Weil's Conjecture for Function Fields

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1 First Lecture: The Mass Formula and Weil's Conjecture

Let R be a commutative ring and let V be an R-module. A *quadratic form* on V is a map $q : V \to R$ satisfying the following conditions:

(a) The construction $(v, w) \mapsto q(v + w) - q(v) - q(w)$ determines an R-bilinear map $V \times V \to R$.

(b) For every element $\lambda \in R$ and every $v \in V$, we have $q(\lambda v) = \lambda^2 q(v)$.

A *quadratic space* over R is a pair (V, q), where V is a finitely generated projective R-module and q is a quadratic form on V.

One of the basic problems in the theory of quadratic forms can be formulated as follows:

Question 1.1. Let R be a commutative ring. Can one classify quadratic spaces over R (up to isomorphism)?

Let us begin by describing some cases where Question 1.1 admits a complete answer.

Example 1.2 (Quadratic Forms over $\mathbf{C}$)**.** Let $R = \mathbf{C}$ be the field of complex numbers (or, more generally, any algebraically closed field of characteristic different from 2). Then every quadratic space over R is isomorphic to $(\mathbf{C}^n, q)$, where q is given by the formula

$$q(x_1, \ldots, x_n) = x_1^2 + \cdots + x_r^2$$

for some $0 \leqslant r \leqslant n$. Moreover, the integer r is uniquely determined: it is an isomorphism-invariant called the *rank* of the quadratic form q.

Example 1.3 (Quadratic Forms over the Real Numbers)**.** Let $R = \mathbf{R}$ be the field of real numbers. Then every quadratic space over R is isomorphic to $(\mathbf{R}^n, q)$, where the quadratic form q is given by

$$q(x_1, \ldots, x_n) = x_1^2 + \cdots + x_a^2 - x_{a+1}^2 - x_{a+2}^2 - \cdots - x_{a+b}^2$$

for some pair of nonnegative integers a and b satisfying $a + b \leqslant n$. Moreover, Sylvester's *invariance of signature theorem* asserts that the integers a and b are uniquely determined. The difference $a - b$ is an isomorphism-invariant of q, called the *signature of* q. We say that a quadratic space (V, q) is *positive-definite* if it has signature n: that is, if it is isomorphic to the standard Euclidean space $(\mathbf{R}^n, q_0)$ with $q_0(x_1, \ldots, x_n) = x_1^2 + \cdots + x_n^2$. Equivalently, a quadratic form q is positive-definite if it satisfies $q(v) > 0$ for every nonzero vector $v \in V$.

Example 1.4 (Quadratic Forms over p-adic Fields)**.** Let $R = \mathbf{Q}_p$ be the field of p-adic rational numbers, for some prime number p. One can show that a *nondegenerate* quadratic space (V, q) over $\mathbf{Q}_p$ is determined up to isomorphism by its *discriminant* (an element of the finite group $\mathbf{Q}_p^\times / \mathbf{Q}_p^{\times 2}$) and its *Hasse invariant* (an element of the group $\{\pm 1\}$). In particular, if p is odd and $n \gg 0$, then there are exactly eight isomorphism classes of nondegenerate quadratic spaces of dimension n over $\mathbf{Q}_p$. When $p = 2$, there are sixteen isomorphism classes.

If (V, q) is a quadratic space over a commutative ring R and $f : R \to S$ is a ring homomorphism, then we can extend scalars along f to obtain a quadratic space (V_S, q_S) over S; here $V_S = S \otimes_R V$ and q_S is the unique extension of q to a quadratic form on V_S. Note that if two quadratic spaces (V, q) and (V', q') over R are isomorphic, then they remain isomorphic after extending scalars to S. In the case $R = \mathbf{Q}$, this assertion has a converse:

Theorem 1.5 (The Hasse Principle)**.** *Let (V, q) and (V', q') be quadratic spaces over the field $\mathbf{Q}$ of rational numbers. Then (V, q) and (V', q') are isomorphic if and only if the following conditions are satisfied:*

- *(a) The quadratic spaces $(V_{\mathbf{R}}, q_{\mathbf{R}})$ and $(V'_{\mathbf{R}}, q'_{\mathbf{R}})$ are isomorphic over $\mathbf{R}$ (this can be tested by comparing signatures).*
- *(b) For every prime number p, the quadratic spaces $(V_{\mathbf{Q}_p}, q_{\mathbf{Q}_p})$ and $(V'_{\mathbf{Q}_p}, q'_{\mathbf{Q}_p})$ are isomorphic.*

Remark 1.6. Theorem 1.5 is known as the *Hasse-Minkowski* theorem: it is originally due to Minkowski, and was later generalized to arbitrary number fields by Hasse.

Remark 1.7. Theorem 1.5 asserts that the canonical map

$$\{\text{Quadratic spaces over } \mathbf{Q}\}/\sim \;\to \prod_K \{\text{Quadratic spaces over } K\}/\sim$$

is injective, where K ranges over the collection of all completions of $\mathbf{Q}$. It is possible to explicitly describe the image of this map (using the fact that the theory of quadratic forms over real and p-adic fields are well-understood; see Examples 1.3 and 1.4 above). We refer the reader to [17] for details.

Let us now specialize to the case $R = \mathbf{Z}$. Note that if (V, q) is a quadratic space over $\mathbf{Z}$, then q determines (and is determined by) a bilinear form

$$b : V \times V \to \mathbf{Z} \qquad b(x, y) = q(x + y) - q(x) - q(y).$$

The bilinear form b has the property that, for each $x \in V$, $b(x, x) = q(2x) - q(x) - q(x) = 2q(x)$ is *even*; conversely, if $b : V \times V \to \mathbf{Z}$ is any symmetric bilinear form with the property that each $b(x, x)$ is even, then we can equip V with a quadratic form given by $q(x) = \frac{b(x,x)}{2}$. For this reason, quadratic spaces over $\mathbf{Z}$ are often called *even lattices*.

We can now ask if the analogue of Theorem 1.5 holds over the integers.

Exercise 1.8. Let (V, q) and (V', q') be quadratic spaces over $\mathbf{Z}$. Show that the following conditions are equivalent:

(a) The quadratic spaces (V, q) and (V', q') become isomorphic after extending scalars along the projection map $\mathbf{Z} \to \mathbf{Z}/N\mathbf{Z}$, for every positive integer N.

(b) The quadratic spaces (V, q) and (V', q') become isomorphic after extending scalars along the projection map $\mathbf{Z} \to \mathbf{Z}/p^k\mathbf{Z}$, for each prime number p and each $k \geqslant 0$.

(c) The quadratic spaces (V, q) and (V', q') become isomorphic after extending scalars along the inclusion $\mathbf{Z} \hookrightarrow \mathbf{Z}_p$ for each prime number p. Here $\mathbf{Z}_p$ denotes the ring of p-adic integers.

(d) The quadratic spaces (V, q) and (V', q') become isomorphic after extending scalars along the inclusion $\mathbf{Z} \hookrightarrow \widehat{\mathbf{Z}}$. Here $\widehat{\mathbf{Z}} = \varprojlim_{N>0}(\mathbf{Z}/N\mathbf{Z}) \simeq \prod_p \mathbf{Z}_p$ is the *profinite completion of* $\mathbf{Z}$.

Definition 1.9. Let (V, q) be a quadratic form over $\mathbf{Z}$. We say that q is *positive-definite* if the quadratic form $q_{\mathbf{R}}$ is positive-definite (Example 1.3). We say that two positive-definite quadratic spaces (V, q) and (V', q') *have the same genus* if they satisfy the equivalent conditions of Exercise 1.8.

Let (V, q) and (V', q') be positive-definite quadratic spaces over $\mathbf{Z}$. If (V, q) and (V', q') are isomorphic, then they are of the same genus. The converse is generally false. However, it is *almost* true in the following sense: for a fixed positive definite quadratic space (V, q) over $\mathbf{Z}$, there are only finitely many quadratic spaces of the same genus (up to isomorphism). In fact, there is even a *formula* for the number of such quadratic spaces, counted with multiplicity. To state it, we first need to introduce some notation.

Notation 1.10. Let (V, q) be a quadratic space over a commutative ring R. We let $\mathrm{O}_q(R)$ denote the automorphism group of (V, q): that is, the group of R-module isomorphisms $\alpha : V \to V$ such that $q = q \circ \alpha$. We will refer to $\mathrm{O}_q(R)$ as the *orthogonal* *group* of the quadratic space (V, q). More generally, if $\phi : R \to S$ is a map of commutative rings, we let $\mathrm{O}_q(S)$ denote the automorphism group of the quadratic space (V_S, q_S) over S obtained from (V, q) by extension of scalars to S.

Example 1.11. Suppose q is a positive-definite quadratic form on an real vector space V of dimension n. Then $\mathrm{O}_q(\mathbf{R})$ can be identified with the usual orthogonal group $\mathrm{O}(n)$, generated by rotations and reflections in Euclidean space. In particular, $\mathrm{O}_q(\mathbf{R})$ is a compact Lie group of dimension $\frac{n^2-n}{2}$.

Example 1.12. Let (V, q) be a positive-definite quadratic space over $\mathbf{Z}$. Then we can identify $\mathrm{O}_q(\mathbf{Z})$ with a subgroup of the orthogonal group $\mathrm{O}_q(\mathbf{R})$: it consists of rotations and reflections in Euclidean space $V_{\mathbf{R}}$ which preserve the lattice $V \subseteq V_{\mathbf{R}}$. In particular, it is a finite group.

Theorem 1.13 (Smith-Minkowski-Siegel Mass Formula)**.** *Let (V, q) be a positive-definite quadratic space over $\mathbf{Z}$ of rank $\geqslant 2$, and let D be the discriminant of q. Then we have*

$$\sum_{q'} \frac{1}{|\mathrm{O}_{q'}(\mathbf{Z})|} = \frac{2|D|^{(n+1)/2}}{\prod_{m=1}^{n} \mathrm{Vol}(S^{m-1})} \prod_p c_p,$$

where the sum on the left hand side is taken over all isomorphism classes of quadratic spaces (V', q') in the genus of (V, q), $\mathrm{Vol}(S^{m-1}) = \frac{2\pi^{m/2}}{\Gamma(m/2)}$ denotes the volume of the standard $(m-1)$-sphere, and the product on the right ranges over all prime numbers p, with individual factors c_p satisfying $c_p = \frac{2p^{kn(n-1)/2}}{|\mathrm{O}_q(\mathbf{Z}/p^k\mathbf{Z})|}$ for $k \gg 0$.

Example 1.14. Let K be an imaginary quadratic field with ring of integers $\mathcal{O}_K$, and let $q : \mathcal{O}_K \to \mathbf{Z}$ be the norm map. Then $(\mathcal{O}_K, q)$ is a positive-definite quadratic space of rank 2 over $\mathbf{Z}$. In this case, Theorem 1.13 reduces to the class number formula for K.

We can greatly simplify the statement of Theorem 1.13 by restricting our attention to a special case.

Definition 1.15. Let (V, q) be a quadratic space over $\mathbf{Z}$. We say that q is *unimodular* if, for every prime number p, the quadratic form $q_{\mathbf{F}_p}$ is nondegenerate. That is, the associated symmetric bilinear form

$$b(x, y) = q_{\mathbf{F}_p}(x + y) - q_{\mathbf{F}_p}(x) - q_{\mathbf{F}_p}(y)$$

is nondegenerate on the $\mathbf{F}_p$-vector space $\mathbf{F}_p^n$.

Warning 1.16. Unimodularity is a very strong condition. Note that the "standard" quadratic form $q(x_1, \ldots, x_n) = x_1^2 + \cdots + x_n^2$ is not unimodular: the associated bilinear form is identically zero modulo 2. One can show that if a quadratic (V, q) is both unimodular and positive-definite, then the rank of V must be divisible by 8.

Let (V, q) and (V', q') be positive-definite quadratic spaces over $\mathbf{Z}$ (of the same rank). It follows immediately from the definition that if (V, q) and (V', q') are of the same genus, then (V, q) is unimodular if and only if (V', q') is unimodular. This assertion has a converse: if (V, q) and (V', q') are both unimodular, then they are of the same genus. In other words, if we fix the number of variables (assumed to be a multiple of 8), then the unimodular quadratic spaces *comprise* a genus. This is in some sense the simplest genus, and for this genus the statement of Theorem 1.13 simplifies: the discriminant D has absolute value 1 (this is equivalent to unimodularity), and the Euler factors c_p can be explicitly evaluated.

Theorem 1.17 (Mass Formula: Unimodular Case)**.** *Let n be an integer which is a positive multiple of 8. Then*

$$\begin{aligned}\sum_q \frac{1}{|\operatorname{O}_q(\mathbf{Z})|} &= \frac{2\zeta(2)\zeta(4)\cdots\zeta(n-4)\zeta(n-2)\zeta(n/2)}{\operatorname{Vol}(S^0)\operatorname{Vol}(S^1)\cdots\operatorname{Vol}(S^{n-1})} \\ &= \frac{B_{n/4}}{n}\prod_{1\leqslant j<n/2}\frac{B_j}{4j}.\end{aligned}$$

Here ζ denotes the Riemann zeta function, B_j denotes the jth Bernoulli number, and the sum is taken over all isomorphism classes of positive-definite unimodular quadratic spaces (V, q) of rank n.

Example 1.18. Let $n = 8$. The right hand side of the mass formula evaluates to $\frac{1}{696729600}$. The integer $696729600 = 2^{14}3^5 5^2 7$ is the order of the Weyl group of the exceptional Lie group E_8, which is also the automorphism group of the root lattice of E_8 (which is an even unimodular lattice). Consequently, the fraction $\frac{1}{696729600}$ also appears as one of the summands on the left hand side of the mass formula. It follows from Theorem 1.17 that no other terms appear on the left hand side: that is, the root lattice of E_8 is the *unique* positive-definite even unimodular lattice of rank 8, up to isomorphism.

Remark 1.19. Theorem 1.17 allows us to count the number of positive-definite even unimodular lattices of a given rank with multiplicity, where a lattice (V, q) is counted with multiplicity $\frac{1}{|\operatorname{O}_q(\mathbf{Z})|}$. If the rank of V is positive, then $\operatorname{O}_q(\mathbf{Z})$ has order at least 2 (since $\operatorname{O}_q(\mathbf{Z})$ contains the group $\langle \pm 1\rangle$), so that the left hand side of Theorem 1.17 is at most $\frac{C}{2}$, where C is the number of isomorphism classes of positive-definite even unimodular lattices. In particular, Theorem 1.17 gives an inequality

$$C \geqslant \frac{\Gamma(\frac{1}{2})\Gamma(\frac{2}{2})\cdots\Gamma(\frac{n}{2})\zeta(2)\zeta(4)\cdots\zeta(n-4)\zeta(n-2)\zeta(\frac{n}{2})}{2^{n-2}\pi^{n(n+1)/4}}.$$

The right hand side of this inequality grows very quickly with n. For example, when $n = 32$, we can deduce the existence of more than eighty million pairwise nonisomorphic (positive-definite) even unimodular lattices of rank n.

We now describe a reformulation of Theorem 1.13, following ideas of Tamagawa and Weil. Suppose we are given a positive-definite quadratic space (V, q) over $\mathbf{Z}$, and that we wish to classify other positive-definite quadratic spaces (V', q') of the same genus. Note that (V, q) and (V', q') must then become isomorphic after extending scalars to $\widehat{\mathbf{Z}}$ (Exercise 1.8). We can therefore choose an isomorphism

$$\alpha : \widehat{\mathbf{Z}} \otimes V \simeq \widehat{\mathbf{Z}} \otimes V'$$

satisfying $q_{\widehat{\mathbf{Z}}} = q'_{\widehat{\mathbf{Z}}} \circ \alpha$. It follows that (V, q) and (V', q') are isomorphic after extending scalars to $\mathbf{Z}_p$, for every prime number p, hence also after extending scalars to $\mathbf{Q}_p = \mathbf{Z}_p[1/p]$. Since (V, q) and (V', q') are also isomorphic over $\mathbf{R}$ (by virtue of our assumption that both are positive-definite), the Hasse principle (Theorem 1.5) guarantees that they are isomorphic over $\mathbf{Q}$. That is, we can choose an isomorphism of rational vector spaces

$$\beta : \mathbf{Q} \otimes V \simeq \mathbf{Q} \otimes V'$$

satisfying $q_{\mathbf{Q}} = q'_{\mathbf{Q}} \circ \beta$.

