READING PROJECTS FOR MORROW 2019 AWS STUDY GROUP

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1. Introduction

This document is meant as a companion to Matthew Morrow's lecture notes "Topological Hochschild homology in arithmetic geometry"[1]. It consits of four "reading projects" each of which focuses on a key topic of the notes. The goal of the reading projects is to give a streamlined path to quickly approach interesting themes and theorems in Morrow's notes. This is not meant to suggest that reading the notes as they are is a bad idea, in fact they are already very streamlined. Rather, the hope is to focus on certain ideas and themes while reading the rest as well.

There is a consistent parallel between the theory of HH / HC^- / HP (and HC) in the rational setting and that of THH / TC^- / TP in positive characteristic. As a result the projects will often start in the characteristic 0 world (with results about HH and friends) before moving to characteristic p to discuss analogous results for THH, etc.

I use the currently most up to date (2nd March) numbering from the version of the notes updated 27th February.

2. Reading projects

Note: The introduction to each project uses very vague language, and is certainly up for discussion.

Reading project 1: The HKR-theorem.

Introduction: A recurring theme in the theory of (Topological) Hochschild homology is what happens when the input is *smooth*. The result is often related to differential geometry in the sense of algebraic de Rham cohomology. The Hochschild-Kostant-Rosenberg theorem states that in the smooth case the Hochschild homology groups are given by differential forms. If one restricts to the characteristic p case then there is a similar result for THH.

Goal: To understand the proof of Theorem 2.8, i.e.

Theorem (Hochschild-Kostant-Rosenberg)**.** *If A is a smooth k-algebra, then the natural maps $\Omega^n_{A/k} \to \mathrm{HH}_n(A/k)$ are isomorphisms of A-modules for all $n \geq 0$.*

Then move on to the analogous result in the topological setting, appearing as Theorem 5.6.

[1] I thank Sean Howe for suggesting that I make such a document. I have followed the style of a similar document that he made for AWS in 2017.

Theorem (Hesselholt's HKR)**.** *Let k be a perfect field of characteristic p and let R be a smooth k-algebra. Then the map of graded rings $\Omega^*_{R/k} \otimes_{\mathbb{F}_p} \mathrm{THH}_*(\mathbb{F}_p) \to \mathrm{THH}_*(R)$ is an isomorphism.*

Suggested reading guide:

(1) Section 2.1: This section contains the HKR theorem (Theorem 2.8), but the proof relies on a result about étale extensions from Section 2.4.
(2) Using the Tor-definition for HH, compute $\mathrm{HH}_*(k[x]/k)$ and check that you get the correct result, as predicted by HKR.
(3) Section 2.4, The cotangent complex and its basic properties proven in Proposition 2.27.
(4) Section 2.4, Prove Proposition 2.28, and apply it as sketched back in Remark 2.6
(5) Section 4.5, the definition of THH
(6) Section 5.1, Corollary 5.5(i). "$\mathrm{THH}(-)$ is a one-parameter deformation of $\mathrm{HH}(-/\mathbb{F}_p)$".
(7) Section 5.1, Proposition 5.6. Hesselholt's HKR theorem, using Corollary 5.5(i).

Reading project 2: The de Rham differential and S^1-actions.

Introduction: The HKR theorem (Theorem 2.8) relates Hochschild homology with differential *forms*. But differential forms assemble into a cochain complex, i.e. we have the de Rham differential $d : \Omega^n \to \Omega^{n+1}$. A natural question is then whether the Hochschild homology groups also admit such a structure. This leads to the theory of cyclic objects and/or mixed complexes. In fact the de Rham differential is a "shadow" of the circle action on Hochschild homology. From this point of view the theory of cyclic homology (and the two variants: negative/periodic cyclic homology) becomes very natural, as opposed to the *ad hoc* bicomplex constructions.

Goal: Get familiar with (various forms of) S^1-actions, and how they lead to differentials.

Suggested reading guide:

(1) Section 2.1. Go through definitions and the statement of HKR (Theorem 2.8). Skip past Remarks 2.6 and 2.7.
(2) Section 2.2. Focus on Remark 2.12.
(3) Section 2.2. Definition 2.17, Cyclic objects. Check that $\mathrm{HH}(A/k)$ (in the simplicial perspective from Remark 2.3) is a cyclic k-module.
(4) Section 2.2. Definition 2.19, Mixed complexes. Remark 2.21 shows how a cyclic object gives rise to a mixed complex.
(5) Pause and remind yourself what the singular homology $H_*(S^1, k)$ of the circle (as a topological group) is. Here k is any commutative ring. Remember to think about the product structure.
(6) Section 4.1. Lemma 4.3, in which the previous step is applied to show that mixed complexes are the same (up to homotopy) as "complexes with S^1-action".

