ARIZONA WINTER SCHOOL: $\mathbb{A}^1$-ENUMERATIVE GEOMETRY

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ABSTRACT. We use $\mathbb{A}^1$-homotopy theory to include arithmetic-geometric information into results from classical enumerative geometry.

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1. LECTURE 1: INTRODUCTION AND AN ARITHMETIC COUNT OF THE LINES ON A SMOOTH CUBIC SURFACE

Enumerative geometry counts algebro-geometric objects satisfying certain conditions. Classically, such counts are performed over an algebraically closed field to obtain "invariance of number" results. For example, the number of solutions in $\mathbf{C}$ of a degree n polynomial is always n, but this is no longer true over $\mathbf{Q}$. We use $\mathbb{A}^1$-homotopy theory to give enumerative results over general fields, including number fields, local fields, finite fields etc. As the number of objects is not itself invariant, we do not obtain formulas for these numbers; rather, information about the fields of definition will be recorded with a bilinear form, and the sum will satisfy an "invariance of bilinear form" principle.

We start with an example from joint work with Jesse Leo Kass [KW17]. A cubic surface X over $\mathbf{C}$ is the space of solutions to a cubic polynomial in three variables. Namely,

$$X = \{(x,y,z) \in \mathbf{C}^3 : f(x,y,z) = 0\}$$

where $f \in \mathbf{C}[x,y,z]$ is a degree 3 polynomial. X is *smooth* if the partials of X do not simultaneously vanish on X, which is equivalent to X being a manifold. We can substitute an arbitrary field k for $\mathbf{C}$, and view X as a variety or scheme over k. When $k = \mathbf{R}$, the real points of a cubic surface can be naturally embedded as a 2-dimensional surface in our 3-dimensional room, and there are pretty models and pictures, for example, the model photographed in Figure 1. For the invariance of number principle invoked above, it is better to compactify, viewing X as the subscheme of $\mathbf{P}^3$ determined by a homogeneous degree 3 polynomial f in $k[w,x,y,z]$.

It is a lovely theorem of Salmon and Cayley proven in 1849 [Cay49] that there are exactly 27 lines on a smooth cubic surface over $\mathbf{C}$.

Theorem 1. *[Salmon, Cayley] Let X be a smooth cubic surface over $\mathbf{C}$. Then X contains exactly 27 lines.*

See [Dol05] for interesting historical remarks.

Example 2. *The polynomial*

$$f(w,x,y,z) = w^3 + x^3 + y^3 + z^3$$

determines the Fermat Cubic Surface, whose lines can be described as follows. Let $[S,T]$ denote homogeneous coordinates on $\mathbf{P}^1 = \operatorname{Proj} \mathbf{C}[S,T]$. The line

$$\{[S,-S,T,-T] : [S,T] \in \mathbf{P}^1\}$$

is on the Fermat Cubic Surface by inspection. Moreover, for any λ, ω such that $\lambda^3 = \omega^3 = -1$, the line

$$\{[S,\lambda S,T,\omega T] : [S,T] \in \mathbf{P}^1\}$$

is also on the Fermat Cubic Surface by the same principle. Permuting the coordinates produces more lines, giving a total of $\frac{\binom{4}{2}}{2} \cdot 3 \cdot 3 = 27$ lines. It can be checked in an elementary manner that these are the only lines; see for instance `https://www.mathematik.uni-kl.de/~gathmann/class/alggeom-2014/alggeom-2014-c11.pdf`.

Here is a modern proof of Salmon and Cayley's theorem [EH16, Chapter 6, especially 6.2 and 6.4].

Proof. Let $\mathrm{Gr}(1,3)$ denote the Grassmannian parametrizing lines in $\mathbf{P}^3$ or equivalently 2-dimensional subspaces W of a 4-dimensional vector space. The *tautological bundle* (not to be confused with the canonical bundle) is the vector bundle $\mathcal{S} \to \mathrm{Gr}(1,3)$ whose fiber over a W is W itself. The third symmetric power of the dual of $\mathcal{S}$ is the vector bundle $\mathrm{Sym}^3 \mathcal{S}^* \to \mathrm{Gr}(1,3)$, whose fiber over the point corresponding to W of the Grassmannian is the space of cubic polynomials on W, i.e., $\mathrm{Sym}^3 W^*$. The cubic surface X is defined by a cubic polynomial f on the entire 4 dimensional vector space. f therefore determines an element of $\mathrm{Sym}^3 W^*$ for every subspace W by restriction. In other words, f determines a section σ_f of $\mathrm{Sym}^3 \mathcal{S}^* \to \mathrm{Gr}(1,3)$ by

$$\sigma_f(W) = f|_W.$$

A line $\mathbf{P}W$ is contained in X exactly when the polynomial f vanishes on W. In other words, the zeros of the section σ_f are in bijection with the lines in X.

Aside on the Euler Class 3. *There is a tool for counting the zeros of a section of a vector bundle. This tool is the* Euler Class *and we use it in the case when generically there are a finite number of zeros, or equivalently, we use it on a (relatively oriented) vector bundle V of rank r on a dimension r (real, respectively) complex manifold M. Choose such a $V \to M$, and then choose a section σ so that all the zeros of σ are* isolated *in the sense that there for every p in M with $\sigma(p) = 0$, there is an open neighborhood U of p such that the only zero of σ in U is p.*

The Euler Class can be defined using the degree *of a map $S^d \to S^d$ between oriented topological spheres of the same dimension. The degree only depends on the homotopy class of the map, namely we have a map*

$$\deg : [S^d, S^d] \to \mathbf{Z}.$$

Now for the Euler class of V, computed using σ. Let p be a point of M such that $\sigma(p) = 0$. By assumption, p is an isolated zero *of σ, so we may choose local coordinates around p and a local trivialization of V (compatible with the relative orientation) and identify σ with a function, which by an abuse of notation we denote $\sigma : \mathbf{C}^n \to \mathbf{C}^n$ (respectively $\sigma : \mathbf{R}^n \to \mathbf{R}^n$) in such a way that the point p corresponds to the origin in the domain. Then we may choose a small ball B around the origin such that p is the only zero of σ in B. We thus obtain a function between oriented spheres*

$$\overline{\sigma} : \partial B \to \partial\{x \in \mathbf{C}^n : \|x\| = 1\}$$

(respectively $\overline{\sigma} : \partial B \to \partial\{x \in \mathbf{R}^n : \|x\| = 1\}$) given by

$$\overline{\sigma}(x) = \frac{\sigma(x)}{\|\sigma(x)\|}.$$

Then set the local degree *or* index *of σ at p to be*

$$\deg_p \sigma = \deg(\overline{\sigma}).$$

Then define the Euler class $e(V)$ by

$$e(V) = \sum_{p:\sigma(p)=0} \deg_p \sigma, \tag{1}$$

and one can show that this is independent of the choice of σ.