Let $\mathbf{A}_f$ denote the ring of *finite adeles*: that is, the tensor product $\widehat{\mathbf{Z}} \otimes_{\mathbf{Z}} \mathbf{Q}$. The isomorphism $\widehat{\mathbf{Z}} \simeq \prod_p \mathbf{Z}_p$ induces an injective map

$$\mathbf{A}_f \simeq \widehat{\mathbf{Z}} \otimes_{\mathbf{Z}} \mathbf{Q} \hookrightarrow \prod_p (\mathbf{Z}_p \otimes_{\mathbf{Z}} \mathbf{Q}) \simeq \prod_p \mathbf{Q}_p,$$

whose image is the *restricted product* $\prod_p^{\mathrm{res}} \mathbf{Q}_p \subseteq \prod_p \mathbf{Q}_p$: that is, the subset consisting of those elements $\{x_p\}$ of the product $\prod_p \mathbf{Q}_p$ such that $x_p \in \mathbf{Z}_p$ for all but finitely many prime numbers p. The quadratic spaces (V, q) and (V', q') become isomorphic after extension of scalars to $\mathbf{A}_f$ in two different ways: via the isomorphism α which is defined over $\widehat{\mathbf{Z}}$, and via the isomorphism β which is defined over $\mathbf{Q}$. Consequently, the composition $\beta^{-1} \circ \alpha$ can be regarded as an element of the orthogonal group $\mathrm{O}_q(\mathbf{A}_f)$. This element depends not only the quadratic space (V', q'), but also on our chosen isomorphisms α and β. However, any other isomorphism between $(V_{\widehat{\mathbf{Z}}}, q_{\widehat{\mathbf{Z}}})$ and $(V'_{\widehat{\mathbf{Z}}}, q'_{\widehat{\mathbf{Z}}})$ can be written in the form $\alpha \circ \gamma$, where $\gamma \in \mathrm{O}_q(\widehat{\mathbf{Z}})$. Similarly, the isomorphism β is well-defined up to right multiplication by elements of $\mathrm{O}_q(\mathbf{Q})$. Consequently, the composition $\beta^{-1} \circ \alpha$ is really well-defined as an element of the set of double cosets

$$\mathrm{O}_q(\mathbf{Q}) \backslash \mathrm{O}_q(\mathbf{A}_f) / \mathrm{O}_q(\widehat{\mathbf{Z}}).$$

Let us denote this double coset by $[V', q']$. Note that $[V', q']$ is the identity double coset if and only if it possible to arrange that $\alpha = \beta$: in this case, the map $\alpha = \beta$ is defined over the subring $\mathbf{Z} = \widehat{\mathbf{Z}} \cap \mathbf{Q}$ (where the intersection is formed in the ring $\mathbf{A}_f$), and therefore defines an isomorphism of quadratic space $(V, q) \simeq (V', q')$ over $\mathbf{Z}$. Using a more elaborate version of the same argument, one obtains the following:

Proposition 1.20. *Fix a positive-definite quadratic space* (V, q) *over* $\mathbf{Z}$. *Then the construction* $(V', q') \mapsto [V', q']$ *induces a bijection*

$$\begin{array}{c}\{\text{Quadratic spaces } (V', q') \text{ in the genus of } (V, q)\}/\text{isomorphism} \\ \downarrow \sim \\ \mathrm{O}_q(\mathbf{Q}) \backslash \mathrm{O}_q(\mathbf{A}_f) / \mathrm{O}_q(\widehat{\mathbf{Z}}).\end{array}$$

Exercise 1.21. Give an explicit description of the inverse bijection (this is actually somewhat easier, because it does not depend on the Hasse principle of Theorem 1.5).

At this point, it will be convenient to modify the preceding discussion. Let $\mathbf{A}$ denote the ring of *adeles*, given by the product $\mathbf{A}_f \times \mathbf{R}$. Then the collection of double cosets appearing in Proposition 1.20 can also be written as

$$\mathrm{O}_q(\mathbf{Q}) \backslash \mathrm{O}_q(\mathbf{A}) / \mathrm{O}_q(\widehat{\mathbf{Z}} \times \mathbf{R}).$$

Note that $\mathbf{A}$ has the structure of a locally compact commutative ring, containing $\mathbf{Q}$ as a closed subring (with the discrete topology). It follows that $\mathrm{O}_q(\mathbf{A})$ inherits the structure of a locally compact topological group, which contains $\mathrm{O}_q(\mathbf{Q})$ as a discrete subgroup and $\mathrm{O}_q(\widehat{\mathbf{Z}} \times \mathbf{R})$ as a compact open group.

Let μ be a Haar measure on the group $\mathrm{O}_q(\mathbf{A})$. One can show that the group $\mathrm{O}_q(\mathbf{A})$ is unimodular: that is, the measure μ is invariant under both right and left translations. In particular, μ determines a measure on the quotient space $\mathrm{O}_q(\mathbf{Q}) \backslash \mathrm{O}_q(\mathbf{A})$, which is invariant under the right action of $\mathrm{O}_q(\widehat{\mathbf{Z}} \times \mathbf{R})$. We will abuse notation by denoting this measure also by μ. Write $\mathrm{O}_q(\mathbf{Q}) \backslash \mathrm{O}_q(\mathbf{A})$ as a union of orbits $\bigcup_{x \in X} O_x$ for the action of the group $\mathrm{O}_q(\widehat{\mathbf{Z}} \times \mathbf{R})$. If $x \in X$ is a double coset represented by an element $\gamma \in \mathrm{O}_q(\mathbf{A})$, then we can identify the orbit O_x with the quotient of $\mathrm{O}_q(\widehat{\mathbf{Z}} \times \mathbf{R})$ by the finite subgroup $\mathrm{O}_q(\widehat{\mathbf{Z}} \times \mathbf{R}) \cap \gamma^{-1} \mathrm{O}_q(\mathbf{Q}) \gamma$. We therefore have

$$\sum_{\gamma} \frac{1}{|\,\mathrm{O}_q(\widehat{\mathbf{Z}} \times \mathbf{R}) \cap \gamma^{-1} \mathrm{O}_q(\mathbf{Q}) \gamma|} = \sum_{x \in X} \frac{\mu(O_x)}{\mu(\mathrm{O}_q(\widehat{\mathbf{Z}} \times \mathbf{R}))} \tag{1}$$

$$= \frac{\mu(\mathrm{O}_q(\mathbf{Q}) \backslash \mathrm{O}_q(\mathbf{A}))}{\mu(\mathrm{O}_q(\widehat{\mathbf{Z}} \times \mathbf{R}))}. \tag{2}$$

Exercise 1.22. Show that, if $\gamma \in \mathrm{O}_q(\mathbf{A})$ is an element representing a double coset $x \in X$, then the intersection $\mathrm{O}_q(\widehat{\mathbf{Z}} \times \mathbf{R}) \cap \gamma^{-1} \mathrm{O}_q(\mathbf{Q}) \gamma$ is the finite group $\mathrm{O}_{q'}(\mathbf{Z})$, where (V', q') is the quadratic space corresponding to x under the bijection of Proposition 1.20.

Combining Proposition 1.20 with Exercise 1.22, we obtain an identity

$$\sum_{q'} \frac{1}{|\operatorname{O}_{q'}(\mathbf{Z})|} = \frac{\mu(\operatorname{O}_q(\mathbf{Q})\backslash \operatorname{O}_q(\mathbf{A}))}{\mu(\operatorname{O}_q(\widehat{\mathbf{Z}} \times \mathbf{R}))}.$$

Here the sum on the left is taken over isomorphism classes of positive-definite quadratic spaces (V', q') in the genus of (V, q), as in the Smith-Minkowski-Siegel mass formula (Theorem 1.13). The right hand side is *a priori* dependent on a choice of Haar measure μ on the locally compact group $\operatorname{O}_q(\mathbf{A})$. Such a measure is not uniquely determined: it is only well-defined up to multiplication by a positive scalar. However, the measure μ appears both in the numerator and denominator on the right hand side, and therefore does not depend on μ.

To proceed further, it will be convenient to make another modification. For each commutative ring R, let $\operatorname{SO}_q(R) \subseteq \operatorname{O}_q(R)$ denote the *special orthogonal group* of q over R: that is, the subgroup of $\operatorname{O}_q(R)$ consisting of automorphisms of V_R which preserves the quadratic form q_R and have determinant 1. Then $\operatorname{SO}_q(\mathbf{A})$ is also a locally compact group, on which we can choose a Haar measure μ'. It is not difficult to show that there is an equality

$$\frac{\mu(\operatorname{O}_q(\mathbf{Q})\backslash \operatorname{O}_q(\mathbf{A}))}{\mu(\operatorname{O}_q(\widehat{\mathbf{Z}} \times \mathbf{R}))} = 2^{k-1} \frac{\mu'(\operatorname{SO}_q(\mathbf{Q})\backslash \operatorname{SO}_q(\mathbf{A}))}{\mu'(\operatorname{SO}_q(\widehat{\mathbf{Z}} \times \mathbf{R}))},$$

where k is the number of primes p for which $\operatorname{SO}_q(\mathbf{Z}_p) = \operatorname{O}_q(\mathbf{Z}_p)$. Consequently, to understand Theorem 1.13, it will suffice to compute the ratio

$$\frac{\mu'(\operatorname{SO}_q(\mathbf{Q})\backslash \operatorname{SO}_q(\mathbf{A}))}{\mu'(\operatorname{SO}_q(\widehat{\mathbf{Z}} \times \mathbf{R}))}$$

appearing on the right hand side. The virtue of this maneuver is that on the group $\operatorname{SO}_q(\mathbf{A})$, there is actually a *canonical* choice for the Haar measure μ' (called *Tamagawa measure*), which will allow us to evaluate the numerator and denominator individually.

Note that the group $\operatorname{SO}_q(\mathbf{A})$ can be identified with the *restricted* product

$$\operatorname{SO}_q(\mathbf{R}) \times \prod_p^{\text{res}} \operatorname{SO}_q(\mathbf{Q}_p) \subseteq \operatorname{SO}_q(\mathbf{R}) \times \prod_p \operatorname{SO}_q(\mathbf{Q}_p),$$

where the superscript indicates that we allow only those tuples $\{\gamma_p \in \operatorname{SO}_q(\mathbf{Q}_p)\}$ where γ_p belongs to $\operatorname{SO}_q(\mathbf{Z}_p) \subseteq \operatorname{SO}_q(\mathbf{Q}_p)$ for all but finitely many prime numbers p. Let us first focus our attention on the group $\operatorname{SO}_q(\mathbf{R})$. Since q is positive-definite, this is a familiar group: it is the connected Lie group of rotations in Euclidean space. In particular, it is a smooth manifold of dimension $d = \frac{n^2-n}{2}$. Every differential form ω of degree d on $\operatorname{SO}_q(\mathbf{R})$ determines a measure on $\operatorname{SO}_q(\mathbf{R})$ (given by integrating the absolute value $|\omega|$). Moreover, this measure is a Haar measure if and only if ω is invariant under translation (and nonzero). The collection of translation-invariant differential forms forms a 1-dimensional vector space $\Omega_{\mathbf{R}}$ over the real numbers, and every nonzero element $\omega \in \Omega_{\mathbf{R}}$ determines a Haar measure on $\mathrm{SO}_q(\mathbf{R})$, which we will denote by $\mu_{\omega,\mathbf{R}}$.

Note that the Lie group $\mathrm{SO}_q(\mathbf{R})$ is rather special: it is an *linear algebraic group* over $\mathbf{R}$. In other words, it can be described explicitly as the group

$$\mathrm{SO}_q(\mathbf{R}) = \{A \in \mathrm{GL}_n(\mathbf{R}) : \det(A) = 1, q \circ A = q\} \subseteq \mathrm{GL}_n(\mathbf{R})$$

of invertible matrices over $\mathbf{R}$ whose matrix coefficients satisfy some polynomial equations. Moreover, these polynomial equations have coefficients in $\mathbf{Q}$ (or even in $\mathbf{Z}$), and therefore define a linear algebraic group over $\mathbf{Q}$ that we will denote by SO_q. (In the language of schemes, we identify SO_q with the $\mathbf{Q}$-scheme representing the functor $R \mapsto \mathrm{SO}_q(R)$ defined above). It follows that the vector space $\Omega_{\mathbf{R}}$ (which is one-dimensional over the field of real numbers) contains a subspace $\Omega_{\mathbf{Q}}$ (which is one-dimensional over the rational numbers) consisting of translation-invariant *algebraic* differential forms on the algebraic group SO_q. Any nonzero element $\omega \in \Omega_{\mathbf{Q}}$ is also a nonzero element of $\Omega_{\mathbf{R}}$, and therefore determines a Haar measure $\mu_{\omega,\mathbf{R}}$ on the compact Lie group $\mathrm{SO}_q(\mathbf{R})$.

Let us now consider the p-adic factors $\mathrm{SO}_q(\mathbf{Q}_p)$. These are not Lie groups in the usual sense. However, it is an example of a *p-adic analytic Lie group.* In the setting of p-adic analytic groups, one can tell a parallel story: there is a 1-dimensional vector space $\Omega_{\mathbf{Q}_p}$ of differential forms of top degree on $\mathrm{SO}_q(\mathbf{Q}_p)$, and any nonzero element $\omega \in \Omega_{\mathbf{Q}_p}$ determines a Haar measure on $\mathrm{SO}_q(\mathbf{Q}_p)$ which we will denote by $\mu_{\omega,\mathbf{Q}_p}$. Moreover, since the p-adic analytic Lie group $\mathrm{SO}_q(\mathbf{Q}_p)$ comes from the same algebraic group SO_q over the rational numbers, we have $\Omega_{\mathbf{Q}_p} \simeq \mathbf{Q}_p \otimes_{\mathbf{Q}} \Omega_{\mathbf{Q}}$, where $\Omega_{\mathbf{Q}}$ is the 1-dimensional $\mathbf{Q}$-vector space described above (consisting of algebraic differential forms of top degree on SO_q). It follows that every nonzero element $\omega \in \Omega_{\mathbf{Q}}$ can also be regarded as a nonzero element of $\Omega_{\mathbf{Q}_p}$, and therefore determines a Haar measure $\mu_{\omega,\mathbf{Q}_p}$ on the group $\mathrm{SO}_q(\mathbf{Q}_p)$.

Construction 1.23 (Tamagawa)**.** Fix a nonzero element $\omega \in \Omega_{\mathbf{Q}}$. We let μ_{Tam} denote the invariant measure on the group $\mathrm{SO}_q(\mathbf{A})$ given informally by the infinite product

$$\mu_{\omega,\mathbf{R}} \times \prod_p \mu_{\omega,\mathbf{Q}_p}.$$

We refer to μ_{Tam} as *Tamagawa measure* on the group $\mathrm{SO}_q(\mathbf{A})$.

Remark 1.24. To define Tamagawa measure more precisely, recall that the group $\mathrm{SO}_q(\mathbf{A})$ contains the group

$$\mathrm{SO}_q(\widehat{\mathbf{Z}} \times \mathbf{R}) \simeq \mathrm{SO}_q(\mathbf{R}) \times \prod_p \mathrm{SO}_q(\mathbf{Z}_p)$$

as a compact open subgroup. Consequently, for every positive real number λ, there is a unique Haar measure μ on $\mathrm{SO}_q(\mathbf{A})$ satisfying

$$\mu(\mathrm{SO}_q(\widehat{\mathbf{Z}} \times \mathbf{R})) = \lambda.$$

To obtain Tamagawa measure $\mu = \mu_{\mathrm{Tam}}$, we take λ to be the real number

$$\mu_{\omega,\mathbf{R}}(\mathrm{SO}_q(\mathbf{R})) \times \prod_p \mu_{\omega,\mathbf{Q}_p}(\mathrm{SO}_q(\mathbf{Z}_p)).$$

To guarantee that this is well-defined, we need to know that the infinite product $\prod_p \mu_{\omega,\mathbf{Q}_p}(\mathrm{SO}_q(\mathbf{Z}_p))$ converges; this is our motivation for working with the group SO_q in place of the larger group O_q (for which the analogous product would not converge).

Remark 1.25. The Tamagawa measure of Construction 1.23 is *a priori* dependent on a choice of nonzero element $\omega \in \Omega_{\mathbf{Q}}$. However, it turns out to be independent of this choice. If λ is any nonzero rational number, we have equalities

$$\mu_{\lambda\omega,\mathbf{R}} = |\lambda| \mu_{\omega,\mathbf{R}} \qquad \mu_{\lambda\omega,\mathbf{Q}_p} = |\lambda|_p \mu_{\omega,\mathbf{Q}_p},$$

where $|\lambda|$ denotes the usual absolute value of λ and $|\lambda|_p$ denotes the p-adic absolute value of λ. Consequently, replacing ω by $\lambda\omega$ will later the choice of Tamagawa measure by an overall factor of

$$|\lambda| \cdot \prod_p |\lambda|_p,$$

which is equal to 1 by the product formula.

