is a morphism of diamonds and $g : Y \to \varphi_X^* Y$ is an isomorphism (using f to define $\varphi_X^* Y$). In general this is not a sheaf for either the pro-étale or v-topologies, but it is a stack over **Pfd** for these topologies. However, if X is "sufficiently nontrivial" then X/φ is a sheaf for the v-topology (and hence a small v-sheaf, since $X \to X/\varphi$ is surjective); for instance, this happens if X arises from an analytic adic space (see [147] for a more general statement about *locally spatial diamonds*).

We now reinterpret the construction of Fargues–Fontaine curves in the language of diamonds and small v-sheaves, starting with a calculation adapted from [145, Proposition 11.2.2].

Lemma 4.3.6. *For $S = \mathrm{Spa}(R, R^+) \in \mathbf{Pfd}$, put $\mathbf{A}_{\mathrm{inf}} := \mathbf{A}_{\mathrm{inf}}(R, R^+)$, let $\overline{x}_1, \dots, \overline{x}_n \in R^+$ be topologically nilpotent elements which generate the unit ideal in R, and put*

$$U_S := \{v \in \mathrm{Spa}(\mathbf{A}_{\mathrm{inf}}, \mathbf{A}_{\mathrm{inf}}) : v([\overline{x}_i]) \neq 0 \text{ for some } i \in \{1, \dots, n\}\}.$$

Then there is a natural (in S) isomorphism of small v-sheaves

$$S^\diamond \times \mathrm{Spd}(\mathbb{Z}_p) \cong U_S^\diamond.$$

Proof. For $Y \in \mathbf{Pfd}$, $(S^\diamond \times \mathrm{Spd}(\mathbb{Z}_p))(Y)$ consists of pairs $(f, Y^\sharp)$ in which $f : Y \to S$ is a morphism in $\mathbf{Pfd}$ and $Y^\sharp$ is an isomorphism class of untilts of Y. For $Y = \mathrm{Spa}(R', R'^+)$, such data correspond to a primitive ideal I of $W^(R'^+)$ for which $Y^\sharp = \mathrm{Spa}(W^b(R')/I, W(R'^+)/I)$ and a morphism $(R, R^+) \to (R', R'^+)$ of Huber rings. The latter induces a map $W(R^+) \to W(R'^+))$ and hence a map $W(R^+) \to W^b(R')/I$ under which the images of $[\overline{x}_1], \dots, [\overline{x}_n]$ generate the unit ideal. We thus obtain a map $Y^\sharp \to U_S$ and hence a morphism $Y^\diamond \to U_S^\diamond$.

In the other direction, $(U_S^\diamond)(Y)$ consists of pairs $(Y^\sharp, f)$ in which $Y^\sharp$ is an isomorphism class of untilts of Y and $f : Y^\sharp \to U_S$ is a morphism of adic spaces. The latter gives rise to a map $W(R^+) \to W^b(R')/I$ under which the images of $[\overline{x}_1], \dots, [\overline{x}_n]$ generate the unit ideal; we may thus tilt to obtain a map $R^+ \to R'$ which extends to R. We thus obtain a morphism $Y^\diamond \to S^\diamond \times \mathrm{Spd}(\mathbb{Z}_p)$. □

Definition 4.3.7. Recall that for $S \in \mathbf{Pfd}$, the relative Fargues–Fontaine curve over S is defined as the quotient

$$\mathrm{FF}_S := Y_S/\varphi_S \tag{4.3.7.1}$$

where φ_S is the map induced by the Witt vector Frobenius. Using Lemma 4.3.6, we have natural isomorphisms of diamonds

$$Y_S \cong S^\diamond \times \mathrm{Spd}(\mathbb{Q}_p), \qquad \mathrm{FF}_S^\diamond \cong Y_S^\diamond \cong (S^\diamond/\varphi) \times \mathrm{Spd}(\mathbb{Q}_p).$$

In particular, there is now a natural projection map $\mathrm{FF}_S^\diamond \to S^\diamond/\varphi$. Since φ acts trivially on the underlying topological space $|S|$ and on the étale site S_{et}, this projection induces the map $|\mathrm{FF}_S| \to |S|$ seen in Remark 3.1.12 and the map $\mathrm{FF}_{S,\mathrm{et}} \to S_{\mathrm{et}}$ of étale sites seen in Definition 3.7.4.

In light of the previous constructions, it is natural to define the *relative Fargues–Fontaine curve* over any small v-sheaf X as the stack

$$\mathrm{FF}_X := (X/\varphi) \times \mathrm{Spd}(\mathbb{Q}_p);$$

in light of (4.3.7.1), FF_X is a small v-sheaf, and even a diamond if X is a diamond. Taking $X = \mathrm{Spd}(\mathbb{F}_p)$ yields an object which one might call the *absolute Fargues–Fontaine curve*.

Recalling the setup of Drinfeld's lemma, we make the following observation and definition.

Definition 4.3.8. Let $X_1, \dots, X_n$ be small v-sheaves and put $X := X_1 \times \cdots \times X_n$. Write φ_i as shorthand for φ_{X_i}. Define the category

$$\mathbf{FEt}(X/\Phi) := \mathbf{FEt}(X/\langle \varphi_1, \dots, \varphi_n \rangle) \times_{\mathbf{FEt}(X/\varphi)} \mathbf{FEt}(X)$$

where $\mathbf{FEt}(X) \to \mathbf{FEt}(X/\varphi)$ is the canonical section of the forgetful functor $\mathbf{FEt}(X/\varphi) \to \mathbf{FEt}(X)$ (see Remark 4.2.9). For any $i \in \{1, \dots, n\}$, there is a canonical equivalence of categories

$$\mathbf{FEt}(X/\Phi) \cong \mathbf{FEt}(X/\langle \varphi_1, \dots, \widehat{\varphi}_i, \dots, \varphi_n \rangle).$$

Definition 4.3.9. For X a small v-sheaf, from Definition 4.3.8 we have

$$\mathbf{FEt}((X \times \operatorname{Spd}(\mathbb{Q}_p))/\Phi) \cong \mathbf{FEt}(\mathrm{FF}_X) \cong \mathbf{FEt}(X \times (\operatorname{Spd}(\mathbb{Q}_p)/\varphi)).$$

The small v-sheaf $X \times (\operatorname{Spd}(\mathbb{Q}_p)/\varphi)$ (which is a diamond if X is, because $\operatorname{Spd}(\mathbb{Q}_p)/\varphi$ is a diamond; see Definition 4.3.5) is an object we have not previously seen; following Fargues [58, Formulation of Fargues' conjecture], we call it the *mirror curve*[16] over X. Note that this object does not project naturally to $\operatorname{Spd}(\mathbb{Q}_p)$ unless X is equipped with such a projection.

We have the following analogue of Lemma 4.2.6.

Lemma 4.3.10. *Let X be a small v-sheaf. Let F be an algebraically closed nonarchimedean field of characteristic p. Then the base extension functor*

$$\mathbf{FEt}(X) \to \mathbf{FEt}(X \times (\operatorname{Spd}(F)/\varphi)) \cong \mathbf{FEt}((X/\varphi) \times \operatorname{Spd}(F))$$

is an equivalence of categories. (The final equivalence comes from Definition 4.3.8.)

Proof. We reduce immediately to the case where $X = \operatorname{Spd}(A, A^+)$ for some perfectoid pair (A, A^+) of characteristic p. Choose an untilt K of F of characteristic 0 (which is itself algebraically closed by Lemma 2.8.9); using the isomorphism

$$\mathbf{FEt}((X/\varphi) \times \operatorname{Spd}(F)) = \mathbf{FEt}(\mathrm{FF}_X^\diamond \times_{\operatorname{Spd}(\mathbb{Q}_p)} \operatorname{Spd}(F))$$

from Definition 4.3.7, we reduce to showing that the functor

$$\mathbf{FEt}(X) \to \mathbf{FEt}(K \times_{\mathbb{Q}_p} \mathrm{FF}_X), \qquad X' \mapsto K \times_{\mathbb{Q}_p} \mathrm{FF}_{X'} \tag{4.3.10.1}$$

is an equivalence of categories. (Recall that $\operatorname{Spd}(K)$ is just $\operatorname{Spd}(F)$ equipped with a particular morphism to $\operatorname{Spd}(\mathbb{Q}_p)$ that identifies the choice of the untilt.) This claim reduces to the case where A is an algebraically closed field: one first applies this reduction to full faithfulness, then using full faithfulness one applies the reduction again to essential surjectivity.

