Tamagawa Numbers in the Function Field Case

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Let G be a semisimple algebraic group defined over the field $\mathbf{Q}$ of rational numbers and let $G(\mathbf{Q})$ denote the group of rational points of G. Then $G(\mathbf{Q})$ can be regarded as a discrete subgroup of the locally compact group $G(\mathbf{A})$ of *adelic points* of G. Moreover, the group $G(\mathbf{A})$ carries a canonical (bi-invariant) measure, called *Tamagawa measure*. A celebrated conjecture of Weil asserts that, if the group G is simply connected, then the Tamagawa measure of the quotient $G(\mathbf{Q})\backslash G(\mathbf{A})$ is equal to 1. Weil's conjecture is now a theorem of Kottwitz, building on earlier work of Langlands and Lai. More recently, Gaitsgory and Lurie proved a version of Weil's conjecture in the setting of function fields, using techniques inspired by algebraic topology (specifically, the theory of factorization homology). The goal of this lecture series is to explain some of the ideas surrounding the proof.

- In the first lecture, we will review the origin of Weil's conjecture as a generalization of the *mass formula* of Smith-Minkowski-Siegel. We will then discuss how to interpret the function field analogue of Weil's conjecture as a mass formula for counting principal G-bundles on algebraic curves (over finite fields).
- When X is an algebraic curve over a field k, principal G-bundles on X are parametrized by an algebraic stack $\mathrm{Bun}_G(X)$, called the *moduli stack of G-bundles*. In the second lecture, we will review the Grothendieck-Lefschetz trace formula, which reduces the problem of counting principal G-bundles (over a finite field) to the problem of understanding the cohomology $\mathrm{H}^*(\mathrm{Bun}_G(X); \mathbf{Q}_\ell)$ (over an algebraically closed field). We then give a heuristic formula for this cohomology as a "continuous tensor product" $\bigotimes_{x \in X} \mathrm{H}^*(\mathrm{Bun}_G(\{x\}); \mathbf{Q}_\ell)$ which explains the mass formula predicted by Weil's conjecture.
- In the third lecture, we introduce an algebro-geometric version of *factorization homology* and use it to give a precise formulation of the heuristic from Lecture 2. This takes the form of a "product formula" characterizing the cochain algebra $C^*(\mathrm{Bun}_G(X); \mathbf{Q}_\ell)$ (rather than the cohomology algebra $\mathrm{H}^*(\mathrm{Bun}_G(X); \mathbf{Q}_\ell)$) by a universal property.
- In the fourth lecture, we specialize to the case where G and X are defined over a finite field $\mathbf{F}_q$, and explain how to use the product formula of Lecture 3 (together with the Grothendieck-Lefschetz trace formula) to count principal G-bundles on X, thereby obtaining a proof of Weil's conjecture.

1 Project

In order to reduce the proof of Weil's conjecture to a statement about the cohomology of $\mathrm{Bun}_G(X)$, one needs to know that $\mathrm{Bun}_G(X)$ satisfies the Grothendieck-Lefschetz trace formula. This does not follow formally from the classical version of the Grothendieck-Lefschetz trace formula, because the moduli stack $\mathrm{Bun}_G(X)$ is not quasi-compact. Nevertheless, the trace formula is still valid. This was proved by Behrend in the case where G is everywhere semisimple, using the Harder-Narasimhan stratification of $\mathrm{Bun}_G(X)$. In [2], this argument is extended to the case of groups of bad reduction by a somewhat ad-hoc method. However, it seems likely that there is a better proof.

In this project, students will study a potential generalization of the Harder-Narasimhan stratification to the case where G is a group scheme over X which is only *generically* reductive, but can have bad (but parahoric) reduction at finitely many points of X. In the case where the generic fiber of G is GL_n, this generalization can be described very explicitly in terms of vector bundles on algebraic curves. Further extensions could be considered, depending on the expertise of the group.

This project is somewhat orthogonal to the material covered in the lectures, and does not require any knowledge of algebraic topology. However, it will require some algebraic geometry: students should be familiar with the classical Harder-Narasimhan theory for vector bundles on algebraic curves.

References

[1] Behrend, K. *The Lefschetz Trace Formula for the Moduli Stack of Principal Bundles.* PhD dissertation.

[2] Gaitsgory, D., and J. Lurie. *Weil's Conjecture for Function Fields I.* Available at http://www.math.harvard.edu/~lurie/papers/tamagawa-abridged.pdf.

[3] Weil, A. *Adeles and algebraic groups.* Progress in Mathematics, 23, 1982.