Returning to our previous discussion, we obtain a formula

$$\sum \frac{1}{|\mathrm{O}_{q'}(\mathbf{Z})|} = 2^{k-1} \frac{\mu_{\mathrm{Tam}}(\mathrm{SO}_q(\mathbf{Q}) \backslash \mathrm{SO}_q(\mathbf{A}))}{\mu_{\mathrm{Tam}}(\mathrm{SO}_q(\widehat{\mathbf{Z}} \times \mathbf{R})};$$

here k (as above) is the number of primes p for which $\mathrm{O}_q(\mathbf{Z}_p) = \mathrm{SO}_q(\mathbf{Q}_p)$. Moreover, the denominator on the right hand side can be evaluated explicitly (Remark 1.24). This allows us to restate the Smith-Minkowski-Siegel mass formula in a much simpler form (for simplicity, let us assume that the number of variables is at least 3):

Theorem 1.26 (Mass Formula, Tamagawa-Weil Version)**.** *Let (V, q) be a nondegenerate quadratic space over $\mathbf{Q}$ of rank $\geqslant 3$. Then the Tamagawa measure*

$$\mu_{\mathrm{Tam}}(\mathrm{SO}_q(\mathbf{Q}) \backslash \mathrm{SO}_q(\mathbf{A}))$$

is equal to 2.

Remark 1.27. In the statement of Theorem 1.26, it is not necessary to assume that (V, q) is defined over $\mathbf{Z}$ or that the quadratic form q is positive-definite.

Theorem 1.26 can be refined a bit further. Let (V, q) be a nondegenerate quadratic space over $\mathbf{Q}$ of rank n. Then the functor $R \mapsto \mathrm{SO}_q(R)$ is (representable by) an algebraic group over $\mathbf{Q}$, which we denote by SO_q. If $n \geqslant 3$, then this algebraic group is semisimple. However, it is not simply connected: instead, it has a simply-connected double cover $\mathrm{Spin}_q \to \mathrm{SO}_q$. We can regard the group of adelic points $\mathrm{Spin}_q(\mathbf{A})$ as a locally compact group, and a variant of Construction 1.23 produces a canonical measure on $\mathrm{Spin}_q(\mathbf{A})$ which we will also denote by μ_{Tam} and refer to as *Tamagawa measure*. We then have the following result (from which one can recover the mass formula of Smith-Minkowski-Siegel).

Theorem 1.28 (Weil)**.** *Let (V, q) be a nondegenerate quadratic space over $\mathbf{Q}$ of rank $\geqslant 3$. Then the Tamagawa measure $\mu_{\mathrm{Tam}}(\mathrm{Spin}_q(\mathbf{Q})\backslash \mathrm{Spin}_q(\mathbf{A}))$ is equal to* 1*.*

More generally, if G is any semisimple algebraic group over $\mathbf{Q}$, then $G(\mathbf{A})$ is a locally compact group which is equipped with a canonical measure μ_{Tam} (given by an analogue of Construction 1.23). One can show that the measure

$$\tau_G := \mu_{\mathrm{Tam}}(G(\mathbf{Q})\backslash G(\mathbf{A}))$$

is finite; we refer to τ_G as the *Tamagawa number* of the group G. Using this terminology, Theorem 1.26 asserts that $G = \mathrm{SO}_q$ has Tamagawa number 2, and Theorem 1.28 asserts that Spin_q has Tamagawa number 1. Motivated by this (another other examples), Weil conjectured the following:

Conjecture 1.29 (Weil's Conjecture on Tamagawa Numbers)**.** Let G be a simply-connected semisimple algebraic group over $\mathbf{Q}$. Then the Tamagawa number $\tau_G := \mu_{\mathrm{Tam}}(G(\mathbf{Q})\backslash G(\mathbf{A}))$ is equal to 1.

Conjecture 1.29 was proven by Weil in a number of special cases. The general case was proven by Langlands in the case of a split group ([12]), by Lai in the case of a quasi-split group ([10]), and by Kottwitz for arbitrary simply connected algebraic groups satisfying the Hasse principle ([9]; this is now known to be all simply-connected semisimple algebraic groups over $\mathbf{Q}$, by work of Chernousov).

2 Second Lecture: The Function Field Case

We now discuss the analogue of Conjecture 1.29 in the function field setting. First, we need to establish some terminology.

Notation 2.1. Let $\mathbf{F}_q$ denote a finite field with q elements, and let X be an algebraic curve over $\mathbf{F}_q$ (which we assume to be smooth, projective, and geometrically connected). We let K_X denote the function field of the curve X (that is, the residue field of the generic point of X). Fields which arise in this manner are known as *function fields*.

We will write $x \in X$ to mean that x is a *closed* point of the curve X. For each point $x \in X$, we let $\kappa(x)$ denote the residue field of X at the point x. Then $\kappa(x)$ is a finite extension of the finite field $\mathbf{F}_q$. We will denote the degree of this extension by $\deg(x)$ and refer to it as the *degree* of x. We let $\mathcal{O}_x$ denote the completion of the local ring of X at the point x: this is a complete discrete valuation ring with residue field $\kappa(x)$, noncanonically isomorphic to a power series ring $\kappa(x)[[t]]$. We let K_x denote the fraction field of $\mathcal{O}_x$.

We let $\mathbf{A}_X$ denote the *adele ring* of X, given by the subset $\mathbf{A}_X = \prod^{\text{res}}_{x \in X} K_x \subseteq \prod_{x \in X} K_x$ consisting of those tuples $\{f_x \in K_x\}_{x \in X}$ such that f_x belongs to $\mathcal{O}_x$ for all but finitely many values of x. Then $\mathbf{A}_X$ has a canonical topology with respect to which it is locally compact. It contains the product $\prod_{x \in X} \mathcal{O}_x$ as a compact open subring and the function field K_X as a discrete subring.

Let G_0 be a semisimple algebraic group over K_X. For every K_X-algebra R, we let $G_0(R)$ denote the group of R-valued points of G_0. The topology on $\mathbf{A}_X$ determines a topology on the group $G_0(\mathbf{A}_X)$. With respect to this topology, $G_0(\mathbf{A}_X)$ is a locally compact topological group, containing $G_0(K_X)$ as a discrete subgroup. Using an analogue of Construction 1.23, one can specify a canonical Haar measure μ_{Tam} on the group $G_0(\mathbf{A}_X)$, allowing us to define the *Tamagawa number*

$$\tau_{G_0} = \mu_{\text{Tam}}(G_0(K_X) \backslash G_0(\mathbf{A}_X)).$$

We can then formulate the function field analogue of Weil's conjecture as follows:

Conjecture 2.2 (Weil's Conjecture, Function Field Case)**.** Let X be an algebraic curve over a finite field $\mathbf{F}_q$ and let G_0 be a linear algebraic group over the function field K_X which is semisimple and simply connected. Then the Tamagawa number $\tau_{G_0} := \mu_{\text{Tam}}(G_0(K_X) \backslash G_0(\mathbf{A}_X))$ is equal to 1.

In the first lecture, we arrived at the problem of computing the Tamagawa number τ_G (in the special case $G = \mathrm{SO}_q$ for some quadratic space (V, q) over $\mathbf{Z}$) as a reformulation of counting the number of quadratic spaces in the genus of (V, q). Conjecture 2.2 admits a similar interpretation. To formulate it, we need to make an additional choice: a *group scheme* G over the entire curve X, having the algebraic group G_0 as its generic fiber. We will assume that the map $G \to X$ is smooth and affine, with connected fibers (this can always be arranged).

Warning 2.3. Let G be a smooth affine group scheme over X with generic fiber G_0. For each closed point $x \in X$, let G_x denote the fiber of G over the point x, which can then be regarded as a linear algebraic group over the finite field $\kappa(x)$. Our assumption that G_0 is semisimple and simply connected guarantees that the algebraic groups G_x are also semisimple and simply connected with finitely many exceptions. However, it is not always possible to arrange that the fibers G_x are semisimple for *all* x: in general, there will be finitely many points of X at which the group scheme G has "bad reduction."

Note that since G_0 is an algebraic group over K_X, it can be evaluated on any K_X-algebra R to obtain a group $G_0(R)$ of R-valued points of G_0. Taking R to be the local field K_x (for any choice of closed point $x \in X$), we obtain a locally compact group $G_0(K_x)$. However, the choice of a group scheme G provides some additional structure: to any map of schemes $f : \mathrm{Spec}(R) \to X$, we can associate a group $G(R)$ of R-valued points of G. In the special case where f factors through the generic point $\mathrm{Spec}(K_X) \hookrightarrow X$, we have $G(R) = G_0(R)$; in particular, $G(K_x) = G_0(K_x)$ is the locally compact group defined above. However, we can also evaluate G on the local ring $\mathcal{O}_x$ to obtain a compact open subgroup $G(\mathcal{O}_x) \subseteq G(K_x)$, and on the residue field $\kappa(x)$ to obtain a finite group $G(\kappa(x))$ (which is a quotient of $G(\mathcal{O}_x)$). Note that the product

$$\prod_{x \in X} G(\mathcal{O}_x) \simeq G(\prod_{x \in X} \mathcal{O}_x)$$

can be regarded as a compact open subgroup of the adelic group $G(\mathbf{A}_X)$. We can then consider the collection of double cosets

$$G(K_X)\backslash G(\mathbf{A}_X)/G(\prod_{x \in X} \mathcal{O}_x).$$

This collection of double cosets can be described by the following analogue of Proposition 1.20:

Proposition 2.4.

$$\begin{array}{c}\{\text{Principal } G\text{-bundles on } X\}/\text{isomorphism} \\ \downarrow \sim \\ G(K_X)\backslash G(\mathbf{A}_X)/G(\prod_{x \in X} \mathcal{O}_x).\end{array}$$

Moreover, if $\mathcal{P}$ is a principal G-bundle on X which corresponds under this bijection to a $G(\prod_{x \in X} \mathcal{O}_x)$-orbit O on $G(K_X)\backslash G(\mathbf{A}_X)$, then the stabilizer group of O can be identified with the automorphism group $\mathrm{Aut}(\mathcal{P})$ of the principal G-bundle $\mathcal{P}$.

Proposition 2.4 is not entirely formal. It relies on two non-obvious facts about principal G-bundles on X:

- If $\mathcal{P}$ is a principal G-bundle on X, then $\mathcal{P}$ is trivial when restricted to every closed point $x \in X$ This follows from a theorem of Lang, which depends on our assumption that the fibers of G are connected; see [11].

- If $\mathcal{P}$ is a principal G-bundle on X, then $\mathcal{P}$ is trivial at the generic point of X. This is a theorem of Harder, which uses the assumption that G_0 is simply connected; it can be regarded a version of the Hasse principle for G-bundles (analogous to Theorem 1.5); see [8].

Using Proposition 2.4, we obtain the following interpretation of the Tamagawa number τ_{G_0}: it is given by

$$\tau_{G_0} = \mu_{\mathrm{Tam}}(G(K_X)\backslash G(\mathbf{A}_X)) = \mu_{\mathrm{Tam}}(\prod_{x \in X} G(\mathcal{O}_x)) \cdot \sum_{\mathcal{P}} \frac{1}{|\operatorname{Aut}(\mathcal{P})|};$$

here the sum is taken over all isomorphism classes of principal G-bundles on X (note that each of the automorphism groups $\operatorname{Aut}(\mathcal{P})$ is finite, by virtue of the fact that X is a proper curve over a finite field). Here the first factor $\mu_{\mathrm{Tam}}(\prod_{x \in X} G(\mathcal{O}_x))$ is easy to evaluate, because Tamagawa measure is essentially *defined* as a product of measures on the individual groups $G(K_x)$. (Of course, to actually evaluate it, we would need to spell out the definition of Tamagawa measure in more detail: see [7] for a discussion). This allows us to reformulate Conjecture 2.2 as follows:

Conjecture 2.5 (Weil's Conjecture, Mass Formula Version)**.** Let d be the dimension of the algebraic group G_0. Then we have an equality

$$\sum_{\mathcal{P}} \frac{1}{|\operatorname{Aut}(\mathcal{P})|} = q^D \prod_{x \in X} \frac{|\kappa(x)|^d}{|G(\kappa(x))|}.$$

Here the sum on the left hand side is taken over all isomorphism classes of principal G-bundles on X, and we have $D = d(g-1) - \deg(\mathfrak{g})$, where g is the genus of the curve X and $\mathfrak{g}$ is the vector bundle on X given by the Lie algebra of the group scheme G (we will give a better interpretation of the integer D below).

Warning 2.6. Neither the sum $\sum_{\mathcal{P}} \frac{1}{|\operatorname{Aut}(\mathcal{P})|}$ nor the product $\prod_{x \in X} \frac{|\kappa(x)|^d}{|G(\kappa(x))|}$ appearing in the statement of Conjecture 2.5 is finite. Hence part of the content of the conjecture is that both of these expressions are convergent.

The assertion of Conjecture 2.5 can be regarded as a function field version of the Smith-Minkowski-Siegel mass formula. More precisely, we have the following table of analogies:

Classical Mass Formula	Conjecture 2.5
Number field $\mathbf{Q}$	Function field K_X
Quadratic space $(V_{\mathbf{Q}}, q_{\mathbf{Q}})$ over $\mathbf{Q}$	Algebraic Group G_0
Quadratic space (V, q) over $\mathbf{Z}$	Group Scheme G
Quadratic space (V', q') of the same genus	Principal G-bundle $\mathcal{P}$
$\sum_{q'} \frac{1}{\lvert \mathrm{O}_{q'}(\mathbf{Z}) \rvert}$	$\sum_{\mathcal{P}} \frac{1}{\lvert \mathrm{Aut}(\mathcal{P}) \rvert}$.

Note however that there is reason to expect that Conjecture 2.5 should be easier to prove than Theorem 1.13, because function fields tend to be "easier" than number fields. More precisely, there are tools available for attacking Conjecture 2.5 that have no analogue in the number field case. If (V, q) is a quadratic space over $\mathbf{Z}$, then isomorphism classes of quadratic spaces in the genus of (V, q) form a set without any obvious additional structure. However, if G is a group scheme over X, then the collection of principal G-bundles admits an algebro-geometric parametrization.

Definition 2.7. If Y is a scheme equipped with a map $Y \to X$, we define a *G-bundle* on Y to be a principal homogeneous space for the group scheme $G_Y = Y \times_X G$ over Y. The collection of G-bundles on Y forms a category (in which all morphisms are isomorphisms).

For every $\mathbf{F}_q$-algebra R, we let $\mathrm{Bun}_G(X)$ denote the category of G-bundles on the relative curve $X_R = \mathrm{Spec}(R) \times_{\mathrm{Spec}(\mathbf{F}_q)} X$. The construction $R \mapsto \mathrm{Bun}_G(X)(R)$ is an example of an *algebraic stack*, which we will denote by $\mathrm{Bun}_G(X)$. We will refer to $\mathrm{Bun}_G(X)$ as the *moduli stack of G-bundles on X*.