In this case, the argument is ultimately due to Fargues–Fontaine [57]; we follow the treatment in [162, Theorem 3.4.3]. We must show that any connected finite étale covering $f : Y \to K \times_{\mathbb{Q}_p} \mathrm{FF}_X$ is an isomorphism. Using the fact that the category $\mathbf{FEt}(K \times_{\mathbb{Q}_p} \mathrm{FF}_X)$ has an interpretation which depends only on F, not on K (which ultimately comes down to Theorem 2.5.9), we may descend f canonically to a finite étale covering $f_0 : Y_0 \to \mathrm{FF}_X$; the vector bundle $f_{0*}\mathcal{O}_{Y_0}$ carries an $\mathcal{O}_{\mathrm{FF}_X}$-algebra structure. Apply Theorem 3.6.13 to the vector bundle $f_{0*}\mathcal{O}_{Y_0}$, and let μ be the largest slope that occurs in the decomposition. If $\mu > 0$, then any element of a copy of $\mathcal{O}(\mu)$ occurring in the decomposition corresponds to a square-zero element of $H^0(Y, \mathcal{O}_Y)$, which does not exist because Y is connected. It follows that $\mu = 0$; similarly, the smallest slope that occurs in the decomposition cannot be negative. Hence $f_{0*}\mathcal{O}_{Y_0}$ is a trivial bundle of rank equal to the degree of f, as then is $f_*\mathcal{O}_Y$. Now $H^0(Y, \mathcal{O}_Y) = H^0(K \times_{\mathbb{Q}_p} \mathrm{FF}_X, f_*\mathcal{O}_Y)$ is a *connected* finite étale K-algebra, and hence isomorphic to K itself because K is algebraically closed; this proves the claim. ∎

[16] As far as I know, this terminology is not meant to refer specifically to *mirror symmetry* in mathematical physics.

Remark 4.3.11. For $S \in \mathbf{Pfd}$, the space FF_S has a family of cyclic finite étale covers corresponding to replacing the quotient by φ with the quotient by a power of φ (Remark 3.1.8). If $S = \mathrm{Spa}(F, \mathfrak{o}_F)$ for F a perfectoid field, these covers are all connected.

However, suppose that K is an algebraically closed perfectoid field over $\mathbb{Q}_p$. Then one consequence of Lemma 4.3.10 is that the corresponding covers of $K \times_{\mathbb{Q}_p} \mathrm{FF}_S$ are all split! This can be seen more explicitly using the fact that for $d \in \mathbb{Q}$, the bundle $\mathcal{O}(d)$ on FF_S is indecomposable (Corollary 3.6.7) but its pullback to $K \times_{\mathbb{Q}_p} \mathrm{FF}_S$ splits as a direct sum of line bundles.

Remark 4.3.12. Lemma 4.3.10 asserts that for any diamond X, the *geometric* profinite fundamental group of FF_X coincides with the profinite fundamental group of X. For $X = \mathrm{Spd}(\mathbb{Q}_p)$, this recovers the interpretation of $G_{\mathbb{Q}_p}$ as the profinite fundamental group of a diamond, as originally described in [162]. A variation on this theme is the interpretation of the absolute Galois groups of certain fields as topological fundamental groups by Kucharczyk–Scholze [116].

Problem 4.3.13. Can Lemma 4.3.10 be used to give an independent proof of Lemma 4.2.6, and hence of Drinfeld's lemma?

Theorem 4.3.14. *Let $X_1, \dots, X_n$ be connected qcqs diamonds. Then for any geometric point $\overline{x}$ of $X := X_1 \times \cdots \times X_n$, the map*

$$\pi_1^{\mathrm{prof}}(X/\Phi, \overline{x}) \to \prod_{i=1}^{n} \pi_1^{\mathrm{prof}}(X_i, \overline{x})$$

is an isomorphism of profinite groups.

Proof. As in Theorem 4.2.12, we rewrite the group on the left as

$$\pi_1^{\mathrm{prof}}(X_1 \times (X_2/\varphi) \times \cdots \times (X_n/\varphi), \overline{x})$$

and then induct on n, the base case $n = 1$ being trivial. Again, to prove the induction step, we use Lemma 4.3.10 to imitate the proof of Corollary 4.1.23; more details (taken from [145, §17]) may be added later. $\square$

Remark 4.3.15. As in Remark 4.2.15, we may reformulate Theorem 4.3.14 to say that

$$\pi_1^{\mathrm{prof}}((X_1/\varphi) \times_{\mathrm{Spd}(\mathbb{F}_p)/\varphi} \cdots \times_{\mathrm{Spd}(\mathbb{F}_p)/\varphi} (X_n/\varphi), \overline{x})$$

may be naturally identified with the limit of the diagram

$$\begin{tikzcd}[column sep=small]
\pi_1^{\mathrm{prof}}(X_1/\varphi, \overline{x}) \arrow[dr] & \cdots \arrow[d] & \pi_1^{\mathrm{prof}}(X_n/\varphi, \overline{x}) \arrow[dl] \\
& \pi_1^{\mathrm{prof}}(\mathrm{Spd}(\mathbb{F}_p)/\varphi, \overline{x}) \cong \widehat{\mathbb{Z}}. &
\end{tikzcd}$$

As a concrete illustration of Drinfeld's lemma, we highlight a corollary relevant to the study of multidimensional (φ, Γ)-modules, as in work of Zábrádi [166, 167].

Corollary 4.3.16. *Let $F_1, \dots, F_n$ be perfectoid fields of characteristic p, each equipped with a multiplicative norm. Let R^+ be the completion of $\mathfrak{o}_{F_1} \otimes_{\mathbb{F}_p} \cdots \otimes_{\mathbb{F}_p} \mathfrak{o}_{F_n}$ for the $(\varpi_1, \dots, \varpi_n)$-adic topology, where $\varpi_i \in \mathfrak{o}_{F_i}$ is a pseudouniformizer, and put*

$$R := R^+[\varpi_1^{-1}, \dots, \varpi_n^{-1}].$$

(Note that the ultimate definitions of R^+ and R do not depend on the choices of the ϖ_i.) Then the category of continuous representations of $G_{F_1} \times \cdots \times G_{F_n}$ on finite-dimensional $\mathbb{F}_p$-vector spaces is equivalent to the category of finite projective R-modules equipped with commuting semilinear actions of $\varphi_{F_1}, \dots, \varphi_{F_n}$.

Proof. Fix algebraic closures $\overline{F}_i$ of F_i, identify G_{F_i} with $\mathrm{Gal}(\overline{F}_i/F_i)$, let $\overline{R}^+$ be the completion of $\mathfrak{o}_{\overline{F}_1} \otimes_{\mathbb{F}_p} \cdots \otimes_{\mathbb{F}_p} \mathfrak{o}_{\overline{F}_n}$ for the $(\varpi_1, \dots, \varpi_n)$-adic topology, and put

$$\overline{R} := \overline{R}^+[\varpi_1^{-1}, \dots, \varpi_n^{-1}].$$

Equip $\overline{R}$ with the obvious action of $G_{F_1} \times \cdots \times G_{F_n}$. The functor in question then takes a representation V to

$$D(V) := (V \otimes_{\mathbb{F}_p} \overline{R})^{G_{F_1} \times \cdots \times G_{F_n}}$$

for the diagonal action on the tensor product, with $D(V)$ inheriting an action of φ_{F_i} from the canonical action on $\overline{R}$ and the trivial action on V. For any given V, we can also write this as

$$D(V) := (V \otimes_{\mathbb{F}_p} S)^{\mathrm{Gal}(E_1/F_1) \times \cdots \times \mathrm{Gal}(E_n/F_n)}$$

for some finite Galois extensions E_i of F_i within $\overline{F}_i$ and $S := E_1 \widehat{\otimes}_{\mathbb{F}_p} \cdots \widehat{\otimes}_{\mathbb{F}_p} E_n$. Note that $\mathrm{Spec}(S \otimes_R S)$ splits into the graphs of the various maps $\mathrm{Spec}(S) \to \mathrm{Spec}(S)$ induced by $\mathrm{Gal}(E_1/F_1) \times \cdots \times \mathrm{Gal}(E_n/F_n)$; consequently, the action of this product on $V \otimes_{\mathbb{F}_p} S$ gives rise to a descent datum with respect to the faithfully flat homomorphism $R \to S$. By faithfully flat descent for modules [152, Tag 023N], we deduce that $D(V)$ is a finite projective R-module and the natural map

$$D(V) \otimes_R S \to V \otimes_{\mathbb{F}_p} S \tag{4.3.16.1}$$

is an isomorphism.