Now return to the lines on a smooth cubic surface X. We saw above that the lines $L = \mathbf{P}W$ on X are in bijection with the zeros L of σ_f. Now, some multivariate calculus and the smoothness of X implies that every zero L of σ_f is isolated and the local index $\deg_L \sigma_f = 1$ is one [EH16, Section 2.4.2, Corollary 6.17] [KW17, Corollary 52]. We therefore conclude that the number of lines on X is $e(\mathrm{Sym}^3 \mathcal{S}^*)$, and in particular, this number is independent of surface! The example of the Fermat Cubic Surface then completes the proof, or alternatively one can use tools from algebraic topology such as the Splitting Principle, cohomology of Grassmannians, etc.

□

We now consider cubic surfaces over $\mathbf{R}$. This is also a classical topic with many beautiful results, among the first of which are those of Schläfli [Sch58] in 1858. For our purposes, it is useful to say that he showed that the number of real lines depends on the chosen cubic surface: there can be 3,7,15, or 27 real lines.

Segre distinguished between two types of real lines: *hyperbolic* and *elliptic* lines [Seg42]. Given a physical model of a real cubic surface, such as the one pictured if Figure 1, one can feel if the line is hyperbolic or elliptic using (s)pin structures as follows. Place your index finger so it lies on the line and let your palm rest on the tangent plane to the surface. Then move your hand along the line. If your hand becomes "twisted" the line is elliptic and if it remains "untwisted" the line is hyperbolic. More precisely, the line is a copy of $\mathbf{RP}^1$ which is topologically a circle. At every point on the line, your index finger, thumb and a perpendicular through your palm give a frame of the tangent space to $\mathbf{P}^3$. This data describes a loop in SO_3 which is either trivial or non-trivial. Alternatively, this data describes a loop in the frame bundle which can either lift or not lift to its double cover corresponding to the nontrivial loop in SO_3.

Example 4. *The three real lines on the Fermat Cubic Surface are hyperbolic. You can check this with the above procedure and Diagram 2, which consists of two views of the real points of Fermat Cubic Surface. The three real lines lie in a plane and the regions of the surface surrounded by the lines are either entirely above the plane (these are marked with a $+$) or entirely below the plane (these are marked with a $-$).*

A more algebraic approach to define elliptic and hyperbolic lines uses two Segre points of the line over the algebraic closure where the tangent space "pauses." More explicitly, suppose L is a real line on X. We define an involution $I : L \to L$ as follows. Suppose x is a point of L. The tangent space T_xX is a 2-dimensional plane, and the intersection $T_xX \cap X$ of the tangent space and the cubic surface is a degree 3 plane curve by Bézout's theorem. This intersection contains L, and therefore must also contain a degree 2 curve C, i.e.,

$$T_xX \cap X = L \cup C.$$

By Bézout's theorem, the intersection $L \cap C$ consists of two points, counted with multiplicity. Note that the tangent space to X at any point of $L \cap C$ contains T_xX, and therefore equals T_xX because X is smooth. The same reasoning works backwards, showing that $L \cap C$ consists of those points p of L such that $T_pX = T_xX$. Thus $L \cap C = \{x, y\}$ for some well-defined point y of L. Define I by $I(x) = y$. Choosing a coordinate on L gives an

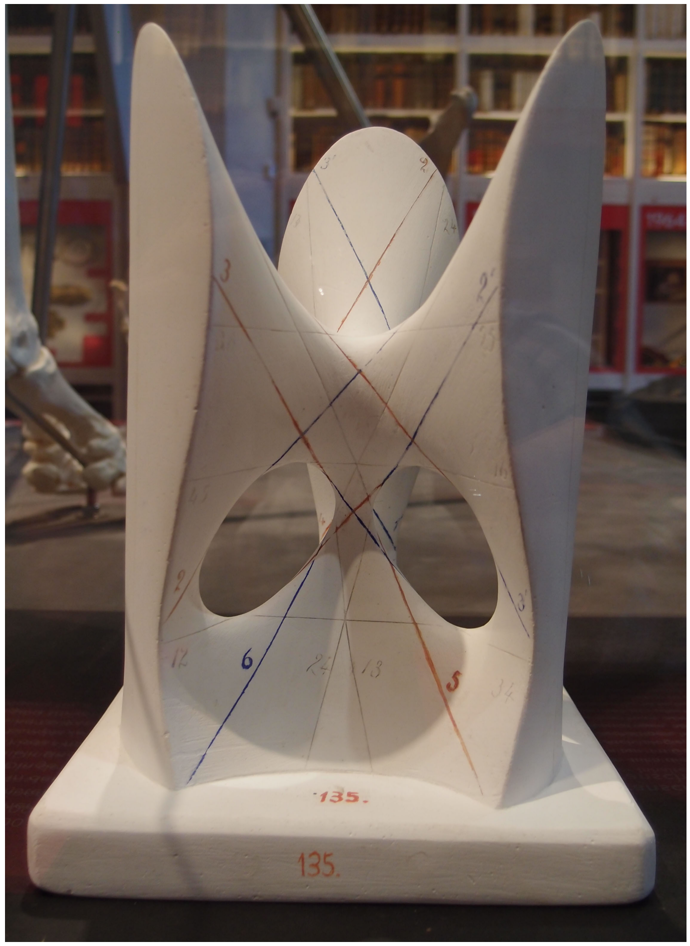

FIGURE 1. $\mathbf{R}$-points of Clebsch Cubic Surface. Model in the collection of mathematical models and instruments, Georg-August-Universität Göttingen. By Oliver Zauzig. Published under CC BY-SA 3.0 on universitaetssammlungen.de

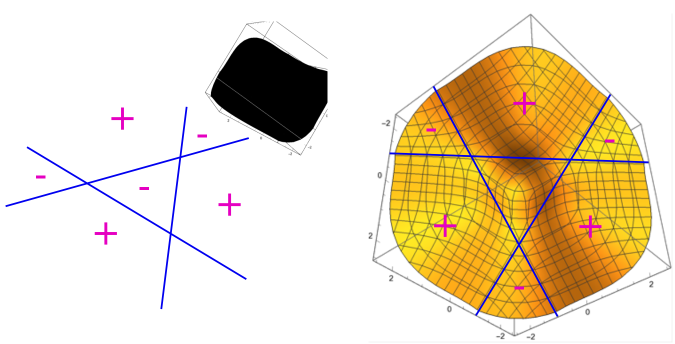

FIGURE 2. $\mathbf{R}$-points of Fermat Cubic Surface

isomorphism $\operatorname{Aut} L = \mathbf{P}\,\mathrm{GL}_2\,\mathbf{R}$, so I can be represented as a matrix

$$I = \begin{bmatrix} a & b \\ c & d \end{bmatrix}, \tag{2}$$

or as the Möbius transformation $z \mapsto \frac{az+b}{cz+d}$. I has two fixed points, namely the roots of the equation $cz^2 + (d-a)z - b = 0$ with real coefficients. These roots are thus either both real or a complex conjugate pair of points. In the first case, the line L is hyperbolic and in the second case the line is elliptic.