Remark 2.8. The algebraic stack $\mathrm{Bun}_G(X)$ is smooth over $\mathbf{F}_q$, and its dimension is the integer $D = d(g-1) - \deg(\mathfrak{g})$ appearing in the statement of Conjecture 2.5. By definition, the category $\mathrm{Bun}_G(X)(\mathbf{F}_q)$ is the category of G-bundles on X. We will denote the sum $\sum_{\mathcal{P}} \frac{1}{\lvert \mathrm{Aut}(\mathcal{P}) \rvert}$ by $\lvert \mathrm{Bun}_G(X)(\mathbf{F}_q) \rvert$: we can think of it as a (weighted) count of the objects of $\mathrm{Bun}_G(X)(\mathbf{F}_q)$, which properly takes into account the fact that $\mathrm{Bun}_G(X)(\mathbf{F}_q)$ is a category rather than a set. With this notation, we can rephrase Conjecture 2.5 as an equality

$$\frac{\lvert \mathrm{Bun}_G(X)(\mathbf{F}_q) \rvert}{q^{\dim(\mathrm{Bun}_G)}} = \prod_{x \in X} \frac{|\kappa(x)|^d}{|G(\kappa(x))|}.$$

Variant 2.9. For each closed point $x \in X$, let G_x denote the fiber $G \times_X \operatorname{Spec}(\kappa(x))$, which we regard as a linear algebraic group over $\kappa(x)$, and let BG_x denote the moduli stack of principal G_x-bundles. Since the fibers of G are connected, every G-bundle on $\operatorname{Spec}(\kappa(x))$ is trivial (by virtue of Lang's theorem), and the trivial G-bundle on $\operatorname{Spec}(\kappa(x))$ has automorphism group $G(\kappa(x))$. We may therefore write

$$\frac{|\kappa(x)|^d}{|G(\kappa(x))|} = \frac{|\mathrm{BG}_x(\kappa(x))|}{|\kappa(x)|^{\dim \mathrm{BG}_x}}.$$

Conjecture 2.5 can therefore be written in the more suggestive form

$$\frac{|\operatorname{Bun}_G(X)(\mathbf{F}_q)|}{q^{\dim \operatorname{Bun}_G(X)}} = \prod_{x \in X} \frac{|\mathrm{BG}_x(\kappa(x))|}{|\kappa(x)|^{\dim \mathrm{BG}_x}}.$$

The problem of counting the number of points on algebraic varieties over $\mathbf{F}_q$ is the subject of another very famous idea of Weil. Let Y be a quasi-projective variety over $\mathbf{F}_q$, so that there exists an embedding $j : Y \hookrightarrow \mathbf{P}^n_{\mathbf{F}_q}$ for some $n \geqslant 0$. Set $\overline{Y} = \operatorname{Spec}(\overline{\mathbf{F}}_q) \times_{\operatorname{Spec}(\mathbf{F}_q)} Y$, so that j determines an embedding $\overline{j} : \overline{Y} \to \mathbf{P}^n_{\overline{\mathbf{F}}_q}$. There is a canonical map from $\mathbf{P}^n_{\overline{\mathbf{F}}_q}$ to itself, given in homogeneous coordinates by

$$[x_0 : \cdots : x_n] \mapsto [x_0^q : \cdots x_n^q].$$

Since $\overline{Y}$ is characterized by the vanishing and nonvanishing of homogeneous polynomials with coefficients in $\mathbf{F}_q$, this construction determines a map $\varphi : \overline{Y} \to \overline{Y}$, which we will refer to as the *geometric Frobenius map*. Then the finite set $Y(\mathbf{F}_q)$ of $\mathbf{F}_q$-points of Y can be identified with the *fixed locus* of the map $\varphi : \overline{Y} \to \overline{Y}$. Weil made a series of conjectures about the behavior of the integers $|Y(\mathbf{F}_q)|$, which are informed by the following heuristic:

Idea 2.10 (Weil)**.** There exists a good cohomology theory for algebraic varieties having the property that the fixed point set $Y(\mathbf{F}_q) = \{y \in \overline{Y}(\overline{\mathbf{F}}_q) : \varphi(y) = y\}$ is given by the Lefschetz fixed point formula

$$Y(\mathbf{F}_q) = \sum_{i \geqslant 0} (-1)^i \operatorname{Tr}(\varphi | \operatorname{H}^i_c(\overline{Y})).$$

Weil's conjectures were eventually proven by Grothendieck and Deligne by introducing the theory of *ℓ-adic cohomology*, which we will denote by $\overline{Y} \mapsto \mathrm{H}^*(\overline{Y}; \mathbf{Q}_\ell)$ (there is also an analogue of ℓ-adic cohomology with compact supports which appears in the fixed point formula above, which we will denote by $\mathrm{H}^*_c(\overline{Y}; \mathbf{Q}_\ell)$). Here ℓ denotes some fixed prime number which does not divide q.

For our purposes, it will be convenient to write the Grothendieck-Lefschetz trace formula in a slightly different form. Suppose that Y is a smooth variety of dimension n

over $\mathbf{F}_q$. Then, from the perspective of ℓ-adic cohomology, Y behaves as if it were a smooth manifold of dimension $2n$. In particular, it satisfies Poincare duality: that is, there is a perfect pairing

$$\mu : \mathrm{H}^i_c(\overline{Y}; \mathbf{Q}_\ell) \otimes_{\mathbf{Q}_\ell} \mathrm{H}^{2n-i}(\overline{Y}; \mathbf{Q}_\ell) \to \mathbf{Q}_\ell .$$

This pairing is not quite φ-equivariant: instead, it fits into a commutative diagram

$$\begin{array}{ccc} \mathrm{H}^i_c(\overline{Y}; \mathbf{Q}_\ell) \otimes_{\mathbf{Q}_\ell} \mathrm{H}^{2n-i}(\overline{Y}; \mathbf{Q}_\ell) & \xrightarrow{\mu} & \mathbf{Q}_\ell \\ \downarrow{\scriptstyle \varphi\otimes\varphi} & & \downarrow{\scriptstyle q^n} \\ \mathrm{H}^i_c(\overline{Y}; \mathbf{Q}_\ell) \otimes_{\mathbf{Q}_\ell} \mathrm{H}^{2n-i}(\overline{Y}; \mathbf{Q}_\ell) & \xrightarrow{\mu} & \mathbf{Q}_\ell . \end{array}$$

It follows formally that

$$q^{-n} \operatorname{Tr}(\varphi | \mathrm{H}^i_c(\overline{Y}; \mathbf{Q}_\ell)) \simeq \operatorname{Tr}(\varphi^{-1} | \mathrm{H}^{2n-i}(\overline{Y}; \mathbf{Q}_\ell)).$$

We may therefore rewrite the Grothendieck-Lefschetz trace formula as an equality

$$\sum_{i \geqslant 0} (-1)^i \operatorname{Tr}(\varphi^{-1} | \mathrm{H}^i(\overline{Y}; \mathbf{Q}_\ell)) = \frac{|Y(\mathbf{F}_q)|}{q^{\dim(Y)}}.$$

This suggests the possibility of breaking Conjecture 2.5 into two parts.

Notation 2.11. In what follows, we fix an embedding $\mathbf{Q}_\ell \hookrightarrow \mathbf{C}$, so that we can regard the trace of any endomorphism of a $\mathbf{Q}_\ell$-vector space as a complex number. If Y is an algebraic stack over a finite field $\mathbf{F}_q$, we write $\overline{Y}$ for the fiber product $\operatorname{Spec}(\overline{\mathbf{F}}_q) \times_{\operatorname{Spec}(\mathbf{F}_q)} Y$, and we write $\operatorname{Tr}(\varphi^{-1} | \mathrm{H}^*(\overline{Y}; \mathbf{Q}_\ell))$ for the alternating sum

$$\sum_{i \geqslant 0} (-1)^i \operatorname{Tr}(\varphi^{-1} | \mathrm{H}^i(\overline{Y}; \mathbf{Q}_\ell).$$

Conjecture 2.12. The algebraic stacks $\mathrm{Bun}_G(X)$ and BG_x satisfy the Grothendieck-Lefschetz trace formulae

$$\frac{|\mathrm{Bun}_G(X)(\mathbf{F}_q)|}{q^{\dim(\mathrm{Bun}_G(X))}} = \operatorname{Tr}(\varphi^{-1} | \mathrm{H}^*(\overline{\mathrm{Bun}}_G(X); \mathbf{Q}_\ell)).$$

$$\frac{|\mathrm{BG}_x(\kappa(x))|}{|\kappa(x)|^{\dim \mathrm{BG}_x}} = \operatorname{Tr}(\varphi_x^{-1} | \mathrm{H}^*(\overline{\mathrm{BG}}_x; \mathbf{Q}_\ell)).$$

Conjecture 2.13. There is an equality

$$\operatorname{Tr}(\varphi^{-1} | \mathrm{H}^*(\overline{\mathrm{Bun}}_G(X); \mathbf{Q}_\ell)) = \prod_{x \in X} \operatorname{Tr}(\varphi_x^{-1} | \mathrm{H}^*(\overline{\mathrm{BG}}_x; \mathbf{Q}_\ell)).$$

Warning 2.14. The ℓ-adic cohomology rings $\mathrm{H}^*(\overline{\mathrm{Bun}}_G(X); \mathbf{Q}_\ell)$ and $\mathrm{H}^*(\mathrm{BG}_x; \mathbf{Q}_\ell)$ are typically nonzero in infinitely many degrees, so the sum

$$\sum_{i \geqslant 0} (-1)^i \operatorname{Tr}(\varphi^{-1} | \mathrm{H}^i(\overline{\mathrm{Bun}}_G(X); \mathbf{Q}_\ell))$$

appearing in Conjectures 2.12 and 2.13 is generally infinite. Consequently, the convergence of this sum is not automatic, but part of the substance of the conjecture.

The proofs of Conjecture 2.12 and 2.13 involve completely different ideas. In the case where G has good reduction (that is, every fiber G_x is a semisimple algebraic group over $\kappa(x)$), the Grothendieck-Lefschetz trace formula for $\mathrm{Bun}_G(X)$ was proved by Behrend in [4]. This is extended to the case of bad reduction in [7] (by a somewhat ad-hoc argument). The Grothendieck-Lefschetz trace formula for BG_x is essentially a formula for the order of the finite group $G(\kappa(x))$ which we will discuss in the next lecture. Along the way, we will develop some ideas which can be applied to Conjecture 2.13 as well.

3 Third Lecture: Steinberg's Formula

Let Y be an algebraic variety over the finite field $\mathbf{F}_q$, and set $\overline{Y} = \operatorname{Spec}(\overline{\mathbf{F}}_q) \times_{\operatorname{Spec}(\mathbf{F}_q)} Y$. In the previous lecture, we discussed the *Grothendieck-Lefschetz fixed point formula*, which asserts that the number of $\mathbf{F}_q$-valued points of Y is given by the trace

$$\operatorname{Tr}(\varphi | \mathrm{H}^*_c(\overline{Y}; \mathbf{Q}_\ell)) := \sum_{n \geqslant 0} (-1)^n \operatorname{Tr}(\varphi | \mathrm{H}^n_c(\overline{Y}; \mathbf{Q}_\ell)).$$

In the case where Y is smooth, this formula can be rewritten as

$$\frac{|Y(\mathbf{F}_q)|}{q^{\dim(Y)}} = \operatorname{Tr}(\varphi^{-1} | \mathrm{H}^*(\overline{Y}; \mathbf{Q}_\ell)).$$

In this lecture, we will discuss a variation on this theme, which relates the number of $\mathbf{F}_q$-valued points of Y to the (ℓ-adic) *homotopy groups* of $\overline{Y}$, rather than to its cohomology.

We begin with some general remarks. Let $\overline{Y}$ be any scheme, and let y denote a *geometric point* of $\overline{Y}$: that is, a map $y : \operatorname{Spec}(k) \to \overline{Y}$, where k is a separably closed field. To the pair $(\overline{Y}, y)$, Grothendieck associated a profinite group $\pi_1(\overline{Y}, y)$, called the *étale fundamental group of* $\overline{Y}$ (with base point y). Assume (for simplicity) that $\overline{Y}$ is connected. Then the fundamental group $\pi_1(\overline{Y}, y)$ is essentially characterized by the requirement that there is an equivalence of categories

$$\begin{array}{c}\{\text{Finite sets with continuous action of } \pi_1(\overline{Y}, y)\ \} \\ \Big\downarrow \sim \\ \{\text{Finite étale covers of } \overline{Y}\}.\end{array}$$

In what follows, it will be convenient to focus our attention on a particular prime number ℓ; we let $\pi_1(\overline{Y}, y)_\ell$ denote the maximal pro-ℓ quotient of $\pi_1(\overline{Y}, y)$ (that is, the quotient $\pi_1(\overline{Y}, y)/N$, where N is the smallest closed normal subgroup of $\pi_1(\overline{Y}, y)$ for which $\pi_1(\overline{Y}, y)/N$ is a pro-ℓ group).

In [2], Artin and Mazur introduced a refinement of the étale fundamental group. To any scheme $\overline{Y}$, they associate an invariant called the *étale homotopy type* of $\overline{Y}$. To simplify the discussion, let us fix a prime number ℓ and assume that $\overline{Y}$ satisfies the following conditions:

($*$) The scheme $\overline{Y}$ is connected, the étale cohomology groups $\mathrm{H}^n(\overline{Y}; \mathbf{Z}/\ell\mathbf{Z})$ are finite for every integer n, and $\mathrm{H}^1(\overline{Y}; \mathbf{Z}/\ell\mathbf{Z})$ vanishes (this last condition is equivalent to the vanishing of the group $\pi_1(\overline{Y}, y)_\ell$, for any choice of geometric point y; that is, it asserts that $\overline{Y}$ is "ℓ-adically simply connected").

In this case, the ℓ-adic version of the Artin-Mazur construction produces a topological space T (or, if you prefer, a simplicial set) with the following properties:

(a) The space T is simply connected, and the homotopy groups of T are finitely generated $\mathbf{Z}_\ell$-modules.

(b) There is a canonical isomorphism of cohomology rings

$$\mathrm{H}^*(T; \mathbf{Z}/\ell\mathbf{Z}) \simeq \mathrm{H}^*(\overline{Y}; \mathbf{Z}/\ell\mathbf{Z}),$$

where the left hand side is denotes the singular cohomology of the topological space T, and the right hand side indicates the étale cohomology of the scheme $\overline{Y}$ (and similarly for coefficients in $\mathbf{Z}/\ell^m\mathbf{Z}$, $\mathbf{Z}_\ell$, or $\mathbf{Q}_\ell$).

Moreover, the space T is essentially characterized (up to weak homotopy equivalence) by (a) together with a stronger form of condition (b). We refer to T as the *ℓ-adic homotopy type* of $\overline{Y}$. For every integer n, we let $\pi_n(\overline{Y})_\ell$ denote the homotopy group $\pi_n(T)$ (note that since T is simply connected, the group $\pi_n(T)$ is canonically independent of the choice of base point). By virtue of (a), this is a free $\mathbf{Z}_\ell$-module of finite rank. We let $\pi_n(\overline{Y})_{\mathbf{Q}_\ell}$ denote the $\mathbf{Q}_\ell$-vector space given by $\pi_n(\overline{Y})_\ell[\ell^{-1}]$.

We will refer to the vector spaces $\pi_n(\overline{Y})_{\mathbf{Q}_\ell}$ as the *rational ℓ-adic homotopy groups of $\overline{Y}$*. They are related to the ℓ-adic cohomology groups of $\overline{Y}$. Recall that, for any topological space T (and any choice of base point $t \in T$), we have a canonical homomorphism

$$\pi_n(T, t) \to \mathrm{H}_n(T; \mathbf{Z}_\ell)$$

called the *Hurewicz map*. Combining this with the canonical pairing $\mathrm{H}^n(T; \mathbf{Z}_\ell) \times \mathrm{H}_n(T; \mathbf{Z}_\ell) \to \mathbf{Z}_\ell$ we obtain a bilinear map

$$\mathrm{H}^n(T; \mathbf{Z}_\ell) \times \pi_n(T, t) \to \mathbf{Z}_\ell.$$

Specializing to the case where T is the ℓ-adic homotopy type of $\overline{Y}$ and inverting the prime number ℓ, we get a bilinear map of $\mathbf{Q}_\ell$-vector spaces

$$b_n : \mathrm{H}^n(\overline{Y}; \mathbf{Q}_\ell) \times \pi_n(\overline{Y})_{\mathbf{Q}_\ell} \to \mathbf{Q}_\ell\,.$$

Summing over all $n > 0$, we obtain a pairing of graded $\mathbf{Q}_\ell$-vector spaces

$$b : \mathrm{H}^*_{\mathrm{red}}(\overline{Y}; \mathbf{Q}_\ell) \times \pi_*(\overline{Y})_{\mathbf{Q}_\ell} \to \mathbf{Q}_\ell\,.$$

Here $\mathrm{H}^*_{\mathrm{red}}(\overline{Y}; \mathbf{Q}_\ell) = \bigoplus_{n>0} \mathrm{H}^n(\overline{Y}; \mathbf{Q}_\ell)$ denotes the *reduced* ℓ-adic cohomology of $\overline{Y}$. Let us denote $\mathrm{H}^*_{\mathrm{red}}(\overline{Y}; \mathbf{Q}_\ell)$ by I, and view it as an ideal in the ring $\mathrm{H}^*(\overline{Y}; \mathbf{Q}_\ell)$. Then the square I^2 is contained in the kernel of the bilinear pairing b. In other words, for $m, n > 0$, every triple of elements $x \in \mathrm{H}^m(\overline{Y}; \mathbf{Q}_\ell)$, $y \in \mathrm{H}^n(\overline{Y}; \mathbf{Q}_\ell)$, $z \in \pi_{m+n}(\overline{Y})_{\mathbf{Q}_\ell}$, we have

$$b(x \cup y, z) = 0 \in \mathbf{Q}_\ell\,.$$

Passing back to topology, it is easy to see why this should be true: we can assume without loss of generality that z comes from a homotopy class of continuous maps $f : S^{m+n} \to T$. In this case, $b(x \cup y, z)$ is obtained by viewing $x \cup y$ a cohomology class on T, pulling it back to the sphere S^{m+n}, and "integrating" (by evaluating on a generator of the group $\mathrm{H}_{m+n}(S^{m+n}; \mathbf{Z}) \simeq \mathbf{Z}$). However, the pullback $f^*(x \cup y) = f^*(x) \cup f^*(y)$ is zero, since the sphere S^{m+n} has vanishing cohomology in degrees m and n. It follows that b restricts to a bilinear map

$$I/I^2 \times \pi_*(\overline{Y})_{\mathbf{Q}_\ell} \to \mathbf{Q}_\ell\,.$$

This pairing is a bit more useful, by virtue of the following:

Proposition 3.1. *Let $\overline{Y}$ satisfy condition $(*)$, suppose that the cohomology ring $\mathrm{H}^*(\overline{Y}; \mathbf{Q}_\ell)$ is a polynomial ring (on generators of even degree), and let $I = \mathrm{H}^*(\overline{Y}; \mathbf{Q}_\ell)$ denote the augmentation ideal. Then the preceding construction gives a* perfect *pairing*

$$I/I^2 \times \pi_*(\overline{Y})_{\mathbf{Q}_\ell} \to \mathbf{Q}_\ell\,.$$

In other words, it identifies $\pi_*(\overline{Y})_{\mathbf{Q}_\ell}$ with the dual of I/I^2 (as a graded $\mathbf{Q}_\ell$-vector space).