To check that this functor is fully faithful, using internal Homs we reduce to checking that

$$V^{G_{F_1} \times \cdots \times G_{F_n}} = D(V)^{\varphi_{F_1}, \dots, \varphi_{F_n}};$$

this follows by taking simultaneous Galois and Frobenius invariants on both sides of (4.3.16.1) and using the equality

$$\overline{R}^{\varphi_{F_1}, \dots, \varphi_{F_n}} = \mathbb{F}_p.$$

To check essential surjectivity, set

$$X_i := \mathrm{Spd}(F_i), \qquad \overline{x}_i := \mathrm{Spd}(\overline{F}_i)$$

and let $\overline{x}$ be a geometric point of $X := X_1 \times \cdots \times X_n$ lying over each $\overline{x}_i$. By Theorem 4.3.14, the map

$$\pi_1^{\mathrm{prof}}(X/\Phi, \overline{x}) \to \prod_{i=1}^{n} \pi_1^{\mathrm{prof}}(X_i, \overline{x}) = G_{F_1} \times \cdots \times G_{F_n}. \tag{4.3.16.2}$$

is an isomorphism of profinite groups. Now let D be a finite projective R-module equipped with commuting semilinear actions of $\varphi_{F_1}, \dots, \varphi_{F_n}$. By composing these actions, we get an action of the absolute Frobenius map φ_R; as in the proof of Lemma 4.2.2, we may invoke [107, Lemma 3.2.6] to see that the sheaf of φ_R-invariants of D on the finite étale site of $\mathrm{Spec}(R)$ is represented by $\mathrm{Spec}(S)$ for some faithfully finite étale R-algebra S. Since D carries actions of $\varphi_{F_1}, \dots, \varphi_{F_n}$ composing to absolute Frobenius, S does likewise.

Now note that there is a natural morphism

$$\mathrm{Spd}(F_1) \times \cdots \times \mathrm{Spd}(F_n) \to \mathrm{Spd}(R, R^+)$$

which identifies the source with the diamond associated to the adic space

$$Y := \{v \in \mathrm{Spa}(R, R^+) : v(\varpi_1), \dots, v(\varpi_n) < 1\};$$

this identification yields additional identifications

$$R^+ = H^0(Y, \mathcal{O}^+), \qquad R = \bigcup_{m_1,\dots,m_n=0}^{\infty} \varpi_1^{-m_1} \cdots \varpi_n^{-m_n} R^+ \tag{4.3.16.3}$$

(see Remark 4.3.18 for an explicit example). Let S^+ be the integral closure of R^+ in S. By pulling back $\mathrm{Spa}(S, S^+)$ from $\mathrm{Spa}(R, R^+)$ to Y and invoking (4.3.16.2), we obtain a representation of $G_{F_1} \times \cdots \times G_{F_n}$ which we claim gives rises to D. By replacing each F_i with a suitable finite extension, we reduce to checking this in the case where this representation is trivial. That is, we may assume that $\mathrm{Spa}(S, S^+) \to \mathrm{Spa}(R, R^+)$ splits completely after pullback to Y and we must check that $R \to S$ itself splits completely.

Let $\mathcal{F}$ be the pullback to Y of the sheaf $\tilde{S}$ associated to S as an R-algebra. Choose a connected affinoid subspace U of Y containing a fundamental domain for the action of Φ; a concrete example would be

$$U = \{v \in Y : v(\varpi_1)^p \leq v(\varpi_i) \leq v(\varpi_1) \qquad (i = 2, \dots, n)\}.$$

We then have a family of idempotents that split $H^0(U, \tilde{S})$ over $H^0(U, \mathcal{O})$, and moreover must be stable under the action of any $\gamma \in \Phi$ (that is, there is agreement among the restrictions to $U \cap \gamma(U)$). By glueing these together, we see that each idempotent in the original family belongs to

$$\bigcup_{m_1,\dots,m_n=0}^{\infty} \varpi_1^{-m_1} \cdots \varpi_n^{-m_n} H^0(Y, \tilde{S}^+) = S,$$

as needed. □

Remark 4.3.17. With notation as in Corollary 4.3.16, one may similarly show that the category of continuous representations of $G_{F_1} \times \cdots \times G_{F_n}$ on finite free $\mathbb{Z}_p$-modules is equivalent to the category of finite projective $W(R)$-modules equipped with commuting semilinear actions of $\varphi_{F_1}, \dots, \varphi_{F_n}$.

Remark 4.3.18. Let us describe the morphism $\mathrm{Spd}(F_1) \times \mathrm{Spd}(F_2) \to \mathrm{Spd}(R, R^+)$ in Corollary 4.3.16 more explicitly in the case where F_1, F_2 are the completed perfected closures of $\mathbb{F}_p((T_1)), \mathbb{F}_p((T_2))$, respectively. In this case, $\mathrm{Spd}(F_1) \times \mathrm{Spd}(F_2)$ is the diamond associated to the adic space

$$Y = \{v \in \mathrm{Spa}(F_1\langle T_2^{p^{-\infty}}\rangle, \mathfrak{o}_{F_1}\langle T_2^{p^{-\infty}}\rangle) : 0 < v(T_2) < 1\};$$

the ring

$$R^+ = H^0(Y, \mathcal{O}^+) = \mathbb{F}_p[\![T_1^{p^{-\infty}}, T_2^{p^{-\infty}}]\!]$$

is the (T_1, T_2)-adic completion of $\mathbb{F}_p[\![T_1, T_2]\!][T_1^{p^{-\infty}}, T_2^{p^{-\infty}}]$; and $H^0(Y, \mathcal{O})$ is the ring of formal sums $\sum_{m_1,m_2 \in \mathbb{Z}[p^{-1}]} c_{m_1 m_2} T_1^{m_1} T_2^{m_2}$ with $c_{m_1 m_2} \in \mathbb{F}_p$ whose support

$$S = \{(m_1, m_2) \in \mathbb{R}^2 : m_1, m_2 \in \mathbb{Z}[p^{-1}], c_{m_1 m_2} \neq 0\}$$

satisfies the following conditions.

- For any $x_0, y_0 \in \mathbb{R}$, the intersection

$$S \cap \{(x,y) \in \mathbb{R}^2 : x \leq x_0, y \leq y_0\}$$

is finite.
- The *lower convex hull* of S, i.e., the convex hull of the set

$$\bigcup_{(m_1,m_2)\in S} \{(x,y) \in \mathbb{R}^2 : x \geq m_1, y \geq m_2\},$$

admits a supporting line of slope $-s$ for each $s > 0$.

Remark 4.3.19. In Remark 4.3.18, the ring R can be interpreted as the subring of $H^0(Y, \mathcal{O})$ consisting of functions which are bounded for $v(T_2)$ close to 1 and of *polynomial growth* for $v(T_2)$ close to 0. This suggests a close relationship between the identification (4.3.16.3) and the perfectoid analogue of the *Riemann extension theorem (Hebbarkeitsatz)* introduced by Scholze [146, §2.3] to study the boundaries of perfectoid Shimura varieties. This result has been (refined and) used by André [5, 6] and Bhatt [15] to resolve a long-standing open problem in commutative algebra, the *direct summand conjecture* of Hochster: if $R \to S$ is a finite morphism of noetherian rings and R is regular, then $R \to S$ splits in the category of R-modules. (See [84] for several equivalent formulations and consequences.)

Remark 4.3.20. One probably cannot hope to have an analogue of Theorem 4.3.14 for étale fundamental groups in the sense of de Jong (Remark 4.1.6). In particular, if F is an algebraically closed perfectoid field of characteristic p and K is an algebraically closed perfectoid field of characteristic 0, then $\mathrm{Spd}(K) \times (\mathrm{Spd}(F)/\varphi) \cong K \times_{\mathbb{Q}_p} \mathrm{FF}^{\diamond}_{\mathrm{Spd}(F)}$ admits no finite étale coverings, but it does admit some nonfinite étale coverings. For example, one can choose two sections of $\mathcal{O}(1)$ on $\mathrm{FF}_{\mathrm{Spa}(F,\mathfrak{o}_F)}$ with distinct zeroes, use these to define a morphism $\mathrm{FF}_{\mathrm{Spa}(F,\mathfrak{o}_F)} \to \mathbf{P}^1_{\mathbb{Q}_p}$, and pull back the Hodge-Tate period mapping; provided that the resulting covering does not split completely (which we have not checked), it gives an example of the desired form.

4.4. **Shtukas in positive characteristic.** We now arrive at the fundamental concept introduced by Drinfeld as a replacement for elliptic curves in positive characteristic; that is to say, the moduli spaces of such objects constitute a replacement for modular curves and Shimura varieties as a tool for studying Galois representations of a global function field in positive characteristic (which we are now prepared to think about as representations of profinite fundamental groups).

Hypothesis 4.4.1. Throughout §4.4, let C be a smooth, projective, geometrically irreducible curve over a finite field $\mathbb{F}_q$ of characteristic p.

Definition 4.4.2. Let S be a scheme over $\mathbb{F}_q$. A *shtuka* over S consists of the following data.

- A finite index set I and a morphism $(x_i)_{i\in I} : S \to C^I$.
- A vector bundle $\mathcal{F}$ over $C \times S$.
- An isomorphism of bundles

$$\Phi : (\varphi_S^* \mathcal{F})|_{(C\times S)\setminus \bigcup_{i\in I} \Gamma_{x_i}} \cong \mathcal{F}|_{(C\times S)\setminus \bigcup_{i\in I} \Gamma_{x_i}},$$

where $\Gamma_{x_i} \subset C \times S$ denotes the graph of x_i.