(7) Section 4.2. On "crossed simplicial groups". Go throug example 4.5.(ii) in detail. Write out what Definition 4.4 says in this case and note that this agrees with the notion of cyclic objects.
(8) Section 4.3, where the statement $|\Lambda| \simeq BS^1$ is used to turn cyclic objects into (∞-categorical) S^1-actions.
(9) Go back to Section 4.1 and read the paragraphs after Lemma 4.3. They describe how (negative/periodic) cyclic homology arise from the algebraic S^1-action on Hochschild homology.

Alternative source: The last step of the reading guide shows how one can view negative cyclic homology as the (derived) fixed points of the circle action on the Hochschild complex. The cited paper by Hoyois gives further results in this direction.

Reading project 3: (T) HH **of** $\mathbb{F}_p$ **and perfectoid rings.**

Introduction: A fundamental calculation, due to Bökstedt, says that $\mathrm{THH}_*(\mathbb{F}_p)$ is a polynomial algebra with generator in degree two. The proof is not covered in the notes, since it requires some sophisticated topological methods. Rather, the approach is to see how (amazingly) far one can get using this as a black box. The corresponding result for $\mathrm{HH}_*(\mathbb{F}_p)$ is more approachable, and it is illuminating to consider the difference between the two.

Goal: Prove that $\mathrm{HH}_*(\mathbb{F}_p)$ is a divided power algebra on a single generator in degree 2.

Suggested reading guide:

(1) Section 2.1. Definitions.
(2) Section 2.4, starting with Example 2.32 and working backwards.
(3) The Krause-Nikolaus notes Proposition 2.6 gives the computation of $\mathrm{HH}_*(\mathbb{F}_p)$ using explicit resolutions.
(4) Compare this result with Bökstedt's result. For example consider the map $\mathrm{THH}_*(\mathbb{F}_p) \to \mathrm{HH}_*(\mathbb{F}_p)$ (e.g. following the remarks after Theorem 3.13 in the Krause-Nikolaus notes).

The perfectoid case: Using $\mathrm{THH}_*(\mathbb{F}_p) \simeq \mathbb{F}_p[u]$ as input, prove that for R perfectoid $\mathrm{THH}_*(R; \mathbb{Z}_p) \simeq R[u]$ where again u is a generator in degree two. This is Theorem 6.1 in Bhatt-Morrow-Scholze's second paper. It is a Nakayama lemma-type argument reducing to the characteristic p case and the characteristic zero case.

Alternative source: You might like to look at the notes by Höning in the Oberwolfach Report for a proof of Bökstedt's result. There is also a proof in the very recent paper by Hesselholt and Nikolaus "Topological cyclic homology", available at Hesselholt's website.

Reading project 4: Filtrations, descent, and crystalline cohomology.

Introduction: The work of Bhatt-Morrow-Scholze shows how one can use THH with its extra structure and nice properties to reconstruct crystalline cohomology. They also reconstruct other p-adic cohomology theories from THH allowing them to make comparisons between the resulting cohomologies. In this reading project we focus on the crystalline side.

Roughly this proceeds in two steps; first one uses the circle action on THH to construct TP. Secondly, one constructs a filtration on TP. This filtration exists because of a useful property of THH (and hence TP) namely *flat descent*. Now one shows that the associated graded pieces of the filtration on TP is given by complexes computing crystalline cohomology.

As usual there is an analogous story with HH (and HP).

Goal: Understand and prove Theorem 3.1 and Theorem 5.9. The latter states the following. Let k be a perfect field of characteristic p.

Theorem (BMS2)**.** *Let R be a smooth k-algebra. Then* $\mathrm{TP}(R)$ *admits a natural, complete, descending $\mathbb{Z}$-indexed filtration whose n^{th} associated graded pieces is given by*

$$gr^n \, \mathrm{TP}(R) \simeq R\Gamma_{crys}(R/W(k))[2n]$$

Suggested reading guide: We will try to prove statements about HH / HP and THH / TP in tandem.

(1) Propostions 2.28 (resp. Corollary 5.7) giving the HKR-filtration on HH (resp. THH).
(2) Using the HKR filtration prove that HH and HP satisfy flat descent, Lemma 3.7. Similarly, using the Hesselholt-HKR-filtration prove that THH and TP satisfy flat descent, Corollary 5.8.
(3) Section 3.1. on quasiregular semiperfect algebras and section 3.4. on the quasisyntomic site.
(4) Lemma 3.8. (resp. Lemma 5.10), dealing with filtrations on HP_0 (resp. TP_0).
(5) Theorem 3.1. (resp. Theorem 5.9), deduced from Lemma 3.8 (resp. Lemma 5.10).

Alternative sources: For additional details and perspectives on some of these results you might like to look at the Oberwolfach Report also cited in the notes. In particular the following sections are relevant:

- Niziol: proves flat descent for THH including a proof of flat descent for the cotangent complex.
- Česnavičius: identifies the associated graded pieces of the filtration on TP_0
- Elmanto: gives a proof of Theorem 5.9 using quasisyntomic descent.