At a first glance, Proposition 3.1 would appear to be useless: its hypotheses are essentially never satisfied. For example, if $\overline{Y}$ is an algebraic variety over an algebraically closed field k, then the ℓ-adic cohomology groups groups $\mathrm{H}^n(\overline{Y};\mathbf{Q}_\ell)$ vanish for $n > 2\dim(\overline{Y})$; in particular, $\mathrm{H}^*(\overline{Y};\mathbf{Q}_\ell)$ can never be a polynomial ring (except in the trivial case $\mathrm{H}^*(\overline{Y};\mathbf{Q}_\ell) \simeq \mathbf{Q}_\ell$). However, the entirety of the preceding discussion can be extended to algebro-geometric objects other than schemes: for example, we can apply it when $\overline{Y}$ is an algebraic stack. In this case, we can find many interesting examples Proposition 3.1 applies.

Example 3.2. Let G be a connected linear algebraic group over a field k, let BG be the classifying stack of G, and set $\overline{\mathrm{BG}} = \mathrm{Spec}(\overline{k}) \times_{\mathrm{Spec}(k)} \mathrm{BG}$ where $\overline{k}$ is an algebraic closure of k. If ℓ is invertible in k, then the polynomial ring $\mathrm{H}^*(\overline{\mathrm{BG}};\mathbf{Q}_\ell)$ is a polynomial ring in r generators, where r is the reductive rank of the group G. It follows that Proposition 3.1 supplies a duality of $\pi_*(\overline{\mathrm{BG}})_{\mathbf{Q}_\ell}$ with I/I^2.

To illustrate this more concretely, let us specialize to the case where $G = \mathrm{SL}_2$. Then G can be viewed as an algebraic group over *any* field k (or even as a *group scheme* over the integers). Let us write G_k for the group SL_2, viewed as an algebraic group over k, and let BG_k denote its classifying stack. One can show that if k is algebraically closed, then the ℓ-adic cohomology $\mathrm{H}^*(\mathrm{BG}_k;\mathbf{Q}_\ell)$ does not depend on k, as long as ℓ is invertible in k (in fact, this is even true for the ℓ-adic homotopy type of BG_k). Consequently, there is no loss of generality in assuming that $k = \mathbf{C}$ is the field of complex numbers. In this case, one can show that the cohomology ring of the *classifying stack* BG_k (in the sense of algebraic geometry) is the same as the cohomology of the *classifying space* of the topological group $G(\mathbf{C}) = \mathrm{SL}_2(\mathbf{C})$ of complex points of $\mathbf{C}$ (in the sense of topology). Here, there is no harm in replacing $\mathrm{SL}_2(\mathbf{C})$ with its maximal compact subgroup $\mathrm{SU}(2)$, which we will identify with the group of *unit quaternions* $\{x \in \mathbf{H} : |x| = 1\}$. The classifying space $\mathrm{BSU}(2)$ can then be defined as the quotient $E/\mathrm{SU}(2)$, where E is a contractible space equipped with a free action of $\mathrm{SU}(2)$. For each $n \geqslant 0$, let $\mathbf{HP}^n$ denote projective space of dimension n over $\mathbf{H}$, given concretely by the quotient

$$(\mathbf{H}^{n+1} - \{0\})/\mathbf{H}^\times \simeq \{(x_0, \ldots, x_n) \in \mathbf{H}^{n+1} : |x_0|^2 + \cdots + |x_n|^2 = 1\}/\mathrm{SU}(2).$$

The description on the right exhibits the sphere S^{2n+3} as a principal $\mathrm{SU}(2)$-bundle over $\mathbf{HP}^n$.

Identifying each $\mathbf{H}^n$ with the subspace $\{(x_0, \ldots, x_n) \in \mathbf{H}^{n+1} : x_n = 0\} \subseteq \mathbf{H}^{n+1}$, we obtain a compatible system of principal $\mathrm{SU}(2)$-bundles

$$\begin{array}{ccccccc} S^3 & \longrightarrow & S^7 & \longrightarrow & S^{11} & \longrightarrow & \cdots \\ \downarrow & & \downarrow & & \downarrow & & \\ \mathbf{HP}^0 & \longrightarrow & \mathbf{HP}^1 & \longrightarrow & \mathbf{HP}^2 & \longrightarrow & \cdots \end{array}$$

Setting $\mathbf{HP}^\infty = \varinjlim \mathbf{HP}^n$, we obtain a principal SU(2)-bundle $E \to \mathbf{HP}^\infty$, where $E \simeq \varinjlim S^{4n+3}$ is contractible. It follows that $\mathbf{HP}^\infty$ is a classifying space for the compact Lie group SU(2), so we have isomorphisms

$$\mathrm{H}^*(\mathrm{BG}_k; \mathbf{Q}_\ell) \simeq \mathrm{H}^*(\mathbf{HP}^\infty; \mathbf{Q}_\ell)$$

The group on the right is easy to compute: it is a polynomial ring $\mathbf{Q}_\ell[t]$, where t is a class in degree 4. One can think of t as playing the role of the *second Chern class*. If X is any k-scheme equipped with a principal SL_2-bundle $\mathscr{E}$ (which we can view as a vector bundle of rank 2 with a trivialization of its determinant), then $\mathscr{E}$ determines a map of algebraic stacks $X \to \mathrm{BG}_k$, hence a map of cohomology rings

$$\mathbf{Q}_\ell[t] \simeq \mathrm{H}^*(\mathrm{BG}_k; \mathbf{Q}_\ell) \to \mathrm{H}^*(X; \mathbf{Q}_\ell),$$

which (for appropriate normalizations) carries t to the Chern class $c_2(\mathscr{E}) \in \mathrm{H}^4(X; \mathbf{Q}_\ell)$.

Remark 3.3. In Example 3.2, we are being a bit sloppy: the second Chern class of the vector bundle $\mathscr{E}$ should really be an element of the cohomology group $\mathrm{H}^4(X; \mathbf{Q}_\ell(2))$, where $\mathbf{Q}_\ell(2)$ denotes the second Tate twist of $\mathbf{Q}_\ell$. In particular, if we regard $G = \mathrm{SL}_2$ as an algebraic group over the finite field $\mathbf{F}_q$, then the geometric Frobenius φ acts on the ℓ-adic cohomology ring $\mathrm{H}^*(\overline{\mathrm{BG}}; \mathbf{Q}_\ell) \simeq \mathbf{Q}_\ell[t]$ by the formula $\varphi(t) = q^2 t$. We therefore have

$$\begin{aligned}
\mathrm{Tr}(\varphi^{-1} | \, \mathrm{H}^*(\overline{\mathrm{BG}}; \mathbf{Q}_\ell)) &= \sum_{n \geqslant 0} \mathrm{Tr}(\varphi^{-1} | \, \mathrm{H}^{4n}(\overline{\mathrm{BG}}; \mathbf{Q}_\ell)) \\
&= \sum_{n \geqslant 0} q^{-2n} \\
&= \frac{1}{1 - q^{-2n}}.
\end{aligned}$$

Consequently, the Grothendieck-Lefschetz trace formula for BG asserts that we have an identity

$$\frac{1}{1 - q^{-2n}} = \frac{q^3}{|\, \mathrm{SL}_2(\mathbf{F}_q)|},$$

or equivalently $|\, \mathrm{SL}_2(\mathbf{F}_q)| = q^3 - q$ (which is easy to verify directly).

Let us now generalize the preceding discussion. Let G be a connected linear algebraic group over a finite field $\mathbf{F}_q$, set $\overline{\mathrm{BG}} = \mathrm{Spec}(\overline{\mathbf{F}}_q) \times_{\mathrm{Spec}(\mathbf{F}_q)} \mathrm{BG}$, and let $I = \mathrm{H}^*_{\mathrm{red}}(\overline{\mathrm{BG}}; \mathbf{Q}_\ell)$ so that Example 3.2 supplies a duality of graded $\mathbf{Q}_\ell$-vector spaces between I/I^2 and $\pi_*(\overline{\mathrm{BG}})_{\mathbf{Q}_\ell}$. The *Grothendieck-Lefschetz trace formula for* BG asserts that we have an equality

$$\mathrm{Tr}(\varphi^{-1} | \, \mathrm{H}^*(\overline{\mathrm{BG}}; \mathbf{Q}_\ell)) := \sum_{i \geqslant 0} (-1)^i \, \mathrm{Tr}(\varphi^{-1} | \, \mathrm{H}^i(\overline{\mathrm{BG}}; \mathbf{Q}_\ell)) = \frac{|\, \mathrm{BG}(\mathbf{F}_q)|}{q^{\dim(\mathrm{BG})}},$$

where $|\mathrm{BG}(\mathbf{F}_q)|$ denotes the sum $\sum_{\mathcal{P}} \frac{1}{|\mathrm{Aut}(\mathcal{P})|}$, taken over all isomorphism classes of principal G-bundles $\mathcal{P}$ on $\mathrm{Spec}(\mathbf{F}_q)$. If G is connected, then every principal G-bundle on $\mathrm{Spec}(\mathbf{F}_q)$ is trivial (by a theorem of Lang [11]), so this sum has only one term: the fraction $\frac{1}{|G(\mathbf{F}_q)|}$. We can therefore rewrite this equality as

$$\mathrm{Tr}(\varphi^{-1}| \mathrm{H}^*(\overline{\mathrm{BG}}; \mathbf{Q}_\ell)) = \frac{q^{\dim(G)}}{|G(\mathbf{F}_q)|}.$$

Here we have fixed an embedding $\mathbf{Q}_\ell \hookrightarrow \mathbf{C}$. Let $\lambda_1, \ldots, \lambda_r$ be the complex numbers given by the generalized eigenvalues of φ on $\pi_*(\mathrm{BG}_{\overline{\mathbf{F}}_q})_{\mathbf{Q}_\ell}$ (listed with multiplicity). These are then the same as the generalized eigenvalues of φ^{-1}, acting on the quotient I/I^2. Since $\mathrm{H}^*(\overline{\mathrm{BG}}; \mathbf{Q}_\ell)$ is a polynomial ring, the generalized eigenvalues of φ^{-1} on $\mathrm{H}^*(\overline{\mathrm{BG}}; \mathbf{Q}_\ell)$ are the same as the generalized eigenvalues of φ on the symmetric algebra $\mathrm{Sym}^*(I/I^2)$: that is, they are given by $\{\lambda_1^{e_1} \cdots \lambda_r^{e_r}\}_{e_1, \cdots, e_r \geqslant 0}$. We can therefore compute

$$\begin{aligned} \mathrm{Tr}(\varphi^{-1}| \mathrm{H}^*(\overline{\mathrm{BG}}; \mathbf{Q}_\ell) &= \sum_{e_1, \ldots, e_r \geqslant 0} \lambda_1^{e_1} \cdots \lambda_r^{e_r} \\ &= \prod_{1 \leqslant i \leqslant r} \frac{1}{1 - \lambda_i} \\ &= \det(1 - \varphi | \pi_*(\overline{\mathrm{BG}})_{\mathbf{Q}_\ell})^{-1}; \end{aligned}$$

here the calculation depends on the fact that each of the eigenvalues λ_i has complex absolute value < 1 (otherwise, the trace on the left hand side would not be well-defined). We can therefore restate the Grothendieck-Lefschetz trace formula for BG as follows:

Proposition 3.4 (Steinberg's Formula)**.** *Let G be a connected linear algebraic group over $\mathbf{F}_q$. Then the order of the finite group $G(\mathbf{F}_q)$ is given by*

$$q^{\dim(G)} \det(1 - \varphi | \pi_*(\overline{\mathrm{BG}})_{\mathbf{Q}_\ell}).$$

Remark 3.5. In the statement of Proposition 3.4, we do not assume that G is reductive. However, there is no real loss in restricting to the reductive case: a general connected linear algebraic group G fits into a short exact sequence

$$0 \to U \to G \to G/U \to 0$$

where U is unipotent and G/U is reductive. We then have a short exact sequence of finite groups

$$0 \to U(\mathbf{F}_q) \to G(\mathbf{F}_q) \to (G/U)(\mathbf{F}_q) \to 0$$

(this follows from Lang's theorem), hence an equality

$$|G(\mathbf{F}_q)| = |(G/U)(\mathbf{F}_q)| \cdot |U(\mathbf{F}_q)| = |(G/U)(\mathbf{F}_q)| \cdot q^{\dim(U)}.$$

Moreover, replacing G by G/U does not change the ℓ-adic homotopy groups $\pi_*(\mathrm{BG}_{\overline{\mathbf{F}}_q})_{\mathbf{Q}_\ell}$, so that the validity of Proposition 3.4 for the group G follows from its validity for the reductive quotient G/U.

Let us now suppose that $\overline{Y}$ is an algebraic stack satisfying condition $(*)$ above, but whose ℓ-adic cohomology $\mathrm{H}^*(\overline{Y};\mathbf{Q}_\ell)$ is not necessarily a polynomial ring. In this case, the relationship between the homotopy and cohomology of $\overline{Y}$ is more subtle. In place of the isomorphism $\mathrm{Sym}^*(\pi_*(\overline{Y})^\vee_{\mathbf{Q}_\ell}) \simeq \mathrm{gr}(\mathrm{H}^*(\overline{Y};\mathbf{Q}_\ell))$, one has a *spectral sequence*

$$\mathrm{Sym}^*(\pi_*(\overline{Y})^\vee_{\mathbf{Q}_\ell}) \Rightarrow \mathrm{H}^*(\overline{Y};\mathbf{Q}_\ell),$$

whose second page is the (graded) symmetric algebra on the dual of $\pi_*(\overline{Y})_{\mathbf{Q}_\ell}$ and whose last page is the associated graded for a suitable filtration on $\mathrm{H}^*(\overline{Y};\mathbf{Q}_\ell)$ (not necessarily the I-adic filtration for $I = \mathrm{H}^*_{\mathrm{red}}(Y;\mathbf{Q}_\ell)$). The existence of this spectral sequence is good enough to draw many of the conclusions mentioned above. Suppose that $\overline{Y} = \mathrm{Spec}(\overline{\mathbf{F}}_q) \times_{\mathrm{Spec}(\mathbf{F}_q)} Y$ for some algebraic stack Y over a finite field $\mathbf{F}_q$, so that the cohomology ring $\mathrm{H}^*(\overline{Y};\mathbf{Q}_\ell)$ and the $\mathbf{Q}_\ell$-homotopy groups $\pi_*(\overline{Y})_{\mathbf{Q}_\ell}$ are equipped with a geometric Frobenius automorphism φ. Assume the following:

$(*')$ The homotopy groups $\pi_n(\overline{Y})_{\mathbf{Q}_\ell}$ vanish for $n \gg 0$, and every eigenvalue of φ on $\pi_*(\overline{Y})_{\mathbf{Q}_\ell}$ has complex absolute value < 1.