The morphisms $x_i : S \to C$ are called the *legs* (in French, *pattes*[17]) of the shtuka.

Remark 4.4.3. For Z a finite set of closed points of C, one may also consider shtukas with *level structure* at C; this amounts to insisting that the legs map S into $C \setminus Z$ and specifying a trivialization of $(\mathcal{F}, \Phi)$ over $Z \times S$.

Remark 4.4.4. For K the function field of C and G a connected reductive algebraic group over K, Varshavsky [159] has introduced the notion of a *G-shtuka*, the previous definition being the case $G = \mathrm{GL}_n$ for $n = \mathrm{rank}(\mathcal{F})$. In the case where G is *split* (i.e., G contains a split maximal torus), then the results of SGA 3 [38, 39, 40] imply that G extends canonically to a group scheme G_C over C, and we then insist that $\mathcal{F}$ be a G_C-torsor and Φ be an isomorphism of G_C-torsors.

Remark 4.4.5. The word *shtuka* (in French, *chtouca*) is a transliteration of the Russian word штука, meaning a generic thing whose exact identity is unknown or irrelevant; it is probably derived from the German word *Stück* (meaning *piece*), although the Russian usage may[18] be influenced by the word что (meaning *what*). Some analogous terms in English are *widget*, *gadget*, *gizmo*, *doodad*, *whatchamacallit*; see Wikipedia on *placeholder names* for more examples.

Remark 4.4.6. As pointed out in [69], one source of inspiration for the definition of shtukas is some work of Krichever on integrable systems arising from the Korteweg–de Vries (KdV) equation. The relationship between these apparently disparate topics has been exposed by Mumford [129].

4.5. **Shtukas in mixed characteristic.** It would be far outside the scope of these lectures to explain in any meaningful detail why shtukas are so important in the study of the Langlands correspondence over global function fields. Instead, we jump straight to the mixed-characteristic analogue, to illustrate a startling[19] convergence between shtukas and Fargues–Fontaine curves.

We take the approach to sheaves on stacks used in [152, Tag 06TF].

Definition 4.5.1. Let $\mathcal{O} : \mathbf{Pfd} \to \mathbf{Ring}$ be the functor taking X to $\mathcal{O}(X)$. By Theorem 3.8.2 this functor is a sheaf of rings for the v-topology. For any small v-sheaf X, we may restrict $\mathcal{O}$ to the arrow category $\mathbf{Pfd}_X$ (i.e., the category of morphisms $S^\diamond \to X$ with $S \in \mathbf{Pfd}$) to obtain the *structure sheaf* on X.

A *vector bundle* on X is a locally finite free $\mathcal{O}_X$-module; let $\mathbf{Vec}_X$ denote the category of such objects. We avoid trying to define a *pseudocoherent sheaf* on a diamond or small v-sheaf due to the issues raised in §3.8.

This category of vector bundles lives entirely in characteristic p; we actually need something slightly different.

[17]This word translates into English variously as *legs* or *paws*, the latter translation being used in [145]. However, the term *paw* is typically used only for mammalian feet, whereas the intended animal metaphor seems to be a caterpillar or millipede. We thus prefer the translation *legs*, following V. Lafforgue in [119].

[18]Beware that I have no evidence to support this speculative claim!

[19]To be fair, this convergence was anticipated well before the technology became available to make it overt. For example, the analogy between Breuil–Kisin modules and shtukas was suggested by Kisin in [112].

Definition 4.5.2. Let $\mathcal{O}^\sharp : \mathbf{Pfd}_{\mathrm{Spd}(\mathbb{Z}_p)} \to \mathbf{Ring}$ be the functor taking X to $\mathcal{O}(X^\sharp)$, where $X^\sharp$ is the untilt of X corresponding to the structure morphism $X \to \mathrm{Spd}(\mathbb{Z}_p)$. By Theorem 3.8.2 again, this functor is a sheaf of rings for the v-topology. For any small v-sheaf X over $\mathrm{Spd}(\mathbb{Z}_p)$, we may restrict $\mathcal{O}^\sharp$ to $\mathbf{Pfd}_X$ to obtain the *untilted structure sheaf* on X. An *untilted vector bundle* on X is a locally finite free $\mathcal{O}^\sharp_X$-module; let $\mathbf{Vec}^\sharp_X$ denote the category of such objects.

As an immediate consequence of Theorem 3.8.7, we have the following.

Theorem 4.5.3. *Let (A, A^+) be a perfectoid pair of characteristic p.*

(a) *The pullback functor $\mathbf{FPMod}_A \to \mathbf{Vec}_{\mathrm{Spd}(A,A^+)}$ is an equivalence of categories.*

(b) *Fix a morphism $\mathrm{Spd}(A, A^+) \to \mathrm{Spd}(\mathbb{Z}_p)$ corresponding to an untilt $(A^\sharp, A^{\sharp+})$ of (A, A^+). Then the pullback functor $\mathbf{FPMod}_{A^\sharp} \to \mathbf{Vec}^\sharp_{\mathrm{Spd}(A,A^+)}$ is an equivalence of categories.*

Remark 4.5.4. Let X be an analytic adic space on which p is topologically nilpotent. In many cases of interest, the pushforward of $\mathcal{O}_{X_{\mathrm{proet}}}$ to X coincides with $\mathcal{O}_X$; in such cases, the base extension functor $\mathbf{Vec}_X \to \mathbf{Vec}^\sharp_{X^\diamond}$ is fully faithful. However, even in such cases, this functor is generally not essentially surjective (unless X is perfectoid, in which case Theorem 3.8.7 applies). For example, if $X = \mathrm{Spd}(\mathbb{Q}_p)$, then the source category consists of finite-dimensional $\mathbb{Q}_p$-vector spaces while the target consists of finite-dimensional $\mathbb{C}_p$-vector spaces equipped with continuous $G_{\mathbb{Q}_p}$-actions.

A related point is that if $X = \mathrm{Spa}(A, A^+)$ is not perfectoid, then objects of $\mathbf{Vec}^\sharp_{X^\diamond}$ need not be acyclic on X.

Remark 4.5.5. The definition of a shtuka over a diamond (Definition 4.5.6) will refer to $\mathrm{Spd}(\mathbb{Z}_p) \times S$, but in order to formulate the definition correctly we must unpack this concept a bit in the case where $S \in \mathbf{Pfd}$. In this case, $\mathrm{Spd}(\mathbb{Z}_p) \times S^\diamond$ descends to an adic space in a canonical way: for $S = \mathrm{Spd}(R, R^+)$, writing $\mathbf{A}_{\mathrm{inf}}$ for $\mathbf{A}_{\mathrm{inf}}(R, R^+)$ we have

$$\mathrm{Spd}(\mathbb{Z}_p) \times S^\diamond \cong W_S^\diamond, \qquad W_S := \mathrm{Spa}(\mathbf{A}_{\mathrm{inf}}, \mathbf{A}_{\mathrm{inf}}) \setminus V([\overline{x}_1], \dots, [\overline{x}_n]),$$

where $\overline{x}_1, \dots, \overline{x}_n \in R$ are topologically nilpotent elements which generate the unit ideal. The space W_S has the property that the pushforward of $\mathcal{O}_{W_S,\mathrm{proet}}$ to W_S coincides with $\mathcal{O}_{W_S}$ (this can be seen from the explicit description given in the proof of Lemma 3.1.3), so we may view the vector bundles on W_S as a full subcategory of the untilted vector bundles on $\mathrm{Spd}(\mathbb{Z}_p) \times S^\diamond$ (with respect to the first projection).

Definition 4.5.6. Let S be a diamond. A *shtuka* over S consists of the following data.

- A finite index set I and a morphism $(x_i)_{i\in I} : S \to \mathrm{Spd}(\mathbb{Z}_p)^I$.
- An untilted vector bundle $\mathcal{F}$ over $\mathrm{Spd}(\mathbb{Z}_p) \times S$ with respect to the first projection which locally-on-S arises from a vector bundle on the underlying adic space W_S (Remark 4.5.5).
- An isomorphism of bundles

$$\Phi : (\varphi_S^* \mathcal{F})|_{(\mathrm{Spd}(\mathbb{Z}_p)\times S)\setminus \bigcup_{i\in I}\Gamma_{x_i}} \cong \mathcal{F}|_{(\mathrm{Spd}(\mathbb{Z}_p)\times S)\setminus \bigcup_{i\in I}\Gamma_{x_i}},$$

  where $\Gamma_{x_i} \subset \mathrm{Spd}(\mathbb{Z}_p) \times S$ denotes the graph of x_i. We also insist that Φ be meromorphic along $\bigcup_{i\in I} \Gamma_{x_i}$, this having been implicit in the schematic case.