Remark 5. *For future reference, note that this procedure generalizes to fields other than* $\mathbf{R}$*. Let* X *be a smooth cubic surface over a field* k*, of characteristic not 2. Let* L *be a line on* X *defined over a field extension* $k(L)$ *of* k*. Then there is an associated involution* I *in* $\mathrm{Aut}\, L \cong \mathbf{P}\,\mathrm{GL}_2\, k(L)$*.*

Although the number of lines on a real cubic surface depends on the surface, a certain signed count does not:

Theorem 6. *[Segre, Benedetti–Silhol, Okonek–Teleman, Finashin–Kharlamov, Horev–Solomon] Let* X *be a smooth cubic surface over* $\mathbf{R}$*. Let* h *denote the number of hyperbolic lines. Let* e *denote the number of elliptic lines. Then*

$$h - e = 3.$$

Interestingly, although Theorem 6 follows from the work of Segre, it does not seem to have been noticed by him [OT14]. The proofs in [OT14] and [FK13] are along similar lines to the proof given above for the complex case (but harder). They replace the complex Grassmannian with the real one, and again compute the Euler class of $\mathrm{Sym}^3 \mathcal{S}^* \to \mathrm{Gr}(1,3)(\mathbf{R})$. The local index $\deg_L \sigma_f$ is shown to be $+1$ (respectively -1) when L is hyperbolic (respectively elliptic), and then the Euler class is computed to be 3. They obtain an analogous result for hypersurfaces of degree $2n-3$ in $\mathbf{P}^n$. Note that the Euler class computation of 3 for $\mathrm{Sym}^3 \mathcal{S}^* \to \mathrm{Gr}(1,3)(\mathbf{R})$ follows from Example 4. The proof in [BS95] uses (s)pin structures, and that of [HS12] uses Gromov–Witten theory.

There is a general principle that a result which is true over $\mathbf{C}$ and $\mathbf{R}$ may be a result in $\mathbf{A}^1$-homotopy theory. $\mathbf{A}^1$-homotopy theory was developed by Morel and Voevodsky in the late 1990's [MV99], and it allows us to treat smooth schemes like manifolds in certain respects. In some sense, there are small spheres equivalent to $\mathbf{P}^n/\mathbf{P}^{n-1}$ around any point of a smooth scheme. Morel defined a degree homomorphism

$$\deg : [\mathbf{P}^n/\mathbf{P}^{n-1}, \mathbf{P}^n/\mathbf{P}^{n-1}] \to \mathrm{GW}(k), \tag{3}$$

whose target is the Grothendieck–Witt group of bilinear forms over a field. Elements of $\mathrm{GW}(k)$ are formal differences of symmetric, non-degenerate, k-valued bilinear forms on finite dimensional vector spaces. We will discuss $\mathrm{GW}(k)$ in detail in Section 2.3, but we claim that this group is both interesting and computable. Since such forms can be (stably) diagonalized, $\mathrm{GW}(k)$ is generated by elements $\langle a \rangle$ for a in $k^*/(k^*)^2$, which correspond to the isomorphism classes of the rank one bilinear forms $k \times k \to k$ defined $(x,y) \mapsto axy$.

Example 7. $\mathrm{GW}(\mathbf{C}) \cong \mathbf{Z}$*. An isomorphism is given by the* rank *homomorphism, taking a bilinear form on a vector space* V *to the dimension of* V*.*

Example 8. *A bilinear form over* $\mathbf{R}$ *can be diagonalized so only* 1*'s and* -1*'s appear on the diagonal. The* signature *is the number of* 1*'s minus the number of* -1*'s.*

$$\mathrm{GW}(\mathbf{R}) \cong \mathbf{Z} \times \mathbf{Z}.$$

An isomorphism is induced by the rank and signature*.*

Example 9.

$$\mathrm{GW}(\mathbf{F}_q) \cong \mathbf{Z} \times \mathbf{F}_q^*/(\mathbf{F}_q^*)^2.$$

An isomorphism is given by the rank and discriminant, *where the discriminant takes a bilinear form* $\beta : V \times V \to k$ *to the determinant of a Gram matrix* $\big(\beta(v_i, v_j)\big)_{i,j}$ *representing* β*, where* $\{v_1, \ldots, v_r\}$ *is a basis of* V*.*

GW also admits transfer maps

$$\mathrm{Tr}_{E/k} : \mathrm{GW}(E) \to \mathrm{GW}(k)$$

for finite degree field extensions $k \subseteq E$. When $k \subset E$ is a separable extension, these can be described using the trace map $\mathrm{Tr}_{E/k} : E \to k$ from Galois theory which takes an element of E to the sum of its Galois conjugates in k. Namely, for a bilinear form $\beta : V \times V \to E$ on an E-vector space V, the image $\mathrm{Tr}_{E/k}(\beta)$ of the isomorphism class of β under $\mathrm{Tr}_{E/k}$ is represented by the bilinear form

$$V \times V \xrightarrow{\beta} E \xrightarrow{\mathrm{Tr}_{E/k}} k$$

where V is now viewed as a k-vector space.

Morel's degree and Equation (1) can be used to define an Euler class in this context, which we will discuss in detail in Section 4. Then one can repeat the previous proof to count lines on a smooth cubic surface over a field k. A line L in $\mathbf{P}^3$ is determined by two linear equations $aw + bx + cy + dz = 0$ and $a'w + b'x + c'y + d'z = 0$. We will allow $a, b, c, d, a', b', c', d'$ to be elements of an algebraic extension of k, so a line determines a closed subscheme of $\mathbf{P}^3_{k(L)}$ isomorphic to $\mathbf{P}^1_{k(L)}$, where $k(L) = k[a, b, c, d, a', b', c', d']$. Equivalently, a line L is a closed point of the Grassmannian. A line L of X determines an involution I n $\mathrm{Aut}\, L \cong \mathbf{P}\,\mathrm{GL}_2\, k(L)$ as above. The two fixed points of I are either two $k(L)$-points or a conjugate pair of points defined over some quadratic extension $k(L)[\sqrt{D}]$ of $k(L)$ for a unique D in $k(L)^*/(k(L)^*)^2$.

Definition 10. *The* type *of* L *is* $\mathrm{Type}(L) = \langle D \rangle$ *in* $\mathrm{GW}(k(L))$

It can be shown that

$$\mathrm{Type}(L) = \langle -1 \rangle \deg I$$

and

$$\mathrm{Type}(L) = \langle ad - bd \rangle$$

when I is represented as the matrix in Equation (2).

The main theorem of [KW17] is:

Theorem 11. *[Kass – W.] Let* k *be a field of characteristic not 2. Let* X *be a smooth cubic surface over* k*. Then*

$$\sum_{\text{lines } L \text{ on } X} \mathrm{Tr}_{k(L)/k}\, \mathrm{Type}(L) = 15\langle 1 \rangle + 12\langle -1 \rangle.$$

Applying invariants of bilinear forms to Theorem 11 gives more traditional counts, such as the following.

- When k is $\mathbf{C}$, or an algebraically closed field, applying Rank shows Theorem 1, i.e., that there are exactly 27 lines on a smooth cubic surface over $\mathbf{C}$.
- When $k = \mathbf{R}$, applying the signature shows Theorem 6. In more detail: a line L on a real cubic surface X is either defined over $\mathbf{R}$ or $\mathbf{C}$. If $k(L) = \mathbf{C}$, then $\mathrm{Type}(L) = \langle 1 \rangle$. Then $\mathrm{Tr}_{k(L)/k} \mathrm{Type}(L) = \langle 1 \rangle + \langle -1 \rangle$. If $k(L) = \mathbf{R}$, then $\mathrm{Type}(L) = \langle 1 \rangle$ if L is hyperbolic and $\mathrm{Type}(L) = \langle -1 \rangle$ if L is elliptic.

