CONSTRUCTION OF EULER SYSTEMS

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1. Course outline

There are a number of conjectures – some proved, some wide open – relating the special values of L-functions to the properties of arithmetic objects; for instance, the main conjecture of Iwasawa theory (which concerns class groups of number fields) and the Birch–Swinnerton-Dyer conjecture (describing points on elliptic curves). One attempt at a unified formulation is the Bloch–Kato conjecture, which relates values of L-functions to the cohomology of global Galois representations.

One of the most important tools that has been used to make progress on these conjectures is the idea of an *Euler system*. These are certain families of Galois cohomology classes, satisfying precise compatibilities ("norm relations"), which can be used to control the sizes of Galois cohomology groups. Unsurprisingly, these objects are rather difficult to construct, and the list of known examples is still relatively short. All of the known constructions rely, in some way, on exploiting the properties of modular and automorphic forms.

The goal of this lecture course will be to explain how some of the known Euler systems are constructed, focussing on three main examples: the Euler system of Kato, associated to a modular form; the Euler system of Beilinson–Flach elements, associated to a pair of modular forms (and its generalisation to Hilbert modular forms); and the Euler system of Lemma–Flach elements, associated to a genus 2 Siegel modular form. All of these are ultimately built up from certain special rational functions on modular curves called *Siegel units*.

Another topic which we will also aim to cover (although in rather less detail) is the relation between Euler systems and the values of L-functions, via p-adic regulator formulae. This is a much more advanced topic, and one where the known results are quite fragmentary; the principal tools involved come from p-adic Hodge theory and arithmetic geometry, notably Besser's rigid syntomic cohomology.

2. Possible projects

2.1. **An Euler system for $GSp(4) \times GL(2)$.** The goal of this project would be to construct an Euler system for the Galois representations appearing in the cohomology of a certain 4-dimensional Shimura variety: namely, the product of a Siegel modular threefold and a modular curve. Recent work of Francesco Lemma [Lem17] shows that there is a supply of interesting cohomology classes available for this variety, so the aim is to show that these can be assembled into an Euler system. This would be closely analogous to the construction of an Euler system for $GSp(4)$ alone, carried out recently by us together with Chris Skinner.

This project has several sub-projects, which are to some extent orthogonal (so it would be suitable for a fairly large team):

- determining which weights (i.e. coefficient systems for cohomology) are accessible;
- writing down the relevant cohomology classes, following Lemma;
- proving norm-compatibility statements in the "vertical" aspect (i.e. for classes over cyclotomic fields $\mathbf{Q}(\mu_m)$ where m is a power of p)
- proving the "horizontal" norm relation (in which we take m to be a prime not dividing the level) – this is a rather more difficult problem, since extra Euler factors make an appearance.

2.2. **Euler systems and Selmer groups in Coleman families.** Modular forms can often be deformed in p-adic families (Hida families of ordinary forms, or the more general finite-slope families constructed by Coleman [Col97]). One knows that the Euler system of Kato, and the Euler system of Beilinson–Flach elements, can be interpolated in Coleman families (see [Wan12, Han15] for the former, and [LZ16] for the latter). This raises (at least) two natural questions.

- In the Beilinson–Flach setting, can one define a Selmer group attached to the family, and use the Euler system to give upper bounds on its size? There is a general approach to defining Selmer groups in families, due to Pottharst [Pot13], and it would be interesting to try to formulate and prove a bound for Pottharst's Selmer groups in this setting.
- Are there analogous interpolation results for other Euler systems, e.g. in the Hilbert or Siegel settings? The existence of p-adic families of finite-slope automorphic forms in these cases is known, by recent works of Andreatta et al; but whether the Euler systems interpolate in these families is an open question.

3. Suggested reading

Bellaïche's notes [Bel09] are an excellent introduction to Galois representations, their cohomology, and the statement of the Bloch–Kato conjecture. For the algebraic side of the theory of Euler systems, Rubin's book [Rub00] is the canonical reference, while some may prefer the alternative account found in [MR04]; however, both of these works concentrate heavily on the algebraic side of the theory – how Euler systems can be used to bound Selmer groups – rather than saying much about how Euler systems are actually constructed, which is the emphasis of this course.

Kato's Euler system is described in detail in [Kat04]. In particular, §2 of this book is an excellent source for the definition and properties of Siegel units. There is also an alternative viewpoint on Kato's construction to be found in Colmez's Bourbaki seminar [Col04]. For the newer constructions – Beilinson–Flach elements for $GL(2) \times GL(2)$, and Lemma–Flach elements for $GSp(4)$ – see [LLZ14] and [LSZ] respectively.

The relations between Euler systems and special values of (complex and p-adic) L-functions are surveyed in [BCD$^+$14][1].

[1] Note that this survey takes a different, and much broader, interpretation of the term "Euler system", so many of the examples considered are not Euler systems in the sense we consider here.

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