Again, the morphisms $x_i : S \to \mathrm{Spd}(\mathbb{Z}_p)$ are called the *legs* of the shtuka.

Remark 4.5.7. For $S \in \mathbf{Pfd}$, we could have defined a shtuka over S directly in terms of a vector bundle over W_S, without reference to untilted vector bundles. The point of the formulation used here is to encode the fact that shtukas satisfy descent for the v-topology, which does not immediately follow from Theorem 3.8.7 because W_S is not a perfectoid space.

To unpack this definition further, let us first consider the case of a shtuka with no legs.

Remark 4.5.8. Suppose that $I = \emptyset$. A shtuka over S with no legs is simply an untilted vector bundle $\mathcal{F}$ over $\mathrm{Spd}(\mathbb{Z}_p) \times S$ (which locally-on-S descends to the underlying adic space W_S) equipped with an isomorphism with its φ-pullback.

In the case where $S = \mathrm{Spd}(R, R^+) \in \mathbf{Pfd}$, by restricting from W_S to Y_S and then quotienting by the action of φ, we obtain a vector bundle on the relative Fargues–Fontaine curve FF_S. However, not all vector bundles can arise in this fashion, for the following reasons.

- The resulting bundle is fiberwise semistable of slope 0.
- The associated étale $\mathbb{Q}_p$-local system (see Theorem 3.7.5) descends to an étale $\mathbb{Z}_p$-local system determined by the shtuka. (For a general étale $\mathbb{Q}_p$-local system, such a descent only exists locally on S; see [107, Corollary 8.4.7].)

In fact, the functor from shtukas over S with no legs to étale $\mathbb{Z}_p$-local systems on S is an equivalence of categories; this follows from a certain analogue of Theorem 3.7.5 (see [107, Theorem 8.5.3]).

Remark 4.5.9. In the case where S is a geometric point, Remark 4.5.8 asserts that shtukas over S with no legs correspond simply to finite free $\mathbb{Z}_p$-modules. In particular, they extend canonically from W_S over all of $\mathrm{Spa}(\mathbf{A}_{\mathrm{inf}}, \mathbf{A}_{\mathrm{inf}})$.

A partial extension of this result is the following.

Theorem 4.5.10 (Kedlaya)**.** *Let (R, R^+) be a perfectoid Huber pair of characteristic p and write $\mathbf{A}_{\mathrm{inf}}$ for $\mathbf{A}_{\mathrm{inf}}(R, R^+)$.*

(a) *Choose topologically nilpotent elements $\overline{x}_1, \dots, \overline{x}_n \in R^+$ generating the unit ideal in R. Then the pullback functor from vector bundles on the scheme*

$$\mathrm{Spec}(\mathbf{A}_{\mathrm{inf}}) \setminus V(p, [\overline{x}_1], \dots, [\overline{x}_n])$$

to vector bundles on the analytic locus of $\mathrm{Spa}(\mathbf{A}_{\mathrm{inf}}, \mathbf{A}_{\mathrm{inf}})$ *is an equivalence of categories.*

(b) *Suppose that $R = F$ is a perfectoid field. Then both categories in (a) are equivalent to the category of finite free $\mathbf{A}_{\mathrm{inf}}$-modules and to the category of vector bundles on* $\mathrm{Spec}(\mathbf{A}_{\mathrm{inf}})$.

Proof. See [105]. □

Remark 4.5.11. Theorem 4.5.10(b) is analogous to the assertion that if R is a two-dimensional local ring, then the pullback functor from vector bundles on $\mathrm{Spec}(R)$ to vector bundles on the complement of the closed point is an equivalence of categories (because reflexive and projective R-modules coincide). By contrast, one does not have a similar assertion comparing, say, vector bundles on $\mathrm{Spec}(k[\![x, y, z]\!])$ (for k a field) with vector bundles on the complement of the locus where x and y both vanish; similarly, Theorem 4.5.10(b) cannot be extended beyond the case where R is a perfectoid field.

Even if R is a perfectoid field, if R is not algebraically closed, then shtukas over S with no legs need not extend as bundles from W_S to the whole analytic locus of $\mathrm{Spa}(\mathbf{A}_{\mathrm{inf}}, \mathbf{A}_{\mathrm{inf}})$. Namely, the only ones that do so are the ones coming from étale local systems on S that extend to $\mathrm{Spa}(R^+, R^+)$, i.e., the ones corresponding to *unramified* Galois representations.

Remark 4.5.12. As per Remark 4.5.11, for $S \in \mathbf{Pfd}$, the restriction functor on φ-equivariant vector bundles from the analytic locus of $\mathrm{Spa}(\mathbf{A}_{\mathrm{inf}}, \mathbf{A}_{\mathrm{inf}})$ to W_S is not essentially surjective. However, one does expect it to be fully faithful; see Lemma 4.5.16 for a special case of a related statement.

We now increase complexity slightly by considering shtukas with one leg.

Lemma 4.5.13. *Suppose that $I = \{1\}$ is a singleton set and that the morphism x_1 factors through $\mathrm{Spd}(\mathbb{Q}_p)$. Then the following categories are canonically equivalent:*

(a) *shtukas over S with leg x_1;*

(b) *data $\mathcal{F}_1 \dashrightarrow \mathcal{F}_2$, where $\mathcal{F}_1$ is a φ-equivariant bundle over $\mathrm{Spd}(\mathbb{Z}_p) \times S$ (which locally-on-S descend to W_S), $\mathcal{F}_2$ is a φ-equivariant bundle over $\mathrm{Spd}(\mathbb{Q}_p) \times S$ (which locally-on-S descend to Y_S), and the arrow denotes a meromorphic φ-equivariant map over Y_S which is an isomorphism away from $\bigcup_{n\in\mathbb{Z}} \varphi^n(\Gamma_{x_1})$.*

Proof. We start with the general idea: if one thinks of Φ as defining an isomorphism from $\varphi_S^*\mathcal{F}$ to $\mathcal{F}$ "up to a discrepancy," then $\mathcal{F}_1, \mathcal{F}_2$ are obtained by resolving the discrepancy respectively in favor of $\varphi_S^*\mathcal{F}, \mathcal{F}$.

We now make this explicit. Since we are constructing a canonical equivalence, we may assume that $S = \mathrm{Spd}(R, R^+)$ where (R, R^+) is a Tate perfectoid pair of characteristic p. Choose a pseudouniformizer $\varpi \in R$. Given a shtuka oven S with leg x_1, we obtain the bundle $\mathcal{F}_1$ by restricting $\varphi_S^*\mathcal{F}$ to $\{v \in W_S : v(p) \leq v([\varpi]^{p^{-n}})\}$ for sufficiently small n (ensuring that Γ_{x_i} does not meet this set; here we use the fact that x_1 factors through $\mathrm{Spd}(\mathbb{Q}_p)$), then using the isomorphism $\varphi_S^*\mathcal{F} \cong \mathcal{F}$ to enlarge n. Similarly, we obtain $\mathcal{F}_2$ by restricting $\mathcal{F}$ to $\{v \in W_S : v(p) \geq v([\varpi]^{p^n})\}$ for sufficiently small n, then using the isomorphism $\varphi_S^*\mathcal{F} \cong \mathcal{F}$ to enlarge n. (Note that in this second case, the union of these spaces is only Y_S, not W_S.) The meromorphic map $\varphi_S^*\mathcal{F} \dashrightarrow \mathcal{F}$ gives rise to the meromorphic map $\mathcal{F}_1 \dashrightarrow \mathcal{F}_2$. One may check that this construction is reversible and does not depend on ϖ. $\square$

Remark 4.5.14. Suppose that $S = \mathrm{Spd}(R, R^+) \in \mathbf{Pfd}$, $I = \{1\}$ is a singleton set, and that the morphism x_1 factors through $\mathrm{Spd}(\mathbb{Q}_p)$. From Lemma 4.5.13, we obtain a pair of vector bundles $\mathcal{F}_1, \mathcal{F}_2$ on FF_S and a meromorphic map $\mathcal{F}_1 \dashrightarrow \mathcal{F}_2$ which is an isomorphism away from the untilt corresponding to x_1. Of these, $\mathcal{F}_1$ arises from a shtuka with no legs, so it is fiberwise semistable of slope 0 and its associated étale $\mathbb{Q}_p$-local system descends to an étale $\mathbb{Z}_p$-local system determined by the shtuka.

Over a point, we may relate this discussion back to previously studied concepts in p-adic Hodge theory.