Over a finite field $k = \mathbf{F}_q$, a line on a smooth cubic surface is defined over some finite field extension $k(L) = \mathbf{F}_{q^d}$. Since $k(L)^*/(k(L)^*)^2 \cong \mathbf{Z}/2$ consists of two elements, it again makes sense to use Segre's elliptic/hyperbolic terminology, i.e., if $\mathrm{Type}(L) = \langle 1 \rangle$, then L is hyperbolic and if $\mathrm{Type}(L) = \langle u \rangle$, where u is the non-identity element of $k(L)^*/(k(L)^*)^2$, then L is elliptic. Applying the discriminant to Theorem 11 gives the following corollary.

Corollary 12. *Let X be a smooth cubic surface over the finite field $\mathbf{F}_q$, with q odd. Then*

$$|\{\textit{elliptic lines defined over } \mathbf{F}_{q^d} : d \textit{ is odd}\}| + |\{\textit{hyperbolic lines defined over } \mathbf{F}_{q^d} : d \textit{ is even}\}| \equiv 0 \mod 2$$

Many enumerative problems over $\mathbf{C}$ can be solved using degrees, characteristic classes, and intersection theory. A lot of wonderful examples are in [EH16] and [Ful98]. Such tools can be defined using homotopy theory. In the classical case, the homotopy theory topological spaces suffices, and to include the desired arithmetic information, we will use $\mathbf{A}^1$-homotopy theory. These notes will introduce enough $\mathbf{A}^1$-homotopy theory to do some first enumerative applications, and then give these applications, followed by project suggestions. This is a new direction of study, including contributions from Candace Bethea [BKW18], Marc Hoyois [Hoy14], Jesse Kass [KW16] [KW17], Marc Levine [Lev17b] [Lev17a] [Lev18a] [Lev18b], Padmavathi Srinivasan [SW18], Matthias Wendt [Wen18], and the lecturer.

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2. Lecture 2: User's guide to $\mathbf{A}^1$-homotopy theory for $\mathbf{A}^1$-enumerative geometry

2.1. **Spaces.** We will use Morel–Voevodsky's $\mathbf{A}^1$-homotopy theory of schemes. In this theory, we allow certain topological operations to be performed on schemes such as gluing, collapsing sub complexes, and finding small spheres around points of smooth schemes. The first two of these is accomplished by adding colimits into schemes. The last is the Purity Theorem of Morel–Voevodsky. It's proof requires properties of the étale or Nisnevich topology, as well as forcing $\mathbf{A}^1$ to be contractible, so it plays the role of the interval $[0,1]$ in classical homotopy theory. Thus, we will add colimits to schemes in such a way that existing colimits coming from open covers remain colimits, and we will then force $\mathbf{A}^1$ to be contractible. These steps are all that is required to form the $\mathbf{A}^1$-homotopy theory of schemes. For the present purposes, it will mostly suffice to view the existence of this theory as a license to treat schemes like topological spaces and smooth schemes like manifolds. In more detail and fairness, we have the following.

The *étale topology* is the Grothendieck topology on smooth k-schemes where a map $U = \coprod_\alpha U_\alpha \to X$ is an *étale cover* if $U \to X$ is étale (meaning for every u in U mapping to x in X the induced map on tangent spaces $T_u U \to T_x X$ is an isomorphism), and surjective on points.

The *Nisnevich topology* is defined so that covers are étale covers satisfying the additional property that for every x in X there is a u in U mapping to x such that the induced map on residue fields $k(x) \to k(u)$ is an isomorphism.

A homotopy theory could mean a simplicial model category. Such a category is enriched in simplicial sets, and has morphisms distinguished as weak equivalences, cofibrations, and fibrations. The category of simplicial sets will be denoted by $\mathbf{sSet}$ and can be thought of as the category of topological spaces. A homotopy theory could alternatively mean an infinity category or quasi-category, which is a simplicial set satisfying an inner horn filling condition. In both contexts there is an associated homotopy category, but one can do more things with the homotopy theory.

Let $\mathbf{Sm}_k$ denote the full subcategory of schemes over k with objects the smooth schemes over k. Then Yoneda embedding gives a functor $\mathbf{Sm}_k \to \mathrm{Fun}(\mathbf{Sm}_k^{\mathrm{op}}, \mathbf{Set}) \to \mathrm{Fun}(\mathbf{Sm}_k^{\mathrm{op}}, \mathbf{sSet})$ to the category of presheaves of simplicial sets on smooth schemes over k. The category $\mathrm{Fun}(\mathbf{Sm}_k^{\mathrm{op}}, \mathbf{sSet})$ can be equipped with the structure of a simplicial model category of infinity category. The passage from $\mathbf{Sm}_k$ to $\mathrm{Fun}(\mathbf{Sm}_k^{\mathrm{op}}, \mathbf{sSet})$ can be thought of as freely adjoining colimits to $\mathbf{Sm}_k$.

There is a formal process called *Bousfield localization* that allows us to force a chosen class of morphisms in a homotopy theory to be weak equivalences. We use this to make the colimits (or gluings) in $\mathbf{Sm}_k$ resulting from open (Nisnevich) covers remain colimits in the localization of $\mathrm{Fun}(\mathbf{Sm}_k^{\mathrm{op}}, \mathbf{sSet})$. Given an open cover $U = \coprod_\alpha U_\alpha \to X$, we obtain a map $\mathrm{Cosk}_X^0 U \to X$ in $\mathrm{Fun}(\mathbf{Sm}_k^{\mathrm{op}}, \mathbf{sSet})$ where $\mathrm{Cosk}_X^0 U$ is the Čech nerve of the cover. Bousfield localizing at these morphisms produces a new homotopy theory that we will denote $\mathbf{Sh}_k$ for "sheaves" and a localization functor

$$\mathrm{L}_{\mathrm{Nis}} : \mathrm{Fun}(\mathbf{Sm}_k^{\mathrm{op}}, \mathbf{sSet}) \to \mathbf{Sh}_k.$$

We Bousfield localize one more so that the maps $X \times \mathbf{A}^1 \to X$ are weak equivalences for all smooth k-schemes X. We obtain the desired $\mathbf{A}^1$ homotopy theory, which we will denote $\mathbf{Spc}_k$ for "spaces" and a localization functor

$$\mathrm{L}_{\mathbb{A}^1} : \mathbf{Sh}_k \to \mathbf{Spc}_k.$$

In summary, we construct $\mathbf{A}^1$-homotopy theory by

$$\mathbf{Sm}_k \xrightarrow{\text{Yoneda}} \mathrm{Fun}(\mathbf{Sm}_k^{\mathrm{op}}, \mathbf{sSet}) \xrightarrow{\mathrm{L}_{\mathrm{Nis}}} \mathbf{Sh}_k \xrightarrow{\mathrm{L}_{\mathbb{A}^1}} \mathbf{Spc}_k.$$

2.2. **Spheres, Thom spaces, and Purity.** Will follow [WW19, 2.3 and 2.4].