One can then calculate

$$\begin{aligned} \mathrm{Tr}(\varphi^{-1}|\,\mathrm{H}^*(\overline{Y};\mathbf{Q}_\ell)) &= \mathrm{Tr}(\varphi^{-1}|\,\mathrm{gr}(\mathrm{H}^*(\overline{Y};\mathbf{Q}_\ell))) \\ &= \mathrm{Tr}(\varphi^{-1}|E_\infty^{*,*}) \\ &= \mathrm{Tr}(\varphi^{-1}|E_2^{*,*}) \\ &= \mathrm{Tr}(\varphi^{-1}|\,\mathrm{Sym}^*(\pi_*(\overline{Y})^\vee_{\mathbf{Q}_\ell}) \\ &= \det(1-\varphi|\pi_*(\overline{Y})_{\mathbf{Q}_\ell})^{-1} \end{aligned} \tag{3}$$

where the last term is shorthand for the alternating product

$$\prod_{n \geqslant 0} \det(1-\varphi|\pi_n(\overline{Y})_{\mathbf{Q}_\ell})^{(-1)^{n+1}}$$

Consequently, if $\overline{Y}$ is a smooth algebraic stack, then the Grothendieck-Lefschetz trace formula for $\overline{Y}$ (in the sense of the previous lecture) can be written as an equality

$$\frac{|Y(\mathbf{F}_q)|}{q^{\dim(Y)}} = \det(1-\varphi|\pi_*(\overline{Y})_{\mathbf{Q}_\ell})^{-1}.$$

Warning 3.6. If $\overline{Y}$ is an ordinary algebraic variety over $\overline{\mathbf{F}}_q$, the assumption that $\pi_n(\overline{Y})_{\mathbf{Q}_\ell}$ vanishes for $n \gg 0$ is usually not satisfied: this is an algebro-geometric analogue of requiring that $\overline{Y}$ is "elliptic" in the sense of rational homotopy theory.

Example 3.7. Let $\mathbf{CP}^n$ be complex projective space of dimension n, regarded as a topological space. Then there are two interesting elements of the homotopy of $\mathbf{CP}^n$:

- A class $x \in \pi_2(\mathbf{CP}^n)$, represented by the inclusion $S^2 \simeq \mathbf{CP}^1 \hookrightarrow \mathbf{CP}^n$.
- A class $y \in \pi_{2n+1}(\mathbf{CP}^n)$, represented by the Hopf map $S^{2n+1} \to \mathbf{CP}^n$ (whose cone is the projective space $\mathbf{CP}^{n+1}$).

One can show that x and y form a basis for the $\mathbf{Q}$-vector space $\pi_*(\mathbf{CP}^n)_{\mathbf{Q}}$.

The situation in algebraic geometry is completely analogous. If k is any algebraically closed field (and ℓ is a prime number which does not vanish in k), then the $\mathbf{Q}_\ell$-homotopy of the algebraic variety $\mathbf{P}^n_k$ is a vector space of dimension 2, generated by classes

$$x \in \pi_2(\mathbf{P}^n_k)_{\mathbf{Q}_\ell} \qquad y \in \pi_{2n+1}(\mathbf{P}^n_k)_{\mathbf{Q}_\ell}.$$

Moreover, when $k = \overline{\mathbf{F}_q}$ is the algebraic closure of a finite field $\mathbf{F}_q$, then the geometric Frobenius acts on $\pi_*(\mathbf{P}^n_k)_{\mathbf{Q}_\ell}$ by the formula

$$\varphi(x) = \frac{x}{q} \qquad \varphi(y) = \frac{y}{q^{n+1}}.$$

Consequently, the Grothendieck-Lefschetz trace formula can be written as

$$\begin{aligned}
\frac{|\mathbf{P}^n(\mathbf{F}_q)|}{q^n} &= \det(1 - \varphi | \pi_*(\mathbf{P}^n_k)_{\mathbf{Q}_\ell})^{-1} \\
&= \frac{\det(1 - \varphi | \pi_{2n+1}(\mathbf{P}^n_k)_{\mathbf{Q}_\ell})}{\det(1 - \varphi | \pi_2(\mathbf{P}^n_k))} \\
&= \frac{1 - q^{-n-1}}{1 - q^{-1}} \\
&= \frac{1}{q^n} \cdot \frac{q^{n+1} - 1}{q - 1}.
\end{aligned}$$

Multiplying both sides by q^n, we obtain the equality $|\mathbf{P}^n(\mathbf{F}_q)| = \frac{q^{n+1}-1}{q-1} = 1 + q + \cdots + q^n$.

4 Fourth Lecture: The Product Formula

Let X be an algebraic variety defined over a finite field $\mathbf{F}_q$. Recall that the *zeta function of* X is a meromorphic function $\zeta_X(s)$ on the complex plane, which for $\Re(s) \gg 0$ is given by the formula

$$\zeta_X(s) = \prod_{x \in X} \frac{1}{1 - |\kappa(x)|^{-s}},$$

where the product is taken over all closed points $x \in X$ (so that the residue field $\kappa(x)$ is a finite extension of $\mathbf{F}_q$). Using the Grothendieck-Lefschetz fixed point formula, one can give an alternate formula for $\zeta_X(s)$ in terms of the action of the geometric Frobenius automorphism on the ℓ-adic cohomology of $\overline{X} = X \times_{\mathrm{Spec}(\mathbf{F}_q)} \mathrm{Spec}(\overline{\mathbf{F}}_q)$: it is given by

$$\zeta_X(s) = \det(1 - q^{-s}\varphi | \mathrm{H}^*_c(\overline{X}; \mathbf{Q}_\ell))^{-1} := \prod_{n \geqslant 0} \det(1 - q^{-s}\varphi | \mathrm{H}^n_c(\overline{X}; \mathbf{Q}_\ell))^{(-1)^{n+1}}.$$

The zeta function $\zeta_X(s)$ has a generalization. Let $\mathscr{F}$ be an ℓ-adic sheaf on X. For each closed point $x \in X$, we let $\mathscr{F}_{\overline{x}}$ denote the stalk of $\mathscr{F}$ at the geometric point $\overline{x} : \mathrm{Spec}(\overline{\kappa(x)}) \to X$ determined by a choice of algebraic closure $\overline{\kappa(x)}$ of the residue field $\kappa(x)$. Then $\mathscr{F}_{\overline{x}}$ is a finite-dimensional $\mathbf{Q}_\ell$-vector space equipped with a local Frobenius operator φ_x. The L-function $L_{X,\mathscr{F}}(s)$ is then a meromorphic function of a complex variable s, whose value for $\Re(s) \gg 0$ is given by the formula

$$L_{X,\mathscr{F}}(s) = \prod_{x \in X} \det(1 - |\kappa(x)|^{-s}\varphi_x | \mathscr{F}_{\overline{x}})^{-1}.$$

By virtue of the Grothendieck-Lefschetz trace formula, this can also be written as

$$L_{X,\mathscr{F}}(s) = \prod_{n \geqslant 0} \det(1 - q^{-s}\varphi | \mathrm{H}^n_c(\overline{X}; \mathscr{F}))^{(-1)^{n+1}}.$$

We will need a slightly more general version of this construction. Let $D^b(X)$ denote the *bounded derived category* of ℓ-adic sheaves on X. Then every object $\mathscr{F} \in D^b(X)$ has *cohomology sheaves* $\mathcal{H}^i(\mathscr{F})$, which are ℓ-adic sheaves on X in the usual sense. Moreover, these sheaves vanish for almost every integer i (since we are working with the *bounded* derived category). We can then define a meromorphic function $L_{X,\mathscr{F}}(s)$ by the formula $L_{X,\mathscr{F}}(s) = \prod_{i \in \mathbf{Z}} L_{X,\mathcal{H}^i(\mathscr{F})}(s)^{(-1)^i}$. This function has an Euler product expansion (convergent for $\Re(s) \gg 0$)

$$L_{X,\mathscr{F}}(s) = \prod_{x \in X} \det(1 - |\kappa(x)|^{-s}\varphi_x | \mathscr{F}_{\overline{x}})^{-1},$$

where $\det(A|\mathscr{F}_{\overline{x}})$ is shorthand for the product $\prod_{i \in \mathbf{Z}} \det(A|\mathcal{H}^i(\mathscr{F})_{\overline{x}})^{(-1)^i}$. We also have a cohomological formula

$$L_{X,\mathscr{F}}(s) = \prod_{n \in \mathbf{Z}} \det(1 - q^{-s}\varphi | \mathrm{H}^n_c(\overline{X}; \mathscr{F}))^{(-1)^{n+1}},$$

where $\mathrm{H}^n_c(\overline{X}; \mathscr{F})$ now denotes the (compactly supported) *hypercohomology* of $\overline{X} = X \times_{\mathrm{Spec}(\mathbf{F}_q)} \mathrm{Spec}(\overline{\mathbf{F}}_q)$ with coefficients in $\mathscr{F}$.

We now describe the context in which we would like to apply these ideas.

Claim 4.1. *Let X be a scheme and let Y be an algebraic stack equipped with a smooth morphism $Y \to X$ and a choice of section $s : X \to Y$. Under certain hypotheses, one can construct an object $\mathscr{F}_{Y/X} \in D^b(X)$ with the property that, for each geometric point $\overline{x} : \mathrm{Spec}(k) \to X$, we have a canonical isomorphism*

$$\mathcal{H}^{-*}(\mathscr{F}_{Y/X})_x \simeq \pi_*(Y_{\overline{x}})_{\mathbf{Q}_\ell};$$

here $Y_{\overline{x}} = Y \times_X \mathrm{Spec}(k)$ denotes the fiber of $Y \to X$ over the geometric point x, and $\pi_*(Y_{\overline{x}})_{\mathbf{Q}_\ell}$ denotes the $\mathbf{Q}_\ell$-homotopy groups of the stack $Y_{\overline{x}}$ in the previous lecture (note that the section s provides a base point $s(\overline{x})$ on $Y_{\overline{x}}$). Moreover, this construction has the following properties

(i) *If we are given a pullback square*

$$\begin{array}{ccc} Y' & \longrightarrow & Y \\ \downarrow & & \downarrow \\ X' & \xrightarrow{f} & X, \end{array}$$

then $\mathscr{F}_{Y'/X'} \in D^b(X')$ can be identified with the pullback $f^*\mathscr{F}_{Y/X}$.

(ii) *For fixed X, the object $\mathscr{F}_{Y/X} \in D^b(X)$ depends functorially on Y: any map $Y' \to Y$ of smooth algebraic stacks over X (satisfying the relevant hypotheses) determines a map $\mathscr{F}_{Y'/X} \to \mathscr{F}_{Y/X}$ in the derived category $D^b(X)$.*

Remark 4.2. The construction of $\mathscr{F}_{Y/X} \in D^b(X)$ appears implicitly in [7]: it can be defined as the Verdier dual of the cotangent fiber of the relative cohomology sheaf $[Y]_X$ which appears in [7].

The hypotheses needed for Claim 4.1 are relative versions of conditions $(*)$ and $(*')$ from the previous lecture, which will guarantee in particular that the ℓ-adic homotopy groups $\pi_*(Y_{\overline{x}})_{\mathbf{Q}_\ell}$ are well-behaved (and vanish for $* \gg 0$). Under these assumptions, we can consider the L-function $L_{Y/X}(s) = L_{X,\mathscr{F}_{Y/X}}(s)$, given for $\Re(s) \gg 0$ by the formula

$$\begin{aligned} L_{Y/X}(s) &= \prod_{x \in X} \det(1 - |\kappa(x)|^{-s}\varphi_x|\mathscr{F}_{Y/X,x})^{-1} \\ &= \prod_{x \in X}\prod_{n>0} \det(1 - |\kappa(x)|^{-s}\varphi_x|\pi_n(Y_{\overline{x}})_{\mathbf{Q}_\ell})^{(-1)^{n+1}}. \end{aligned}$$

If the eigenvalues of Frobenius on the homotopy groups $\pi_n(Y_x)_{\mathbf{Q}_\ell}$ are sufficiently small, then this product will converge when $s = 0$ and (by virtue of calculation (3) from Lecture 3) yield the identity

$$L_{Y/X}(0) = \prod_{x \in X} \mathrm{Tr}(\varphi_x^{-1}|\,\mathrm{H}^*(Y_{\overline{x}}; \mathbf{Q}_\ell)).$$

If, in addition, we assume that each fiber $Y_{\overline{x}}$ satisfies the Grothendieck-Lefschetz trace formula, we can rewrite this formula as

$$L_{Y/X}(0) = \prod_{x \in X} \frac{|Y_x(\kappa(x))|}{|\kappa(x)|^{\dim(Y_x)}}.$$

Let us now specialize to the situation which is relevant to Weil's conjecture. Let X be an algebraic curve over $\mathbf{F}_q$ (assumed to be smooth, projective, and geometrically connected), and let G be a smooth affine group scheme over X. Let BG_X denote the classifying stack of G. More precisely, BG_X is an algebraic stack whose R-valued points can be identified with pairs $(\eta, \mathcal{P})$, where $\eta : \mathrm{Spec}(R) \to X$ is a morphism of schemes and $\mathcal{P}$ is a principal G-bundle on $\mathrm{Spec}(R)$. Then the construction $(\eta, \mathcal{P}) \mapsto \eta$ determines a projection map $\mathrm{BG}_X \to X$. If the fibers of G are connected, then each fiber of the map $\mathrm{BG}_X \to X$ can be identified with the classifying stack of a connected linear algebraic group. It therefore satisfies conditions $(*)$ and $(*')$ of Lecture 3, as well as the Grothendieck-Lefschetz trace formula (Proposition 3.4). We can then apply the construction of Claim 4.1 to obtain an object $\mathscr{F}_{\mathrm{BG}_X/X} \in D^b(X)$. If we further assume that the generic fiber of G is semisimple, then Euler product for $L_{\mathrm{BG}_X/X}(s)$ converges at $s = 0$, and therefore yields an identity

$$L_{\mathrm{BG}_X/X}(0) = \prod_{x \in X} \mathrm{Tr}(\varphi_x^{-1} | \, \mathrm{H}^*(\overline{\mathrm{BG}}_x; \mathbf{Q}_\ell))$$

where the right hand side is the infinite product appearing in the statement of Conjecture 2.13. Since each of the classifying stacks BG_x satisfies the Grothendieck-Lefschetz trace formula (Proposition 3.4), we can also write this as

$$L_{\mathrm{BG}_X/X}(0) = \frac{|\,\mathrm{BG}_x(\kappa(x))|}{|\kappa(x)|^{\dim(\mathrm{BG}_x)}} = \prod_{x \in X} \frac{|\kappa(x)|^{\dim(G_x)}}{|G_x(\kappa(x))|}.$$

To relate the product expansion to the left hand side of Conjecture 2.13, we note that the classifying stack BG_X is closely related to the moduli stack of G-bundles $\mathrm{Bun}_G(X)$. Note that there is a canonical *evaluation map*

$$e : X \times_{\mathrm{Spec}(\mathbf{F}_q)} \mathrm{Bun}_G(X) \to \mathrm{BG}_X\,.$$

An R-valued point of the left hand side is given by a pair $(\eta, \mathcal{P})$ where $\eta : \mathrm{Spec}(R) \to X$ is an R-valued point of X and $\mathcal{P}$ is a G-bundle on the *entire* product $X_R = \mathrm{Spec}(R) \times_{\mathrm{Spec}(\mathbf{F}_q)} X$, and in these terms the map e is given by $e(\eta, \mathcal{P}) = (\eta, \eta^* \mathcal{P})$. Letting $f : X \to \mathrm{Spec}(\mathbf{F}_q)$ denote the projection map, we see that properties (i) and (ii) above supply a comparison map

$$\rho : f^* \mathscr{F}_{\mathrm{Bun}_G(X)/\mathrm{Spec}(\mathbf{F}_q)} \simeq \mathscr{F}_{X \times_{\mathrm{Spec}(\mathbf{F}_q)} \mathrm{Bun}_G(X)/X} \to \mathscr{F}_{\mathrm{BG}_X/X}$$

in the derived category $D^b(X)$. We then have the following:

Theorem 4.3 (Gaitsgory-L). *Let X be a smooth projective geometrically connected curve over $\mathbf{F}_q$ and let G be a smooth affine group scheme over X. Assume that the fibers of G are connected and that the generic fiber of G is semisimple and simply connected. Then the map ρ above induces an isomorphism*

$$\mathscr{F}_{\mathrm{Bun}_G(X)/\operatorname{Spec}(\mathbf{F}_q)} \to Rf_* \mathscr{F}_{\mathrm{BG}_X/X}$$

in the derived category $D^b(\mathbf{F}_q)$. In other words, it induces an isomorphism of graded $\mathbf{Q}_\ell$-vector spaces

$$\pi_*(\overline{\mathrm{Bun}}_G(X))_{\mathbf{Q}_\ell} \to \mathrm{H}^{-*}(\overline{X}; \mathscr{F}_{\mathrm{BG}_X/X}),$$

where $\overline{\mathrm{Bun}}_G(X)_{\mathbf{Q}_\ell}$ denotes the product $\operatorname{Spec}(\overline{\mathbf{F}}_q) \times_{\operatorname{Spec}(\mathbf{F}_q)} \mathrm{Bun}_G(X)$.