Definition 4.5.15. Let F be a perfectoid field of characteristic p and write $\mathbf{A}_{\mathrm{inf}}$ for $\mathbf{A}_{\mathrm{inf}}(F, \mathfrak{o}_F)$. Also fix a primitive element z of $\mathbf{A}_{\mathrm{inf}}$ corresponding to an untilt $F^\sharp$ of F of characteristic 0 (that is, z is not divisible by p). A *Breuil–Kisin module*[20] over $\mathbf{A}_{\mathrm{inf}}$ (with respect to z)

[20]The term *Breuil–Kisin module* originally referred to similar objects defined not over $\mathbf{A}_{\mathrm{inf}}$, but over a certain power series ring; see [112]. The relationship between this construction and the one we are now discussing is analogous to the relationship between *imperfect* and *perfect* (φ, Γ)-modules described in [110].

is a finite free $\mathbf{A}_{\text{inf}}$-module D equipped with an isomorphism $\Phi : (\varphi^* D)[z^{-1}] \cong D[z^{-1}]$. Let $x_1 : \operatorname{Spd}(F, \mathfrak{o}_F) \to \operatorname{Spd}(\mathbb{Z}_p)$ be the morphism corresponding to the untilt $F^\sharp$ of F.

The following result is due to Fargues [55], though the proof we obtain here is slightly different; it first appears in [145].

Lemma 4.5.16. *With notation as in Definition 4.5.15, suppose that F is algebraically closed. Then restriction of φ-equivariant vector bundles along the inclusion*

$$Y_S \subset \{v \in \operatorname{Spa}(\mathbf{A}_{\text{inf}}, \mathbf{A}_{\text{inf}}) : v(p) \neq 0\}$$

is an equivalence of categories.

Proof. Full faithfulness follows from a calculation using Newton polygons, which does not depend on F being algebraically closed or even a field (compare [145, Proposition 13.3.2]). Essential surjectivity is a consequence of Theorem 3.6.13. $\square$

Theorem 4.5.17 (Fargues)**.** *With notation as in Definition 4.5.15, suppose that F is algebraically closed. Then the category of Breuil–Kisin modules over $\mathbf{A}_{\text{inf}}$ is equivalent to the category of shtukas over $\operatorname{Spd}(F, \mathfrak{o}_F)$ with the single leg x_1.*

Proof. By Lemma 4.5.13, a shtuka with one leg corresponds to a datum $\mathcal{F}_1 \dashrightarrow \mathcal{F}_2$. By Lemma 4.5.16, $\mathcal{F}_2$ extends uniquely over the point $v([\varpi]) = 0$ (for $\varpi \in F$ a pseudouniformizer). By glueing with $\mathcal{F}_1$, we obtain a vector bundle over the analytic locus of $\operatorname{Spa}(\mathbf{A}_{\text{inf}}, \mathbf{A}_{\text{inf}})$, which by Theorem 4.5.10 arises from a finite free $\mathbf{A}_{\text{inf}}$-module. This proves the claim. $\square$

Remark 4.5.18. With notation as in Definition 4.5.15, Breuil–Kisin modules appear naturally in the study of crystalline representations. In fact, the crystalline comparison isomorphism in p-adic Hodge theory can be exhibited by giving a direct cohomological construction of suitable Breuil–Kisin modules from which the étale, de Rham, and crystalline cohomologies can be functorially recovered (the étale cohomology arising as in Remark 4.5.14). This is the approach taken in the work of Bhatt–Morrow–Scholze [17] (see also [127] and [16, Lecture 4]).

As noted above, the idea to formulate Definition 4.5.15 and relate it to shtukas with one leg as in Theorem 4.5.17 is due to Fargues [55]. This development was one of the primary triggers for both the line of inquiry discussed in this lecture and for [17].

Remark 4.5.19. In light of the second part of Remark 4.5.11, Theorem 4.5.17 does not extend to the case where F is a general perfectoid field; the extra structure imposed by the existence of the Breuil–Kisin module restricts the étale $\mathbb{Z}_p$-local system arising from $\mathcal{F}_1$ by forcing the associated Galois representation to be *crystalline* in the sense of Fontaine. One can also try to consider *relative Breuil–Kisin modules* over more general base spaces, but then the first part of Remark 4.5.11 also comes into play.

4.6. **Affine Grassmannians.** The concept of an *affine Grassmannian* plays a central role in geometric Langlands, in enabling the construction of moduli spaces of shtukas. We describe three different flavors of the construction here; while these constructions operate with respect to more general algebraic groups, we restrict to the case of GL_n for the sake of exposition.

We start with the original affine Grassmannian of Beauville–Laszlo [11]. See [168, Lecture 1] for a detailed treatment.

Definition 4.6.1. Fix a field k and a positive integer n. For R a k-algebra, a *lattice* in $R((t))^n$ is a finite projective $R[\![t]\!]$-submodule Λ such that the induced map

$$\Lambda \otimes_{R[\![t]\!]} R((t)) \to R((t))^n$$

is an isomorphism. The functor[21] Gr taking R to the set of lattices in $R((t))^n$ is a sheaf for the Zariski topology, so it extends to a sheaf on the category of k-schemes.

Theorem 4.6.2 (Beauville–Laszlo)**.** *The functor* Gr *is represented by an ind-projective k-scheme. More precisely, for each N, the subfunctor of lattices lying between $t^{-N}R[\![t]\!]^n$ and $t^N R[\![t]\!]^n$ is represented by a projective k-scheme* $\mathrm{Gr}^{(N)}$*, and the transition maps* $\mathrm{Gr}^{(N)} \to \mathrm{Gr}^{(N+1)}$ *are closed immersions.*

Proof. See [168, Theorem 1.1.3]. □

The following analogue of the previous construction was originally considered at the pointwise level by Haboush [74], and in the following form by Kreidl [113].

Definition 4.6.3. Let k be a perfect field of characteristic p. For R a perfect k-algebra, a *lattice* in $W(R)[p^{-1}]^n$ is a finite projective $W(R)$-submodule Λ such that the induced map

$$\Lambda \otimes_{W(R)} W(R)[p^{-1}] \to W(R)[p^{-1}]^n$$

is an isomorphism. Again, the functor Gr^{W} taking R to the set of lattices in $W(R)[p^{-1}]^n$ is a sheaf for the Zariski topology, so it extends to a sheaf on the category of perfect k-schemes.

The following statement is due to Bhatt–Scholze [19, Theorem 1.1], improving an earlier result of Zhu [169] which asserts $\mathrm{Gr}^{\mathrm{W},(N)}$ is represented by a proper algebraic space over k.

Theorem 4.6.4 (Zhu, Bhatt–Scholze)**.** *For each N, the functor* $\mathrm{Gr}^{W,(N)}$ *of lattices in $W(R)[p^{-1}]^n$ lying between $p^{-N}W(R)[p^{-1}]^n$ and $p^N W(R)[p^{-1}]^n$ is represented by the perfection of a projective k-scheme. The transition maps* $\mathrm{Gr}^{\mathrm{W},(N)} \to \mathrm{Gr}^{\mathrm{W},(N+1)}$ *are closed immersions.*

In the context of perfectoid spaces, it is natural to introduce the following variant of the previous construction.

Definition 4.6.5. Define the presheaves $\mathbf{B}_{\mathrm{dR}}^+, \mathbf{B}_{\mathrm{dR}}$ on the category of perfectoid Huber pairs whose values on (A, A^+) equal, respectively, the completion of $W^b(A^\flat)$ with respect to the principal ideal $\ker(\theta : W^b(A^\flat) \to A)$, and the localization of this ring with respect to a generator z of $\ker(\theta)$. These extend to sheaves on the category of perfectoid spaces with respect to the analytic topology, the étale topology, the pro-étale topology, and the v-topology.

For A a completed algebraic closure of $\mathbb{Q}_p$ and $A^+ = A^\circ$, $\mathbf{B}_{\mathrm{dR}}(A, A^+)$ is Fontaine's *ring of de Rham periods* [59, §2].

Definition 4.6.6. For (A, A^+) a perfectoid pair, a *lattice* in $\mathbf{B}_{\mathrm{dR}}(A, A^+)^n$ is a finite projective $\mathbf{B}_{\mathrm{dR}}^+(A, A^+)$-submodule Λ such that the induced map

$$\Lambda \otimes_{\mathbf{B}_{\mathrm{dR}}^+(A,A^+)} \mathbf{B}_{\mathrm{dR}}(A, A^+) \to \mathbf{B}_{\mathrm{dR}}^+(A, A^+)^n$$

[21] This use of the notation Gr conflicts with the notation for graded rings used in Definition 1.5.3, but we will not be using the latter in this lecture.

is an isomorphism. The functor $\mathrm{Gr}^{\mathrm{dR}}$ taking (A, A^+) to the set of lattices in $\mathbf{B}_{\mathrm{dR}}(A, A^+)^n$ is a sheaf for the analytic topology, so it extends to a sheaf on the category of perfectoid spaces. It is also a sheaf for the pro-étale topology (and even the v-topology), so it further extends to a sheaf on the category of small v-sheaves over $\mathrm{Spd}(\mathbb{Z}_p)$.