2.3. **Grothendieck–Witt group** $\mathrm{GW}(k)$ **and Milnor–Witt** K**-theory.**

Will follow [WW19, 4.1 and 4.3].

2.4. **Degree.**

2.5. **Oriented Chow Groups or Chow–Witt Groups.**

One can also define generalized cohomology theories and spectra in $\mathbf{A}^1$-homotopy theory. Motivic cohomology recovers the Chow group CH^i of a smooth scheme X by the formula $\mathrm{H}^{2i}(X, \mathbf{Z}(i)) \cong \mathrm{CH}^i(X)$. As Chow groups are useful in classical intersection theory and enumerative geometry, we first say a word about them.

Let X be a smooth scheme over k of dimension d. Let $X^{(i)}$ denote the set of subvarieties of X of codimension i, i.e. reduced irreducible subschemes of dimension $d-i$. The group of cycles on X is the free abelian group generated by $X^{(i)}$. Rational equivalence is the equivalence relation generated by declaring $V \cap (X \times \{0\})$ equivalent to $V \cap (X \times \{1\})$ for any subvariety V of $X \times \mathbf{P}^1$, and the Chow group $\mathrm{CH}^i(X)$ is the quotient of the group of cycles of codimension i by rational equivalence. More information on Chow groups is available in, for example, [EH16].

Intersection theory over $\mathbf{C}$ is simplified by the existence of canonical orientations. Oriented Chow groups $\widetilde{\mathrm{CH}}^i$ input information about orientations to produce a more refined theory. They are also called Chow–Witt groups. They were introduced by Barge and Morel in [BM00] and further developed by Fasel [Fas08]. Elements of $\widetilde{\mathrm{CH}}^i$ can be represented by certain formal sums of subvarieties Z of codimenison i equipped with an element β of $\mathrm{GW}(k(Z))$.

In analogy with Bloch's formula $\mathrm{CH}^i \cong \mathrm{H}^i(X, \mathrm{K}_i^{\mathrm{M}})$, oriented Chow groups can be defined by

$$\widetilde{\mathrm{CH}}^i(X) = \mathrm{H}^i(X, \mathrm{K}_i^{\mathrm{MW}}).$$

For example, $\widetilde{\mathrm{CH}}^0(\operatorname{Spec} k) \cong \mathrm{GW}(k)$. This cohomology can be computed by the Rost–Schmidt complex described in [Mor12, 5]. Let E be a field extension of k. For a 1-dimensional E-vector space Λ, let

$$\mathrm{K}_n^{\mathrm{MW}}(E; \Lambda) := \mathrm{K}_n^{\mathrm{MW}}(E) \otimes_{\mathbf{Z}[E^*]} \mathbf{Z}[\Lambda^*].$$

The Rost–Schmidt complex gives $\mathrm{H}^i(X, \mathrm{K}_i^{\mathrm{MW}})$ as the kernel mod the image of a sequence

(4)

$$\to \oplus_{z \in X^{(i-1)}} \mathrm{K}_1^{\mathrm{MW}}(k(z), \det_{k(z)} T_z X) \to \oplus_{z \in X^{(i)}} \mathrm{GW}(k(z), \det_{k(z)} T_z X) \to \oplus_{z \in X^{(i+1)}} \mathrm{K}_{-1}^{\mathrm{MW}}(k(z), \det_{k(z)} T_z X) \to$$

For a line bundle L on X, there is a sheaf $\mathrm{K}_n^{\mathrm{MW}}(L)$ on X constructed from the groups $\mathrm{K}_n^{\mathrm{MW}}(E; L(E))$, where E is the function field of a smooth scheme mapping to X. Using this sheaf, we define the oriented Chow groups $\widetilde{\mathrm{CH}}^i(X, L)$ twisted by a line bundle L by

$$\widetilde{\mathrm{CH}}^i(X, L) = \mathrm{H}^i(X, \mathrm{K}_i^{\mathrm{MW}}(L)).$$

The Rost–Schmidt complex becomes

$$\to \oplus_{z\in X^{(i-1)}} K_1^{MW}(k(z), \det_{k(z)} T_z X \otimes_{k(z)} L(k(z))) \to \oplus_{z\in X^{(i)}} GW(k(z), \det_{k(z)} T_z X \otimes_{k(z)} L(k(z))) \to$$
$$\oplus_{z\in X^{(i+1)}} K_{-1}^{MW}(k(z), \det_{k(z)} T_z X \otimes_{k(z)} L(k(z))) \to$$

Since squares act trivially on K_n^{MW}, there is a canonical isomorphism

$$\widetilde{CH}^i(X, M \otimes L^{\otimes 2}) \cong \widetilde{CH}^i(X, M)$$

for line bundles M and L.

There are pullback and proper pushfoward maps on oriented Chow groups. For a map $f : X \to Y$ in $\mathbf{Sm}_k$ we have a pullback map $f^* : \widetilde{CH}^i(Y, L) \to \widetilde{CH}^i(X, f^*L)$. The canonical line bundle $\omega_{X/k}$ on a smooth scheme X is $\omega_{X/k} = \det T^*X$ the determinant of the cotangent bundle. When f is proper of relative dimension $r = \dim X - \dim Y$, there is a pushfoward map $f_* : \widetilde{CH}^i(X, \omega_{X/k} \otimes f^*L) \to \widetilde{CH}^{i-r}(Y, \omega_{Y/k} \otimes L)$.

There is a (non-commutative to pick up on orientations!) ring structure on oriented Chow groups $\oplus_i \widetilde{CH}^i(X, L^{\otimes i})$ [Fas07], giving an oriented intersection theory.

3. Lecture 3: Local degree

3.1. **Local degree.**

3.2. **Eisenbud–Khimshiashvili–Levine signature formula.**

3.3. **$\mathbf{A}^1$-Milnor numbers.**

4. Lecture 4: Euler class

Let X be a connected $\mathbf{R}$-manifold of dimension d and let $V \to X$ be a rank r vector bundle on X.

V is *oriented* by the choice of a *Thom class* u in $H^r(\mathrm{Th}(V), \mathbf{Z})$ such that for each x in X the pullback of u to the Thom space $\mathrm{Th}(V_x)$ of the fiber V_x of V at x generates $H^r(\mathrm{Th}(V_x), \mathbf{Z}) \cong \mathbf{Z}$. The manifold X is oriented if its tangent bundle TX is.

Given an oriented vector bundle $V \to X$, the *Euler class* $e(V)$ in $H^r(X, \mathbf{Z})$ is the pullback of u by any section $s : X \to V$, for example, s could be the zero section,

$$e(V) = S^*u.$$

When X is oriented of dimension $d = r$ equal to the rank of V, Poincaré duality provides an isomorphism $H^r(X, \mathbf{Z}) \cong \mathbf{Z}$, and we may view $e(V)$ as an integer.