Warning 4.4. In the statement of Theorem 4.3, we cannot omit the hypothesis that the generic fiber of G is simply connected. Note that the statement of Theorem 4.3 implicitly assumes that the sheaf $\mathscr{F}_{\mathrm{Bun}_G(X)/\operatorname{Spec}(\mathbf{F}_q)}$ is well-defined: this requires that the moduli stack $\overline{\mathrm{Bun}}_G(X)$ satisfies conditions $(*)$ and $(*')$ of the previous lecture. If the generic fiber of G is not simply connected, then these conditions can fail: for example, $\overline{\mathrm{Bun}}_G(X)$ can fail to be connected.

Assuming Theorem 4.3, we can complete the proof of Conjecture 2.13. Applying the Grothendieck-Lefschetz trace formula to the object $\mathscr{F} \in D^b(X)$, we obtain the identity

$$\begin{aligned} L_{\mathrm{BG}_X/X}(s) &= \prod_{n \in \mathbf{Z}} \det(1 - q^{-s}\varphi |\, \mathrm{H}^n_c(\overline{X}; \mathscr{F}_{\mathrm{BG}_X/X}))^{(-1)^{n+1}} \\ &= \prod_{n \in \mathbf{Z}} \det(1 - q^{-s}\varphi | \pi_{-n}(\overline{\mathrm{Bun}_G(X)})_{\mathbf{Q}_\ell})^{(-1)^{n+1}} \\ &= \det(1 - q^{-s}\varphi | \pi_*(\overline{\mathrm{Bun}}_G(X))_{\mathbf{Q}_\ell}). \end{aligned}$$

Evaluating at $s = 0$ and using calculation (3) of Lecture 3, we obtain the identity

$$L_{\mathrm{BG}_X/X}(0) = \mathrm{Tr}(\varphi^{-1} |\, \mathrm{H}^*(\overline{\mathrm{Bun}}_G(X); \mathbf{Q}_\ell)).$$

Remark 4.5. The L-functions appearing the above discussion are a bit of a red herring, since we do not use them for any purpose other than evaluation at $s = 0$.

Let us close this lecture with a few remarks concerning Theorem 4.3. Theorem 4.3 is essentially a *geometric* statement, rather than an arithmetic one: it is valid more generally for an algebraic curve X defined over any field k (in which ℓ is invertible), and the proof can be immediately reduced to the case where k is algebraically closed. It can be formulated in terms of cohomology, rather than homotopy: roughly speaking, it corresponds to the assertion that the ℓ-adic cochain complex $C^*(\mathrm{Bun}_G(X); \mathbf{Q}_\ell)$ admits a "continuous Künneth decomposition"

$$C^*(\mathrm{Bun}_G(X); \mathbf{Q}_\ell) \simeq \bigotimes_{x \in X}^{\mathrm{cont}} C^*(\mathrm{BG}_x; \mathbf{Q}_\ell).$$

We refer the reader to [7] for a precise statement, where the right hand side is defined using an algebro-geometric version of *factorization homology*. When $k = \mathbf{C}$ is the field of complex numbers, the existence of this decomposition is essentially equivalent to the fact that principal G-bundles on X are governed by an *h-principle*: more precisely, the moduli stack $\mathrm{Bun}_G(X)$ of algebraic G-bundles on X is homotopy equivalent (in a suitable sense) to the classifying space for *topological* G-bundles on X. When k has positive characteristic, the proof requires a different set of ideas.

5 Project: Parahoric Harder-Narasimhan Theory

For motivation, let us begin with a few remarks about Conjecture 2.12. Let Y be a smooth algebraic stack over a finite field $\mathbf{F}_q$, and set $\overline{Y} = \mathrm{Spec}(\overline{\mathbf{F}}_q) \times_{\mathrm{Spec}(\mathbf{F}_q)} Y$. We let $|Y(\mathbf{F}_q)|$ denote the sum $\sum_\eta \frac{1}{|\mathrm{Aut}(\eta)|}$, where η ranges over the collection of isomorphism classes of $\mathbf{F}_q$-valued points of Y (each of which has a finite automorphism group $\mathrm{Aut}(\eta)$). Fix a prime number ℓ which is invertible in $\mathbf{F}_q$ and an embedding $\mathbf{Q}_\ell \hookrightarrow \mathbf{C}$. We say that Y *satisfies the Grothendieck-Lefschetz trace formula* if there is an equality

$$\frac{|Y(\mathbf{F}_q)|}{q^{\dim(Y)}} = \mathrm{Tr}(\varphi^{-1} | \, \mathrm{H}^*(Y; \mathbf{Q}_\ell)) := \sum_{n \geqslant 0} (-1)^n \, \mathrm{Tr}(\varphi^{-1} | \, \mathrm{H}^n(Y; \mathbf{Q}_\ell))$$

(which requires, in particular, that the sums on both sides are convergent).

If Y is a smooth algebraic variety over $\mathbf{F}_q$, then it satisfies the Grothendieck-Lefschetz trace formula. More generally, if Y is given as a *global quotient stack* $Z//H$, where Z is a smooth variety over $\mathbf{F}_q$ and H is a linear algebraic group acting on Z, then one can deduce that the Grothendieck-Lefschetz trace formula holds for Y by knowing that it holds for Z and H (for a more general statement, see [5]). Conjecture 2.12 asserts that the Grothendieck-Lefschetz trace formula holds for $\mathrm{Bun}_G(X)$, where X is an algebraic curve over $\mathbf{F}_q$ and G is a smooth affine group scheme over X with semisimple generic fiber. One can show that $\mathrm{Bun}_G(X)$ is *locally* a global quotient stack: that is, every quasi-compact open substack $U \subseteq \mathrm{Bun}_G(X)$ can be written as a quotient $Z//H$ as above. However, there is still work to be done, because the moduli stack $\mathrm{Bun}_G(X)$ is almost never quasi-compact.

In [4], Behrend shows that $\mathrm{Bun}_G(X)$ satisfies the Grothendieck-Lefschetz trace formula when the group scheme G is *constant*: that is, when it comes from a semisimple algebraic group over $\mathbf{F}_q$. The proof makes use of the *Harder-Narasimhan* stratification of $\mathrm{Bun}_G(X)$: this is a decomposition of $\mathrm{Bun}_G(X)$ into (infinitely many) locally closed substacks which are quasi-compact, and therefore global quotient stacks for which the Grothendieck-Lefschetz trace formula is known. In [7], this approach is generalized to prove Conjecture 2.12 in general, including in cases where G fails to be semisimple at finitely many points. The general strategy of argument is the same: one constructs a suitable stratification of $\mathrm{Bun}_G(X)$, and deduces the trace formula for $\mathrm{Bun}_G(X)$ from the fact that the trace formula is known for each stratum. However, the stratification used in [7] is a bit *ad hoc*. It is probably possible to give a better proof, using an appropriate generalization of the Harder-Narasimhan stratification to the setting of *parahoric* group schemes over curves. In a series of guided exercises, we consider how this might work in the special case where G is a group scheme with generic fiber GL_n (beware that in this case one does not expect the Grothendieck-Lefschetz trace formula to hold for the entire stack $\mathrm{Bun}_G(X)$, because $\mathrm{Bun}_G(X)$ will have infinitely many connected components). We will also work over an algebraically closed ground field k, rather than a finite field $\mathbf{F}_q$.

Remark 5.1. There is a bit of recent literature on this subject, at least over fields of characteristic zero: see [3] and the final section of [1].

5.1 Normed Vector Spaces

We begin with some standard facts about normed vector spaces over nonarchimedean fields. Let $\widehat{K}$ be a field which is complete with respect to a nonarchimedean absolute value $|\bullet|_{\widehat{K}} : \widehat{K} \rightarrow \mathbf{R}_{\geqslant 0}$. We will assume that the absolute value $|\bullet|_{\widehat{K}}$ is nontrivial and discretely valued, so that

$$\widehat{\mathcal{O}} = \{x \in \widehat{K} : |x|_K \leqslant 1\}$$

is a discrete valuation ring with residue field $\widehat{K}$.

Definition 5.2. Let V be a vector space over $\widehat{K}$. A *norm* on E is a function

$$|\bullet|_V : V \rightarrow \mathbf{R}_{\geqslant 0}$$

satisfying the following conditions:

- For $v \in V$, we have $|v|_V = 0$ if and only if $v = 0$.
- For $v \in V$ and $f \in \widehat{K}$, we have $|fv|_V = |f|_{\widehat{K}} \cdot |v|_V$.
- For $u, v \in V$, we have $|u + v|_V \leqslant \max(|u|_V, |v|_V)$.

Exercise 5.3. Let V be a finite-dimensional vector space over $\widehat{K}$ equipped with a norm $|\bullet|_V$. Prove the following:

(a) Every subspace $V' \subseteq V$ is closed (with respect to the topology defined by the metric $d(u,v) = |u - v|_V$).

(b) For every subspace $V' \subseteq V$, the quotient V/V' inherits a norm, given by the formula
$$|\overline{v}|_{V/V'} = \inf_{v \to \overline{v}} |v|_V;$$
here the infimum is taken over all elements $v \in V$ which map to $\overline{v}$ in the quotient V/V'.

(c) For every element $\overline{v} \in V/V'$, the infimum of (b) is actually achieved: that is, one can choose an element $v \in V$ representing $\overline{v}$ such that $|\overline{v}|_{V/V'} = |v|_V$.

(d) The vector space V has a basis $v_1, v_2, \ldots, v_n$ with respect to which the norm $| \bullet |_V$ is given by
$$|f_1 v_1 + \cdots + f_n v_n|_V = \max(c_1 |f_1|_{\widehat{K}}, \cdots, c_n |f_n|_{\widehat{K}}),$$
for some positive real numbers $c_1, c_2, \ldots, c_n \in \mathbf{R}$ (which are then given by given by $c_i = |v_i|_V$).

(e) The norm $| \bullet |_V$ makes V into a Banach space: that is, V is complete with respect to the metric $d(u,v) = |u - v|_V$.

Hint: prove all of these assertions simultaneously by induction on the dimension of V.

Exercise 5.4. Let V be a finite-dimensional vector space over $\widehat{K}$ equipped with a norm $| \bullet |_V$. For each positive real number λ, set $V^{\leqslant \lambda} = \{v \in V : |v|_V \leqslant \lambda\}$. Show that $V^{\leqslant \lambda}$ is a $\widehat{\mathcal{O}}$-lattice in V. That is, $V^{\leqslant \lambda}$ is a free $\widehat{\mathcal{O}}$-module of rank $\dim_{\widehat{K}}(V)$ satisfying $V \simeq \widehat{K} \times_{\widehat{\mathcal{O}}} V^{\leqslant \lambda}$.

Exercise 5.5. Let V be a finite-dimensional vector space over $\widehat{K}$ and let $V_0 \subseteq V$ be a $\widehat{\mathcal{O}}$-lattice in V. For each $v \in V$, set
$$|v|_V = \inf\{|f|_{\widehat{K}} : v \in f V_0\}.$$
Show that $| \bullet |_V$ is a metric on V, and that $V_0 = V^{\leqslant 1}$.

Exercise 5.6. Let Γ denote the *value group* of $\widehat{K}$: that is, the set of all positive real numbers which have the form $|f|_{\widehat{K}}$. Let V be a finite-dimensional vector space over $\widehat{K}$. Show that the construction of Exercise 5.5 induces a bijection from the set of $\widehat{\mathcal{O}}$-lattices in V to the set of all norms
$$| \bullet |_V : V \to \mathbf{R}_{\geqslant 0}$$
which take values in $\Gamma \cup \{0\}$.

Exercise 5.7. Let V be a finite-dimensional vector space over $\widehat{K}$, let

$$|\bullet|_V : V \to \mathbf{R}_{\geqslant 0}$$

be a norm on V, and let $V_0 \subseteq V$ be a $\widehat{\mathcal{O}}$-lattice. Show that there exists a $\widehat{\mathcal{O}}$-basis $v_1, \ldots, v_n$ for V_0 and positive real numbers $c_1, \ldots, c_n \in \mathbf{R}$ such that

$$|f_1 v_1 + \cdots + f_n v_n|_V = \max(c_1 |f_1|_{\widehat{K}}, \cdots, c_n |f_n|_{\widehat{K}}).$$

Exercise 5.8. In the situation of Exercise 5.7, show that the real numbers $c_1, \ldots, c_n \in \mathbf{R}$ are uniquely determined (by the choice of norm $|\bullet|_V$ the choice of lattice $V_0 \subseteq V$). In particular, the product $\prod_{i=1}^n c_n$ is a well-defined positive real number, which we will denote by $\mathrm{Vol}(V_0)$ and refer to as the *volume* of V_0.

5.2 Generalized Vector Bundles

Now suppose that X is an algebraic curve defined over an *algebraically closed* field k. Let K_X denote the fraction field of X. We write $x \in X$ to indicate that x is a *closed* point of X (or, equivalently, a k-valued point of x). We let $\widehat{\mathcal{O}}_{X,x}$ denote the completed local ring of X at the point x, and $\widehat{K}_{X,x}$ its fraction field. Then $\widehat{K}_{X,x}$ is complete with respect to the absolute value given by

$$|f|_{\widehat{K}_{X,x}} = \exp(-\mathrm{ord}_x(f)),$$

where $\mathrm{ord}_x(f)$ denotes the order of vanishing of f at the point x.

Notation 5.9. Let V be a finite-dimensional vector space over the fraction field K_X. For each closed point $x \in X$, we let V_x denote the tensor product $\widehat{K}_{X,x} \otimes_{K_X} V$.

Definition 5.10. A *generalized vector bundle on* X is a pair $(V, \{|\bullet|_{V_x}\}_{x \in X})$, where V is a finite-dimensional vector space over the fraction field K_X and each $|\bullet|_{V_x}$ is a norm on the vector space $V_x = \widehat{K}_{X,x} \otimes_{K_X} V$, satisfying the following condition:

- $(*)$ Let $v_1, \ldots, v_n$ be a basis for the vector space V. Then, for all but finitely many closed points $x \in X$, the norm $|\bullet|_{V_x}$ is given by the formula

  $$|f_1 v_1 + \cdots + f_n v_n|_{V_x} = \max(|f_i|_{\widehat{K}_{X,x}})$$

  (see Exercise 5.5).

Exercise 5.11. Show that condition $(*)$ of Definition 5.10 does not depend on the choice of basis $v_1, \ldots, v_n \in V$.

Definition 5.12. Let $(V, \{|\bullet|_{V_x}\}_{x\in X})$ and $(V', \{|\bullet|_{V'_x}\}_{x\in X})$ be generalized vector bundles on X. A *morphism of generalized vector bundles* from $(V, \{|\bullet|_{V_x}\}_{x\in X})$ to $(V', \{|\bullet|_{V'_x}\}_{x\in X})$ is a K_X-linear map $F : V \to V'$ such that, for each point $x \in X$, the induced map $F_x : V_x \to V'_x$ is a contraction: that is, we have $|F_x(v)|_{V'_x} \leqslant |v|_{V_x}$ for each $v \in V_x$.

Exercise 5.13. Let $\mathscr{E}$ be a vector bundle on X. Then $\mathscr{E}$ determines a generalized vector bundle $(V, \{| \bullet |_{V_x}\}_{x\in X})$, by taking V to be the stalk of $\mathscr{E}$ at the generic point of X and each $| \bullet |_{V_x}$ to be the norm determined by the lattice given by the completed stalk $\widehat{\mathscr{E}}_x$ (see Exercise 5.5). Show that this construction induces a fully faithful embedding

$$\{\text{Vector bundles on } X\} \to \{\text{Generalized vector bundles on } X\}.$$

Moreover, a generalized vector bundle $(V, \{| \bullet |_{V_x}\}_{x\in X})$ belongs to the essential image of this embedding if and only if each of the norms $| \bullet |_{V_x}$ takes values in the set $\exp(\mathbf{Z}) \cup \{0\}$ (Exercise 5.6).