Theorem 4.6.7 (Scholze)**.** *For each N, the functor $\mathrm{Gr}^{\mathrm{dR},(N)}$ of lattices in $\mathbf{B}_{\mathrm{dR}}(A, A^+)^n$ lying between $z^{-N}\mathbf{B}^+_{\mathrm{dR}}(A, A^+)^n$ and $z^N\mathbf{B}^+_{\mathrm{dR}}(A, A^+)$ is represented by a small v-sheaf whose base extension from $\mathrm{Spd}(\mathbb{Z}_p)$ to any diamond is a diamond. The transition maps $\mathrm{Gr}^{\mathrm{dR},(N)} \to \mathrm{Gr}^{\mathrm{dR},(N+1)}$ are closed immersions.*

Proof. See [145, §21]. □

We conclude by giving a very brief indication of how such results can be used to construct some moduli spaces of shtukas, and what additional results along the same lines are needed.

Remark 4.6.8. Using the Beauville–Laszlo glueing theorem (Remark 1.9.9) for its original purpose, one obtains an alternate moduli interpretation of the classical affine Grassmannians. To wit, let C be a curve over k, let V be a vector bundle of rank n on C, let $z \in C$ be a k-rational point, fix an identification of $\widehat{\mathcal{O}}_{C,z}$ with $k[\![t]\!]$, and fix a basis of V over $k[\![t]\!]$. Then the functor Gr may be identified with the functor taking a k-algebra R to the set of meromorphic morphisms $V \dashrightarrow V'$ from V to another vector bundle on C which are isomorphisms away from z. This interpretation leads to moduli spaces of shtukas with one leg, or with multiple *disjoint* legs; however, for crossing legs a more sophisticated construction is needed, the *Beilinson–Drinfeld affine Grassmannian* [168, Lecture III].

By the same token, via Remark 4.5.14 one may interpret $\mathrm{Gr}^{\mathrm{dR}}$ as the moduli space of mixed-characteristic shtukas with one leg; this gives rise to *local Shimura varieties* in the sense of [139]. Again, for multiple crossing legs one needs an analogue instead of the Beilinson–Drinfeld affine Grassmannian; such an object can also be constructed in the category of diamonds, as described in [145, §21].

This construction can be thought of as vaguely analogous to the construction of classical moduli spaces in algebraic geometry using geometric invariant theory. As our earlier invocations of this analogy may suggest, the general strategy is to consider a suitable moduli space of vector bundles on relative Fargues–Fontaine curves, apply Theorem 3.7.2 to identify an open subspace of semistable bundles, apply Theorem 3.7.5 to upgrade these bundles to shtukas, then take a suitable quotient to remove unwanted rigidity. This quotient operation behaves poorly on the full moduli space of vector bundles, but somewhat better on the semistable locus.

For more discussion along these lines, see [163, Lecture 4] and [58].

Appendix A. Project descriptions

A.1. **Extensions of vector bundles and slopes (proposed by David Hansen).** The primary project revolves around the following problem.

Problem A.1.1. Let F be a perfectoid field. (Optionally, assume also that F is algebraically closed.) Determine the set of values taken by the triple $(\mathrm{HN}(V), \mathrm{HN}(V'), \mathrm{HN}(V''))$ as $0 \to V' \to V \to V'' \to 0$ varies over all short exact sequences of vector bundles on the FF-curve X_F over F.

One part of this problem is of a combinatorial nature.

Problem A.1.2. Determine the combinatorial constraints on $(\mathrm{HN}(V), \mathrm{HN}(V'), \mathrm{HN}(V''))$ imposed by the slope formalism (e.g., the statement of Lemma 3.4.17).

In the other direction, we will consider some intermediate steps, such as the following.

Problem A.1.3. Let V', V'' be two semistable vector bundles on X_F with $\mu(V') < \mu(V'')$. Show that a bundle V occurs in a short exact sequence $0 \to V' \to V \to V'' \to 0$ if and only if $\mathrm{HN}(V)$ lies between $\mathrm{HN}(V' \oplus V'')$ and the straight line segment with the same endpoints as $\mathrm{HN}(V' \oplus V'')$.

Addressing the problems discussed above requires some basic familiarity with Banach–Colmez spaces, which are described in [163, Lecture 4]. David Hansen has prepared some supplementary material on this topic, which may be incorporated into these notes later; for the moment, see [75].

A.2. *G*-**bundles.** We next formulate a more general form of Problem A.1.1 (Problem A.2.5) in terms of algebraic groups. This requires giving a general description of G-objects in an exact tensor category.

Definition A.2.1. For G an algebraic group over $\mathbb{Q}_p$, let $\mathbf{Rep}_{\mathbb{Q}_p}(G)$ denote the category of (algebraic) representations of G on finite-dimensional F-vector spaces.

Let $\mathcal{C}$ be an $\mathbb{Q}_p$-linear tensor category. (That is, $\mathcal{C}$ is an exact category where the morphism spaces are not just abelian groups but $\mathbb{Q}_p$-vector spaces, composition is not just additive but $\mathbb{Q}_p$-linear, $\mathcal{C}$ carries a symmetric monoidal structure which is yet again $\mathbb{Q}_p$-linear, and $\mathcal{C}$ carries a rank function which adds in short exact sequences and multiplies in tensor products.) By a *G-object in $\mathcal{C}$*, we will mean a covariant, $\mathbb{Q}_p$-linear, rank-preserving tensor functor $\mathbf{Rep}_{\mathbb{Q}_p}(G) \to \mathcal{C}$.

Example A.2.2. Let $\mathcal{C}$ be the category of finite-dimensional $\mathbb{Q}_p$-vector spaces.

- For $G = \mathrm{GL}_n$, a G-object in $\mathcal{C}$ is the same as a vector space of dimension n. (This includes the case $G = \mathbf{G}_m$ by taking $n = 1$.)
- For $G = \mathrm{SL}_n$, a G-object in $\mathcal{C}$ is the same as a vector space V of dimension n plus a choice of generator of the one-dimensional space $\wedge^n V$.
- For $G = \mathrm{O}_n$ (resp. $G = \mathrm{Sp}_n$), a G-object in $\mathcal{C}$ is the same as a vector space V of dimension n plus the choice of a nondegenerate orthogonal (resp. symplectic) form on n.

Example A.2.3. Let $\mathcal{C}$ be the category of vector bundles on an abstract curve C over $\mathbb{Q}_p$.

- For $G = \mathrm{GL}_n$, a G-object in $\mathcal{C}$ is the same as a vector bundle of rank n.

- For $G = \mathrm{SL}_n$, a G-object in $\mathcal{C}$ is the same as a vector bundle V of rank n plus a trivialization of $\wedge^n V$.
- For $G = \mathrm{O}_n$ (resp. $G = \mathrm{Sp}_n$), a G-object in $\mathcal{C}$ is the same as a vector bundle V of rank n plus a nondegenerate orthogonal (resp. symplectic) pairing $V \times V \to \mathcal{O}_C$.

Remark A.2.4. The idea behind the definition of a G-object is that vector bundles on a scheme X (or for that matter, on a manifold X) correspond to elements of the pointed set $H^1(X, \mathrm{GL}_n)$. By contrast, if one replaces GL_n with a smaller group, the resulting vector bundle is not entirely generic: its construction respects certain extra structure, and the exact nature of that extra structure is encoded in the structure of the category $\mathbf{Rep}_{\mathbb{Q}_p}(G)$. This is closely related to the Tannaka–Krein duality theorem, which asserts that the group G can be reconstructed from the data of the category $\mathbf{Rep}_{\mathbb{Q}_p}(G)$ plus the fiber functor taking representations to their underlying $\mathbb{Q}_p$-vector spaces, by taking the automorphism group of the functor (just as in the definition of profinite fundamental groups).

We can now formulate a group-theoretic variant of Problem A.1.1; the statement of Problem A.1.1 constitutes the case of the following problem in which $G = \mathrm{GL}_n$ and H is a certain parabolic subgroup. For this problem, some relevant background is the classification of G-isocrystals by Kottwitz [114] (see also [115, 138]).

Problem A.2.5. Suppose that F is algebraically closed. Let $H \to G$ be an inclusion of connected reductive algebraic groups over $\mathbb{Q}_p$. For a given H-bundle V on X_F, determine which (isomorphism classes of) G-bundles admit a reduction of structure to H.

Remark A.2.6. There is a (perhaps fanciful) resemblance between Problem A.2.5 and some classic questions about numerical invariants (e.g., eigenvalues, singular values) of triples A, B, C of square matrices satisfying $A + B = C$. See [99, Chapter 4] as a starting point.

A.3. **The open mapping theorem for analytic rings.**

Problem A.3.1. Write out a detailed proof of Theorem 1.1.9 for analytic rings, by modifying the argument in [83] for Tate rings.

The key substep is the following extension of [83, Proposition 1.9].

Definition A.3.2. Let X be a topological space. A subset Z of X is *nowhere dense* if for every nonempty open subset U of X, there is a nonempty open subset V of U which is disjoint from Z. A subset Z of X is *meager* if it can be written as a countable union of nowhere dense subsets.