$\mathbb{A}^1$-enumerative geometry: k field

Goal to record arithmetic information about geometric objects whose # is fixed over $\bar{k}$, (but not over k)

tool: $\mathbb{A}^1$-htpy thy

L1: User's guide to $\mathbb{A}^1$ htpy thy

$$S^n = \{(x_0,\dots,x_n) \mid \sum x_i^2 = 1\} = \mathbb{P}^n(\mathbb{R})/\mathbb{P}^{n-1}(\mathbb{R})$$

$$\deg : \underbrace{[S^n, S^n]}_{\text{pointed homotopy classes}} \longrightarrow \mathbb{Z}$$

Given $f: S^n \longrightarrow S^n$ and $p \in S^n$ s.t. $f^{-1}(y) = \{q_1, \dots, q_N\}$

$$\deg f = \sum_{i=1}^{N} \deg_{q_i} f$$

where $\deg_{q_i} f$ is the local degree:

Let $V \ni p$ be a small ball

Let $f^{-1}(V) \supset U \ni q_i$ be a small ball s.t. $f^{-1}(p) \cap U = q_i$

$$\begin{array}{ccc} U/\partial U \simeq S^n & & V/\partial V \simeq S^n \\ \wr\! \| & & \wr\! \| \\ U/(U - q_i) & \xrightarrow{\bar{f}} & V/(V - p) \end{array}$$

$\deg_{q_i} f := \deg \bar{f}$

Formula from differential topology

Let $(x_1, \ldots, x_n)$ be oriented coordinates near q_i

Let $(y_1, \ldots, y_n)$ —— " —— p

Then $f = (f_1, \ldots, f_n) : \mathbb{R}^n \longrightarrow \mathbb{R}^n$

Let $J = \det\left(\frac{\partial f_i}{\partial x_j}\right)$

$$\deg_{q_i} f = \begin{cases} +1 & J(q_i) > 0 \\ -1 & J(q_i) < 0 \\ ? & J(q_i) = 0 \end{cases}$$

Eisenbud - H. Levine / Khimshiashvili signature formula

Lannes / Morel : k field

degree for $f : \mathbb{P}^1 \longrightarrow \mathbb{P}^1$
valued in $GW(k)$

$GW(k)$ = Grothendieck–Witt group of k

:= group completion under $\oplus$ of semi-ring $\otimes \oplus$
of non-degenerate, symmetric,
bilinear forms

generators: $\langle a \rangle \qquad a \in k^*/(k^*)^2$

$\langle a \rangle$ element associated to bilinear form

$$\langle a \rangle : k \times k \longrightarrow k$$
$$(x, y) \longmapsto axy$$

relations: (1) $\langle a \rangle \langle b \rangle = \langle ab \rangle$

(2) $\langle u \rangle + \langle v \rangle = \langle uv(u+v) \rangle + \langle u+v \rangle \qquad u+v \neq 0$

(3) $\langle u \rangle + \langle -u \rangle = \langle 1 \rangle + \langle -1 \rangle = \mathbb{H}$ ← hyperbolic form

Exercise: (2) $\Rightarrow$ (3) $\qquad$ char $k \neq 2$

$W(k) = \dfrac{GW(k)}{\mathbb{Z}\mathbb{H}}$

Ex: $GW(\mathbb{C}) \underset{\cong}{\xrightarrow{\text{rank}}} \mathbb{Z}$

$(B : V \times V \to \mathbb{C}) \mapsto \dim V$

$\langle a \rangle \mapsto 1$

Ex: $GW(\mathbb{R}) \cong \mathbb{Z} \times \mathbb{Z}$

Silvester's theorem

$B : V \times V \to \mathbb{C}$, there is a basis $\{v_1, \dots, v_r\}$ of V s.t.

Gram matrix $B(v_i, v_j)$ is
$$\begin{bmatrix} 1 & & & & & \\ & \ddots & & & & \\ & & 1 & & & \\ & & & -1 & & \\ & & & & \ddots & \\ & & & & & -1 \end{bmatrix}$$

Signature $B = \#1\text{'s} - \#(-1)\text{'s}$

$$\begin{array}{ccc} GW(\mathbb{R}) & \xrightarrow{\text{rank} \times \text{sign}} & \mathbb{Z} \times \mathbb{Z} \\ & \searrow^{\cong} & \cup \\ & & \{(r,s) \in \mathbb{Z} \times \mathbb{Z} \mid r+s \equiv 0 \bmod 2\} \cong \mathbb{Z} \times \mathbb{Z} \end{array}$$

<u>Ex</u>: $GW(\mathbb{F}_q) \xrightarrow[\cong]{\text{rank} \times \text{disc}} \mathbb{Z} \times \mathbb{F}_q^* / (\mathbb{F}_q^*)^2$

<u>Ex</u>: (Springer's theorem) K complete discrete valued field
k residue field

e.g. $K = \mathbb{Q}_p$ $K = \mathbb{F}_p((t))$, $k = \mathbb{F}_p$

Assume char $k \neq 2$

$$GW(K) \cong \frac{GW(k) \oplus GW(k)}{\mathbb{Z}(\mathbb{H}, -\mathbb{H})}$$

Back to Lannes's formula:

$f: \mathbb{P}^1_k \to \mathbb{P}^1_k$ $p \in \mathbb{P}^1(k)$ $f^{-1}(p) = \{q_1, \dots, q_N\}$

s'pose $J(q_i) = f'(q_i) \neq 0$ $\forall i$

$$\deg f = \sum_{i=1}^{N} \langle J(q_i) \rangle$$

This doesn't depend on p!

<u>Exercise</u>: 1) $\deg\left(\mathbb{P}^1 \xrightarrow[z\mapsto az]{[x:y]\mapsto[ax:y]} \mathbb{P}^1\right) = \langle a \rangle$

2) $\deg\left(\mathbb{P}^1 \xrightarrow[z\mapsto z^2]{} \mathbb{P}^1\right) = \langle 1 \rangle + \langle -1 \rangle$

Cazanavé: studied naïve homotopy classes of maps $\mathbb{P}^1 \to \mathbb{P}^1$ (Karoubi–Villamayor)

$\mathbb{A}^1 = \operatorname{Spec} k[t]$ replaces $[0,1]$

<u>Def</u>: A naïve homotopy h between to maps of schemes

$f, g : X \to Y$ is a map

$h : X \times \mathbb{A}^1 \longrightarrow Y$ s.t. $h|_{X\times\{0\}} = f$ and $h|_{X\times\{1\}} = g$

maps $\mathbb{P}^1 \to \mathbb{P}^1$ are rational functions $\frac{f}{g}$

$\deg\left(\frac{f}{g}\right) = \max(\deg f, \deg g)$ relatively prime

Bézout associated a bilinear form $\text{Béz}\left(\frac{f}{g}\right)$ to f/g constructed as follows:

$$\frac{f(X)g(Y) - f(Y)g(X)}{X - Y} = \sum_{1 \le i,j \le \deg\left(\frac{f}{g}\right)} B_{ij} X^{i-1} Y^{j-1}$$