5.3 Degrees of Generalized Vector Bundles

We maintain the notations of §5.2.

Construction 5.14. Let $\mathscr{V} = (V, \{| \bullet |_{V_x}\}_{x\in X})$ be a generalized vector bundle on X, and choose a basis $v_1, \ldots, v_n$ for the vector space V. For each closed point $x \in X$, let $V_{x0} \subseteq V_x$ be the $\widehat{\mathcal{O}}_{X,x}$-lattice generated by the elements $v_1, \ldots, v_n$. We set

$$\deg(\mathscr{V}) = -\sum_{x\in X} \log(\mathrm{Vol}(V_{x0})),$$

where $\mathrm{Vol}(V_{x0})$ denotes the volume of the lattice V_{x0} with respect to the norm $| \bullet |_{V_x}$ (see Exercise 5.8). We will refer to $\deg(\mathscr{V})$ as the *degree* of the generalized vector bundle $\mathscr{V}$.

Exercise 5.15. Show that the degree $\deg(\mathscr{V})$ of Construction 5.14 is well-defined. That is, the sum $\sum_{x\in X} \log(\mathrm{Vol}(V_{x0}))$ is finite, and does not depend on the choice of basis $v_1, \ldots, v_n \in V$.

Exercise 5.16. Let $\mathscr{E}$ be a vector bundle on X, and let $\mathscr{V}$ be the generalized vector bundle associated to $\mathscr{E}$ in Exercise 5.13. Show that the degree of $\mathscr{V}$ (in the sense of Construction 5.14) is equal to the degree of $\mathscr{E}$ (in the usual sense).

Warning 5.17. If $\mathscr{E}$ is a vector bundle on X, then the degree $\deg(\mathscr{E})$ is an integer. However, the degree of a generalized vector bundle on X can be any real number.

Notation 5.18. Let $\mathscr{V} = (V, \{| \bullet |_{V_x}\}_{x\in X})$ be a generalized vector bundle on X. A *subbundle* of $\mathscr{V}$ is a pair $\mathscr{V}' = (V', \{| \bullet |_{V'_x})$, where $V' \subseteq V$ is a subspace and each of the norms $| \bullet |_{V'_x}$ is given by the restriction of $| \bullet |_{V_x}$ to the subspace $V'_x \subseteq V_x$. If this condition is satisfied, we let $\mathscr{V}/\mathscr{V}'$ denote the generalized vector bundle $(V/V', \{|\bullet|_{V_x/V'_x}\}_{x\in X})$, where each $|\bullet|_{V_x/V'_x}$ is defined as in Exercise 5.3. In this case, we will refer to the diagram

$$0 \to \mathscr{V}' \to \mathscr{V} \to \mathscr{V}/\mathscr{V}' \to 0$$

as a *short exact sequence* of generalized vector bundles.

Exercise 5.19. Show that for every short exact sequence of generalized vector bundles

$$0 \to \mathscr{V}' \to \mathscr{V} \to \mathscr{V}'' \to 0,$$

we have $\deg(\mathscr{V}) = \deg(\mathscr{V}') + \deg(\mathscr{V}'')$.

Exercise 5.20. Let $F : \mathscr{V} \to \mathscr{W}$ be a morphism of generalized vector bundles, and suppose that the underlying map of K_X-vector spaces $V \to W$ is an isomorphism. Show that $\deg(\mathscr{V}) \leqslant \deg(\mathscr{W})$, and that equality holds if and only if F is an isomorphism of generalized vector bundles.

5.4 Slopes and Semistability

We maintain the notations of §5.2.

Definition 5.21. Let $\mathscr{V} = (V, \{|\bullet|_{V_x}\}_{x\in X})$ be a generalized vector bundle on X. The *rank* of $\mathscr{V}$ is the dimension of V as a vector space over K_X. We denote the rank by $\mathrm{rk}(\mathscr{V})$. If $V \neq 0$, we set $\mathrm{slope}(\mathscr{V}) = \frac{\deg(\mathscr{V})}{\mathrm{rk}(\mathscr{V})}$. We refer to $\mathrm{slope}(\mathscr{V})$ as the *slope* of $\mathscr{V}$.

Exercise 5.22. Suppose we have a short exact sequence of (nonzero) generalized vector bundles

$$0 \to \mathscr{V}' \to \mathscr{V} \to \mathscr{V}'' \to 0.$$

Show that:

- If $\mathrm{slope}(\mathscr{V}') = \mathrm{slope}(\mathscr{V}'')$, then $\mathrm{slope}(\mathscr{V}') = \mathrm{slope}(\mathscr{V}) = \mathrm{slope}(\mathscr{V}'')$.
- If $\mathrm{slope}(\mathscr{V}') < \mathrm{slope}(\mathscr{V}'')$, then $\mathrm{slope}(\mathscr{V}') < \mathrm{slope}(\mathscr{V}) < \mathrm{slope}(\mathscr{V}'')$.
- If $\mathrm{slope}(\mathscr{V}') > \mathrm{slope}(\mathscr{V})$, then $\mathrm{slope}(\mathscr{V}') > \mathrm{slope}(\mathscr{V}) > \mathrm{slope}(\mathscr{V}'')$.

Definition 5.23. Let $\mathscr{V} = (V, \{|\bullet|_{V_x}\}_{x\in X})$ be a generalized vector bundle on X. We say that $\mathscr{V}$ is *semistable* if, for every subbundle $\mathscr{V}' \subseteq \mathscr{V}$, we have $\mathrm{slope}(\mathscr{V}') \leqslant \mathrm{slope}(\mathscr{V})$.

Exercise 5.24. Suppose we have a short exact sequence of generalized vector bundles

$$0 \to \mathscr{V}' \to \mathscr{V} \to \mathscr{V}'' \to 0.$$

Show that, if $\mathscr{V}$ is semistable of slope λ, then $\mathrm{slope}(\mathscr{V}') \leqslant \lambda \leqslant \mathrm{slope}(\mathscr{V}'')$

Exercise 5.25. Let $\mathscr{V}$ and $\mathscr{W}$ be generalized vector bundles on X which are semistable of slopes λ and μ, respectively. Show that if $\lambda > \mu$, then every map of generalized vector bundles $\mathscr{V} \to \mathscr{W}$ is zero.

Exercise 5.26. Fix a real number λ, and let $\mathcal{C}$ be the category whose objects are generalized vector bundles of slope λ (where, by convention, we include the zero object). Show that $\mathcal{C}$ is an abelian category (beware that the analogous statement is *not* true for the entire category of generalized vector bundles, or for the usual category of vector bundles on X).

Exercise 5.27. Suppose we have a short exact sequence of generalized vector bundles

$$0 \to \mathscr{V}' \to \mathscr{V} \to \mathscr{V}'' \to 0.$$

Show that, if $\mathscr{V}'$ and $\mathscr{V}''$ are semistable of the same slope λ, then $\mathscr{V}$ is also semistable of slope λ.

5.5 The Harder-Narasimhan Filtration

We maintain the notations of §5.2.

Definition 5.28. Let $\mathscr{V}$ be a generalized vector bundle on X. We say that a sequence of generalized subbundles

$$0 = \mathscr{V}_0 \subsetneq \mathscr{V}_1 \subseteq \mathscr{V}_2 \subsetneq \cdots \subseteq \mathscr{V}_m = \mathscr{V}$$

is a *Harder-Narasimhan filtration* if the following conditions are satisfied:

- Each of the quotients $\mathscr{V}_i / \mathscr{V}_{i-1}$ is semistable of some slope λ_i.
- The slopes λ_i are strictly decreasing: that is, we have $\lambda_1 > \lambda_2 > \cdots > \lambda_m$.

Exercise 5.29. Let $\mathscr{V}$ be a generalized vector bundle on X. Show that a Harder-Narasimhan filtration of $\mathscr{V}$ is unique if it exists.

Exercise 5.30. Let $\mathscr{V}$ be a generalized vector bundle on X, and let S be the collection of all real numbers of the form $\mathrm{slope}(\mathscr{V}')$, where $\mathscr{V}'$ is a nonzero subbundle of $\mathscr{V}$. Show that:

(a) The set S is discrete (with respect to the usual topology on $\mathbf{R}$).

(b) The set S is bounded above.

(c) The set S has a largest element.

Hint: to prove (b), choose a filtration

$$0 = \mathscr{V}_0 \subseteq \mathscr{V}_1 \subseteq \cdots \subseteq \mathscr{V}_n = \mathscr{V}$$

where each successive quotient $\mathscr{V}_m/\mathscr{V}_{m-1}$ has rank 1, and show that $\max(\deg(\mathscr{V}_m/\mathscr{V}_{m-1}))$ is an upper bound for S.

Exercise 5.31. Show that every generalized vector bundle $\mathscr{V}$ on X admits a Harder-Narasimhan filtration. Hint: choose a nonzero subbundle $\mathscr{V}' \subseteq \mathscr{V}$ of the largest possible slope (and largest possible rank for that slope), and proceed by induction.

5.6 Parahoric Group Schemes: Pointwise Case

We return to the notations of §5.1. Let V be a finite-dimensional vector space over a discretely valued field $\widehat{K}$. We let $\mathrm{GL}(V)$ denote the algebraic group over $\widehat{K}$ whose R-valued points are given by

$$\mathrm{GL}(V)(R) = \mathrm{Aut}_R(R \otimes_{\widehat{K}} V).$$

Note that any choice of basis $v_1, \ldots, v_n \in V$ determines an isomorphism of algebraic groups $\mathrm{GL}(V) \simeq \mathrm{GL}_n$.

If $V_0 \subseteq V$ is a $\widehat{\mathcal{O}}$-lattice, we let $\mathrm{GL}(V_0)$ denote the group scheme over $\widehat{\mathcal{O}}$ whose R-valued points are given by

$$\mathrm{GL}(V_0)(R) = \mathrm{Aut}_R(R \otimes_{\widehat{\mathcal{O}}} V_0).$$

Then $\mathrm{GL}(V_0)$ is again isomorphic to GL_n (this time as group schemes over $\widehat{\mathcal{O}}$), with an isomorphism determined by any choice of basis for V_0 as a $\widehat{\mathcal{O}}$-module. Moreover, we can identify the algebraic group $\mathrm{GL}(V)$ with the generic fiber of $\mathrm{GL}(V_0)$; in particular, we have a canonical map of $\widehat{\mathcal{O}}$-schemes $\mathrm{GL}(V) \to \mathrm{GL}(V_0)$.

Definition 5.32. Let V be a finite-dimensional vector space over $\widehat{K}$ and let $|\bullet|_V$ be a norm on V. Let $S \subseteq \mathbf{R}_{>0}$ be a finite set of real numbers with the property that for each nonzero element $v \in V$, we have $|fv|_V \in S$ for some $f \in \widehat{K}$ (exercise: show that S exists). We let $\mathrm{GL}(V, |\bullet|_V)$ denote the scheme-theoretic image of the map

$$\mathrm{GL}(V) \to \prod_{\lambda \in S} \mathrm{GL}(V^{\leqslant \lambda});$$

here the product on the right hand side is formed in the category of $\mathcal{O}$-schemes.

Exercise 5.33. In the situation of Definition 5.32, show that $\mathrm{GL}(V, |\bullet|_V)$ is a group scheme over $\widehat{\mathcal{O}}$, with generic fiber $\mathrm{GL}(V)$. (More precisely, it is a closed subgroup of the product $\mathrm{GL}(V^{\leqslant\lambda})$). In other words, it is an *integral model* for the group scheme $\mathrm{GL}(V)$. Integral models which arise in this way (that is, which come from norms on V) are called *parahoric*.

Exercise 5.34. In the situation of Definition 5.32, show that the map $\mathrm{GL}(V, |\bullet|_V) \to \mathrm{Spec}(\widehat{\mathcal{O}})$ is smooth and has connected fibers.

Exercise 5.35. Let V be a finite-dimensional vector space over $\widehat{K}$ equipped with a norm $|\bullet|_V$. Show that the group $\mathrm{GL}(V, |\bullet|_V)(\widehat{\mathcal{O}})$ of $\widehat{\mathcal{O}}$-valued points of $\mathrm{GL}(V, |\bullet|_V)$ can be identified with the subgroup of $\mathrm{GL}(V)(\widehat{K}) = \mathrm{Aut}_{\widehat{K}}(V)$ consisting of those automorphisms of V which preserve the norm $|\bullet|_V$ (that is, $\widehat{K}$-linear maps $F : V \to V$ satisfying $|F(v)|_V = |v|_V$ for each $v \in V$).

5.7 Parahoric Group Schemes over a Curve

We now return to the notations of §5.2.

Exercise 5.36. Let $\mathscr{V} = (V, \{|\bullet|_{V_x}\}_{x \in X})$ be a generalized vector bundle on X. Show that there is an essentially unique group scheme G over X equipped with isomorphisms

$$\mathrm{Spec}(K_X) \times_X G \simeq \mathrm{GL}(V)$$

$$\mathrm{Spec}(\widehat{\mathcal{O}}_{X,x}) \times_X G \simeq \mathrm{GL}(V_x, |\bullet|_{V_x})$$

which are compatible with restricted to $\widehat{K}_{X,x}$, for each closed point $x \in X$. We will denote the group scheme G by $\mathrm{GL}(\mathscr{V})$.

Exercise 5.37. Let $\mathscr{V} = (V, \{|\bullet|_{V_x}\}_{x \in X})$ be a generalized vector bundle on X, and let $\mathrm{GL}(\mathscr{V})$ be the group scheme of Exercise 5.36. Show that the groupoid of principal $\mathrm{GL}(\mathscr{V})$-bundles on X can be identified with the groupoid of generalized vector bundles $\mathscr{W} = (W, \{|\bullet|_{W_x}\}_{x \in X})$ having the property that, for each closed point $x \in X$, there exists an isomorphism of normed vector spaces $(V_x, |\bullet|_{V_x}) \simeq (W_x, |\bullet|_{W_x})$.

From this point forward, we fix a generalized vector bundle $\mathscr{V}$ on X of rank n. Let $\mathrm{Bun}_{\mathrm{GL}(\mathscr{V})}(X)$ denote the moduli stack of principal $\mathrm{GL}(\mathscr{V})$-bundles, where $\mathrm{GL}(\mathscr{V})$ is the group scheme of Exercise 5.36. By virtue of Exercise 5.37, we can identify k-valued points of $\mathrm{Bun}_{\mathrm{GL}(\mathscr{V})}(X)$ with generalized vector bundles $\mathscr{W}$ which are locally isomorphic to $\mathscr{V}$.

Question 5.38. What can one say about the locus of principal $\mathrm{GL}(\mathscr{V})$-bundles on X which correspond to *semistable* generalized vector bundles of some fixed slope λ? Are they parametrized by an open substack $\mathrm{Bun}^{\mathrm{ss},\lambda}_{\mathrm{GL}(\mathscr{V})}(X) \subseteq \mathrm{Bun}_{\mathrm{GL}(\mathscr{V})}(X)$? If so, is it quasi-compact?

Question 5.39. Suppose we are given real numbers $d_1, \ldots, d_m$ and positive integers $n_1, n_2, \ldots, n_m$ satisfying

$$\frac{d_1}{n_1} > \frac{d_2}{n_2} > \cdots > \frac{d_m}{n_m} \qquad n = n_1 + \cdots + n_m.$$

What can one say about the locus of principal $\mathrm{GL}(\mathscr{V})$-bundles on X which correspond to generalized vector bundles $\mathscr{W}$ which admit a Harder-Narasimhan filtration

$$0 = \mathscr{W}_0 \subsetneq \mathscr{W}_1 \subsetneq \cdots \subsetneq \mathscr{W}_m = \mathscr{W}$$

where each quotient $\mathscr{W}_i/\mathscr{W}_{i-1}$ is a generalized vector bundle of rank n_i and degree d_i. Are they parametrized by a locally closed substack of $\mathrm{Bun}_{\mathrm{GL}(\mathscr{V})}(X)$? If so, is it quasi-compact? Can it be described in terms of moduli stacks of the form $\mathrm{Bun}_{\mathrm{GL}(\mathscr{U}_1)}(X) \times \cdots \times \mathrm{Bun}_{\mathrm{GL}(\mathscr{U}_m)}(X)$, for suitable generalized vector bundles $\mathscr{U}_i$ of rank n_i?

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