Problem A.3.3. Let A be an analytic Huber ring. Let M, N be topological R-modules. Let $u : M \to N$ be an R-linear morphism whose image is not meager. Then for every neighborhood V of 0 in M, the closure of $u(V)$ is a neighborhood of 0 in N.

Remark A.3.4. The key steps of [83, Proposition 1.9] are that if $x \in A$ is a topologically nilpotent unit, then for every neighborhood W of 0 in M

$$\bigcup_{n=1}^{\infty} x^{-n} W = M,$$

and each set $x^{-n}\overline{u(W)}$ is closed in N. For topologically nilpotent elements $x_1, \dots, x_k \in A$ which generate the unit ideal, the correct analogue of the first statement is that

$$\bigcup_{n=1}^{\infty} W_n = M, \qquad W_n = \{m \in M : x_1^n m, \dots, x_k^n m \in W\}.$$

The correct analogue of the second statement is that for each n,

$$\{m \in N : x_1^n m, \dots, x_k^n m \in \overline{u(W)}\}$$

is closed in N. (If $\{m_i\}$ is a sequence in this set with limit m, then $x_j^n m_i \in \overline{u(W)}$ converges to $x_j^n m$ and so the latter is in $\overline{u(W)}$.)

Problem A.3.5. Adapt the previous arguments to show that all of the results of [83], which are proved for topological rings containing a null sequence consisting of units, remain true for first-countable topological rings (not necessarily Huber rings) in which every open ideal is trivial. (Note that Remark 1.1.1 no longer applies, but this turns out not to be relevant to this argument.) What happens if one drops the first-countable hypothesis?

A.4. **The archimedean Fargues–Fontaine curve (proposed by Sean Howe).**

Definition A.4.1. Let $\tilde{\mathbf{P}}$ be the projective curve in $\mathbf{P}^2_{\mathbb{R}}$ defined by the equation $x^2+y^2+z^2$. This is the unique nontrivial Brauer–Severi curve over $\mathbb{R}$. This object plays a fundamental role in archimedean Hodge theory (e.g., in the study of *mixed twistor $\mathcal{D}$-modules*).

We explore the analogy between $\tilde{\mathbf{P}}$ and the Fargues–Fontaine curve over an algebraically closed perfectoid field.

Problem A.4.2. For an algebraic variety X over $\mathbb{R}$, write $\mathrm{FF}_X \times \mathbb{C}$ for the topological space $(X(\mathbb{C}) \times \mathbf{P}^1(\mathbb{C}))/c$, where c acts on $X(\mathbb{C})$ by the usual conjugation (fixing $X(\mathbb{R})$) and on $\mathbf{P}^1(\mathbb{C})$ by the antipode map $z \mapsto -\overline{z}^{-1}$. Can you formulate a precise archimedean analogue of Lemma 4.3.10? (Note that $\tilde{\mathbf{P}}$ is an algebraic analogue of $\mathbf{P}^1(\mathbb{C})/c$.)

Definition A.4.3. Let $\tilde{W}$ be the Weil group of $\mathbb{R}$ modulo its center $\mathbb{R}^\times$: concretely, this group is a semidirect product $S^1 \rtimes \mathbb{Z}/2\mathbb{Z}$ where $\mathbb{Z}/2\mathbb{Z}$ acts by inversion on S^1. We view $\tilde{\mathbf{P}}$ as the projectized cone over the (scheme of) trace-zero, norm-zero elements in the quaternions $\mathbb{H}$, and identify $\tilde{W}$ with $\mathbb{C}^\times \sqcup j\mathbb{C}^\times \subset \mathbb{H}$, so that $\tilde{\mathbf{P}}$ has a natural action of $\tilde{W}$ with a unique fixed point p with residue field $\mathbb{C}$ on which $\tilde{W}$ acts through conjugation by $\mathbb{Z}/2\mathbb{Z}$.

For X an algebraic variety over $\mathbb{R}$, let $H^i(X(\mathbb{C}), \mathbb{R})$ denote the real singular cohomology of the topological space $X(\mathbb{C})$, equipped with its Hodge decomposition as $\bigoplus_{p+q=i} h^{p,q}$. We equip this $\mathbb{R}$-vector space with a representation of $\tilde{W}$ where S^1 acts as z^{-p+q} on $h^{p,q}$ and c acts by the automorphism induced by conjugation on $X(\mathbb{C})$; using this action, we equip the trivial vector bundle $\mathcal{O} \otimes H^i(X(\mathbb{C}), \mathbb{R})$ on $\tilde{\mathbf{P}^1}$ with a $\tilde{W}$-equivariant structure. We equip the algebraic de Rham cohomology $H^i_{\mathrm{dR}}(X)$ with the trivial $\tilde{W}$-action.

Problem A.4.4. Retain notation as in Definition A.4.3.

(a) Prove the following de Rham comparison theorem: there is a natural identification

$$(\mathcal{O} \otimes H^i(X(\mathbb{C}), \mathbb{R}))_{\mathrm{Spec}(\mathrm{Frac}(\widehat{\mathcal{O}_p}))} \cong H^i_{\mathrm{dR}}(X) \otimes \mathrm{Frac}(\widehat{\mathcal{O}_p}),$$

as $\tilde{W}$-equivariant bundles over $\mathrm{Spec}(\mathrm{Frac}(\widehat{\mathcal{O}_p}))$, and in particular

$$(H^i(X(\mathbb{C}), \mathbb{R}) \otimes \mathrm{Frac}(\widehat{\mathcal{O}_p}))^{\tilde{W}} = H^i_{\mathrm{dR}}(X)$$

with the Hodge filtration corresponding to the filtration by order of poles (up to a change in the numbering). Compare with the p-adic de Rham comparison theorem.

(b) What are the corresponding modifications?

(c) Comparing with [156], you will find that we are not using the standard representation of the Weil group attached to a Hodge structure. For even weight, we've simply taken a Tate twist to land in weight 0, but for odd weight we've taken something genuinely different: our construction factors through the split version of the Weil group, while the the original representation does not. Can we fix this and/or should we want to? On a related note, is there a way to modify this construction so that we obtain the correct numbering on the Hodge filtration and the "natural" slopes for the modifications? In general, what can we do to make a stronger analogy with the p-adic case, and if we can't, how should we understand the difference?

A.5. **Finitely presented morphisms.**

Definition A.5.1. Define a Huber ring A to be *strongly sheafy* if $A\langle T_1, \dots, T_n\rangle$ is sheafy for every nonnegative integer n. For example, if A is strongly noetherian, then A is strongly sheafy by Theorem 1.2.11. For another example, if A is perfectoid, then we may see that A is strongly sheafy by applying Corollary 2.5.5 to the map $A\langle T_1, \dots, T_n\rangle \to A\langle T_1^{p^{-\infty}}, \dots, T_n^{p^{-\infty}}\rangle$.

Definition A.5.2. Suppose A is strongly sheafy. A homomorphism $A \to B$ is *affinoid* if:

- it factors through a surjection $A\langle T_1, \dots, T_n\rangle \to B$; and
- for some such factorization, $B \in \mathbf{PCoh}_{A\langle T_1,\dots,T_n\rangle}$. The same is then true for any such factorization (as in the proof of Theorem 1.4.19).

By Theorem 1.4.20, B is again strongly sheafy. For example, by Theorem 1.4.19, any rational localization is an affinoid morphism. Also, any finite flat morphism, and in particular any finite étale morphism, is affinoid.

Problem A.5.3. We previously gave an *ad hoc* definition of an étale morphism of adic spaces (Hypothesis 1.10.3). Use the concept of an affinoid morphism to give a definition in the strongly sheafy case closer to the one given by Huber in the strongly noetherian case [87, Definition 1.6.5].

Problem A.5.4. Similarly, use the concept of an affinoid morphism to define *unramified* and *smooth* morphisms in the strongly sheafy case.

A.6. **Additional suggestions.**

Problem A.6.1. Prove that for any (analytic) Huber pair (A, A^+), $\mathrm{Spa}(A\langle T\rangle, A^+\langle T\rangle) \to \mathrm{Spa}(A, A^+)$ is an open map.

Problem A.6.2. Verify that for any perfectoid Huber pair (R, R^+) of characteristic p, the ring $\mathbf{A}_{\mathrm{inf}} := \mathbf{A}_{\mathrm{inf}}(R, R^+)$ is sheafy. See Remark 3.1.10 for the case where R is a nonarchimedean field. One possible approach is to show that $\mathbf{A}_{\mathrm{inf}}$ admits a split (in the category of topological $\mathbf{A}_{\mathrm{inf}}$-modules) embedding into a perfectoid (and hence sheafy) ring.

Problem A.6.3. Find a "reasonable" (i.e., as small as possible) category of algebraic stacks in which Remark 4.2.14 can be interpreted.

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