$\deg\left(\frac{f}{g}\right) = n$

$\text{Béz}(f/g) : k^n \times k^n \longrightarrow k$

Gram matrix $(\text{B\'ez}(f/g)) = [B_{ij}]$

__Exercise__: (1) $\text{B\'ez}(\mathbb{P}^1 \xrightarrow{z \mapsto z^n} \mathbb{P}^1) = \begin{bmatrix} & & 1 \\ & \iddots & \\ 1 & & \end{bmatrix} \simeq \begin{cases} \frac{n}{2}\mathbb{H} & n \text{ even} \\ \frac{n-1}{2}\mathbb{H} + \langle 1 \rangle & n \text{ odd} \end{cases}$

(2) $\text{B\'ez}(\mathbb{P}^1 \xrightarrow{z \mapsto az^n} \mathbb{P}^1) \quad a \in k^*$

$$\boxed{\text{B\'ez}\left(\frac{f}{g}\right) = \deg\left(\frac{f}{g}\right)}$$

Cazanave gave formula for addition of naive homotopy classes of maps and showed $[\mathbb{P}^1, \mathbb{P}^1]_N \to [\mathbb{P}^1, \mathbb{P}^1]_{\mathbb{A}^1}$ is a group completion

__Morel__: $\deg^{\mathbb{A}^1}: [\mathbb{P}^n/\mathbb{P}^{n-1}, \mathbb{P}^n/\mathbb{P}^{n-1}]_{\mathbb{A}^1} \longrightarrow GW(k)$

To make sense of this we need: $\mathbb{P}^n/\mathbb{P}^{n-1}$

$[-,-]_{\mathbb{A}^1}$

$\mathbb{P}^n/\mathbb{P}^{n-1}$ is $\operatorname{colim}\left(\begin{tikzcd} \mathbb{P}^{n-1} \arrow[r] \arrow[d] & \mathbb{P}^n \\ * & \end{tikzcd} \right) \simeq \mathbb{P}^n/\mathbb{P}^{n-1}$

i.e. maps out of $\mathbb{P}^n/\mathbb{P}^{n-1}$ are maps out of $\mathbb{P}^n$ & $*$ which agree on $\mathbb{P}^{n-1}$

__ex__: $\begin{tikzcd} U \cap V \arrow[r] \arrow[d] & U \arrow[d] \\ V \arrow[r] & U \cup V \end{tikzcd} \simeq$ colim or pushout

But colimits don't always exist in Schemes

We want colimits and we want colimits of $\simeq$ schemes to produce $\simeq$ colimits. So colimit $\rightsquigarrow$ hocolim

Sm_k = smooth schemes $/k$

simplicial sets or think top spaces

$Pre(Sm_k)$ = Presheaves on Sm_k = Functors $(Sm_k^{op}, sSet)$

$$Sm_k \xrightarrow{\text{Yoneda}} Pre(Sm_k)$$ has colimits

$$X \longmapsto Map(-, X)$$

homotopy theory: Simplicial model category
or
∞-cat (= quasicategory)

notion of weak equivalence or homotopy equivalence

$$Sm_k \xrightarrow{\text{Yoneda}} Pre(Sm_k)$$

freely adding colimits.

Problem: We want colimits from "open covers" of schemes to be colimits.

Notion of an "open cover" of a Scheme is a Grothendieck topology

Let τ be a Grothendieck topology

Fix: Bousfield localization imposes additional w.e.

$$\begin{tikzcd} Sm_k \arrow[r] & Pre(Sm_k) \arrow[r, "L_\tau"'] & Sh_k \arrow[l, bend right, "T"'] \arrow[r, "L_{\mathbb{A}^1}"'] & Spc_k \arrow[l, bend right, "T"'] \end{tikzcd}$$

$\mathrm{Cosk}^0 \coprod_\alpha U_\alpha \to X$ is a weak equivalence

$X \times \mathbb{A}^1 \to X$ is a weak equivalence

Topologies: Zariski, Nisnevich, étale

$\mathbb{A}^1$-homotopy thy

$\longrightarrow$ more open sets

Def: $f: X \to Y$ is étale at x if $(f^* \Omega_{Y/k})_x \xrightarrow{\sim} \Omega_{X/k,x}$

cotangent space

$\iff$ $\mathrm{Jac}\, f \neq 0$ in $k(x)$

Def: $\coprod_{\alpha \in A} U_\alpha \to X$ is an étale cover if it is étale and surjective

Def: $\coprod_{\alpha \in A} U_\alpha \to X$ is a Nisnevich cover if it is étale, surjective, and for every $x \in X$ $\exists$ $\alpha \in A$ and $y \in U_\alpha$ s.t. $y \mapsto x$ and $k(x) \xrightarrow{\cong} k(y)$

Facts: • Any $Z \hookrightarrow X$ in Sm_k is Nisnevich locally equivalent to $\mathbb{A}^d_k \hookrightarrow \mathbb{A}^{d+c}_k$ (like étale top)

- Nisnevich coh can be computed using Čech cochains (like étale top)

- Alg K-thy satisfies Nisnevich descent (like Zariski top)

- Nis coh dim = Krull dim (like Zariski top)

Notation: $X \wedge Y = X \times Y / (X \times *) \cup (* \times Y)$

ex: In top, $S^n \wedge S^m \simeq S^{n+m}$

L2: Local degree and $\mathbb{A}^1$-Milnor numbers

Def Given pointed spaces X and Y, the *smash product* is $X \wedge Y = X \times Y / (X \times *) \cup (* \times Y)$

Ex: In classical alg top, $S^n \wedge S^m \simeq S^{n+m}$

Spheres:

$$\mathbb{G}_m = \operatorname{Spec} k[\tfrac{1}{z}, z] = \mathbb{A}^1 - \{0\}$$

$$S^{p+q\alpha} = (S^1)^{\wedge p} \wedge (\mathbb{G}_m)^{\wedge q} \simeq S^{p+q, q}$$

Ex:

$$\begin{tikzcd}
\mathbb{G}_m \arrow[r, "z \mapsto z"] \arrow[d, "z \mapsto \frac{1}{z}"'] & \mathbb{A}^1 \simeq * \arrow[d, "\cap"] \\
\mathbb{A}^1 \simeq * \arrow[r, "\subseteq"] & \mathbb{P}^1
\end{tikzcd}$$

pushout $\Rightarrow$ $\Sigma \mathbb{G}_m \simeq \mathbb{P}^1$, where $\Sigma \mathbb{G}_m = S^1 \wedge \mathbb{G}_m$

Ex: $\mathbb{A}^n - \{0\} \simeq (S^1)^{\wedge n-1} \wedge (\mathbb{G}_m)^{\wedge n}$

pf: Induction and

$$\begin{tikzcd}
(\mathbb{A}^1 - \{0\}) \times (\mathbb{A}^{n-1} - \{0\}) \arrow[r] \arrow[d] & \mathbb{A}^1 \times (\mathbb{A}^{n-1} - \{0\}) \simeq \mathbb{A}^{n-1} - \{0\} \arrow[d] \\
(\mathbb{A}^1 - \{0\}) \times \mathbb{A}^n \arrow[r] & \mathbb{A}^n - \{0\}
\end{tikzcd}$$

pushout

$$(\mathbb{A}^1 - \{0\}) \times \mathbb{A}^n \simeq \mathbb{A}^1 - \{0\}$$

$$\begin{tikzcd}
X \times Y \arrow[r] \arrow[d] & X \arrow[d] \\
Y \arrow[r] & \Sigma X \wedge Y = S^1 \wedge X \wedge Y
\end{tikzcd}$$

pushout

$$\Rightarrow \quad \mathbb{A}^n - \{0\} \simeq \Sigma (\mathbb{A}^n - \{0\}) \wedge (\mathbb{A}^1 - \{0\})$$

