AWS 2019: Topological cyclic homology in arithmetic geometry

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The goal of this course will be to introduce the audience to certain aspects of the theory of topological Hochschild and cyclic homology, particularly those of interest in algebraic and arithmetic geometry.

The classical Hochschild homology of an algebra A over a field k is defined via the explicit Hochschild complex

$$HH(A/k) := \qquad A \leftarrow A \otimes_k A \leftarrow A \otimes_k A \otimes_k A \leftarrow \cdots$$

The extra data of cyclic permutations on each term $A^{\otimes_k n}$ gives $HH(A/k)$ the structure of a so-called *cyclic complex* and was used by Connes and Feĭgin–Tsygan in the 1980s to construct the cyclic homology, negative cyclic homology, and period cyclic homology of A; these fit together into a fibre sequence in the derived category of k-modules

$$HC(A/k)[1] \longrightarrow HC^-(A/k) \longrightarrow HP(A/k).$$

In the case in which k is of characteristic zero and A is smooth over k, they established natural decompositions

$$HP(A/k) = \prod_{n\in\mathbb{Z}} \Omega^\bullet_{A/k}[2n], \qquad HC^-(A/k) = \prod_{n\in\mathbb{Z}} \Omega^{\geq n}_{A/k}[2n]$$

In short, periodic and negative cyclic homology encode the de Rham cohomology of A and its Hodge filtration; similarly statements apply in the case of smooth varieties over k.

A profound idea of Goodwillie was to transport the above constructions to the setting of structured ring spectra, replacing the base k by the sphere spectrum $\mathbb{S}$ and A (now any commutative ring) by its associated Eilenberg–Maclane spectrum HA:

$$THH(A) := |HA \leftarrow HA \otimes_{\mathbb{S}} HA \leftarrow HA \otimes_{\mathbb{S}} HA \otimes_{\mathbb{S}} HA \leftarrow \cdots|$$

This again inherits extra structure, now that of a so-called *cyclotomic spectrum*. The theory of cyclotomic spectra has been substantially clarified by recent work of Nikolaus–Scholze [4], whose point of view we will adopt in the course. Essentially, $THH(A)$ is not merely equipped with an action by the circle S^1, but moreover with a certain Frobenius map for each prime number p

$$\varphi_p : THH(A) \to THH(A)^{tC_p},$$

where the right side denotes the Tate construction associated to the action of the cyclic group $C_p \subseteq S^1$. From this data one builds the topological cyclic homology, negative topological cyclic homology, and periodic cyclic homology of A, fitting into a fibre sequence of spectra

$$TC(A) \longrightarrow TC^-(A) \xrightarrow{\varphi-1} TP(A)$$

(we warn the reader that this does not quite correspond to the fibre sequence in the classical theory).

The style of result which will interest us in the course is the following analogue of the de Rham calculation from the classical theory:

Theorem 0.1 ([2]). *Let A be a smooth algebra over a perfect field k of characteristic p. Then $TP(A)$ and $TC^-(A)$ admit $\mathbb{Z}$-indexed complete filtrations whose graded pieces are respectively shifts of*

$$W\Omega^\bullet_{A/k}, \qquad \mathcal{N}^{\geq i} W\Omega^\bullet_{A/k}.$$

Here $W\Omega^\bullet_{A/k}$ denotes the de Rham–Witt complex of Bloch–Deligne–Illusie while $\mathcal{N}^{\geq i} W\Omega^\bullet_{A/k}$ is its Nygaard filtration, defined as the largest subcomplex on which the absolute Frobenius is divisible by p^i. In other words, the classical de Rham calculation in characteristic zero has a natural analogue in characteristic p in terms of crystalline cohomology, except that the direct sum decomposition is no longer split, i.e., there is merely a complete filtration. This is analogous to the motivic filtration on algebraic K-theory.

A similar theorem holds in mixed characteristic, e.g., A smooth over the ring of integers of a perfectoid field, where the graded steps of the filtrations are now described in terms of $A\Omega_A$, the $\mathbb{E}_\infty$-ring constructed in [1] (and re-constructed as prismatic cohomology in [3]), which simultaneously interpolates certain de Rham, crystalline, and p-adic étale cohomologies.

References

[1] Bhatt, B., Morrow, M., and Scholze, P. Integral p-adic Hodge theory. `arXiv:1602.03148` (2016).

[2] Bhatt, B., Morrow, M., and Scholze, P. Topological Hochschild homology and integral p-adic Hodge theory. `arXiv:1802.03261` (2018).

[3] Bhatt, B., and Scholze, P. Prismatic cohomology. *In preparation*.

[4] Nikolaus, T., and Scholze, P. On topological cyclic homology. `arXiv:1707.01799` (2017).