<u>Ex</u>: $\mathbb{P}^n / \mathbb{P}^{n-1} \simeq (S^1)^{\wedge n} \wedge (\mathbb{G}_m)^{\wedge n}$

<u>pf</u>: $\mathbb{P}^n / \mathbb{P}^{n-1} \simeq \mathbb{P}^n / \mathbb{P}^n - \{0\} \simeq \mathbb{A}^n / \mathbb{A}^n - \{0\}$

$$\simeq * / \mathbb{A}^n - \{0\} \simeq \operatorname{colim} \begin{tikzcd} \mathbb{A}^n - \{0\} \arrow[r] \arrow[d] & * \\ * & \end{tikzcd}$$

$$\simeq \Sigma (\mathbb{A}^n - \{0\})$$

Thom spaces: $V \longrightarrow X$ vector bundle

$X \hookrightarrow V$ zero section

trivial bundle

$$\mathrm{Thom}(V,X) = \mathrm{Th}(V) = X^V = V/_{V-X} \simeq_{\mathbb{A}^1} \frac{\mathbb{P}(V \oplus \mathcal{O})}{\mathbb{P}(V)}$$

Purity: (Morel-Voevodsky) Let $Z \hookrightarrow X$ be a closed immersion in Sm_k.

$$X/_{X-Z} \simeq_{\mathbb{A}^1} \mathrm{Th}(N_Z X)$$

where $N_Z X \longrightarrow Z$ is the normal bundle

Morel: $\deg^{\mathbb{A}^1} : [\mathbb{P}^n/\mathbb{P}^{n-1}, \mathbb{P}^n/\mathbb{P}^{n-1}] \longrightarrow GW(k)$

$$\begin{tikzcd}
{[S^n, S^n]} \arrow[d, "\deg"] & {[\mathbb{P}^n_{\mathbb{R}}/\mathbb{P}^{n-1}_{\mathbb{R}}, \mathbb{P}^n_{\mathbb{R}}/\mathbb{P}^{n-1}_{\mathbb{R}}]} \arrow[l, "\mathbb{R}\text{-pts}"'] \arrow[r, "\mathbb{C}\text{-pts}"] \arrow[d, "\deg^{\mathbb{A}^1}"] & {[S^{2n}, S^{2n}]} \arrow[d, "\deg"] \\
\mathbb{Z} & GW(\mathbb{R}) \arrow[l, "\text{signature}"'] \arrow[r, "\text{rank}"] & \mathbb{Z}
\end{tikzcd}$$

Local $\mathbb{A}^1$-degree: following joint work with Jesse Kass

Thanks to M. Hoyois and F. Morel!

Suppose $f: \mathbb{A}^n \longrightarrow \mathbb{A}^n$, $x \in \mathbb{A}^n(k)$,

x is isolated in $f^{-1}(f(x))$, meaning $\exists$ $U \subset \mathbb{A}^n$ Zariski open, $x \in U$ s.t. $f^{-1}(f(x)) \cap U = x$

The <u>local $\mathbb{A}^1$-degree</u> $\deg_x f$ of f at x is defined to be the degree of

$$\mathbb{P}^n/\mathbb{P}^{n-1} \simeq Th\, T_x\mathbb{A}^n \underset{\text{Purity}}{\simeq} U/U-\{x\} \xrightarrow{f} \mathbb{A}^n/\mathbb{A}^n-\{f(x)\} \simeq \mathbb{P}^n/\mathbb{P}^{n-1}$$

Now, drop the assumption that $k(x) = k$, but assume $k(f(x)) = k$

<u>Def</u>: $\deg_x f$ is the degree of the composite

$$\mathbb{P}^n_k/\mathbb{P}^{n-1}_k \to \mathbb{P}^n_k/\mathbb{P}^n_k-\{x\} \simeq U/U-\{x\} \xrightarrow{\bar{f}} \mathbb{A}^n/\mathbb{A}^n-f(x) \simeq \mathbb{P}^n_k/\mathbb{P}^{n-1}_k$$

<u>Prop</u>: These two definitions agree

<u>Prop</u>: (global degree is sum of local degrees) Let $f: \mathbb{P}^n_k \longrightarrow \mathbb{P}^n_k$ be a finite map s.t. $f^{-1}(\mathbb{A}^n_k) = \mathbb{A}^n_k$

Let $\bar{f}: \mathbb{P}^n/\mathbb{P}^{n-1} \longrightarrow \mathbb{P}^n/\mathbb{P}^{n-1}$ denote the induced map. Then for any $y \in \mathbb{A}^n(k)$

$$\deg \bar{f} = \sum_{x \in f^{-1}(y)} \deg_x f$$

Computing $\deg_x f$:

In top, $f: \cancel{S^n \to} S^n$ $\quad \deg_x f = \begin{cases} +1 & \operatorname{Jac} f(x) > 0 \\ -1 & \operatorname{Jac} f(x) < 0 \\ ? & \operatorname{Jac} f(x) = 0 \end{cases}$

$f: \mathbb{R}^n \longrightarrow \mathbb{R}^n$

Eisenbud-Levine/Khimshiashvili Signature formula :

$$\deg_0 f = \text{Signature } \omega^{EKL}$$

where ω^{EKL} is the isomorphism class of the following bilinear form:

$f = (f_1, \dots, f_n)$ $\qquad Q := \dfrac{\mathbb{R}[x_1, \dots, x_n]_0}{\langle f_1, \dots, f_n \rangle}$ $\quad$ finite dimensional complete intersection

$\Rightarrow Q$ Gorenstein, $\operatorname{Hom}_k(Q, k) \cong Q$ $\quad$ Better: distinguished iso, coming from distinguished socle element $\rightsquigarrow$ bilinear form

Explicitly: $\quad \operatorname{Jac} f = \det\left(\dfrac{\partial f_i}{\partial x_j}\right)$

Choose $\eta: Q \longrightarrow \mathbb{R}$ s.t. $\eta(\operatorname{Jac} f) = \dim_k Q$

$$\omega^{EKL}: Q \times Q \longrightarrow k \qquad \omega^{EKL}(a,b) = \eta(ab)$$

isomorphism class of ω^{EKL} does not depend on choice of η.

Ex: $f: \mathbb{A}^1 \to \mathbb{A}^1 \quad f(z) = z^2$

Eisenbud Q: ω^{EKL} defined over a field k, and acts as "degree." Does ω^{EKL} have top interpretation?

Thm: (Kass-W) $\quad \deg_x f = \omega^{EKL} \quad$ in $GW(k)$

reference for proof: "The local $\mathbb{A}^1$-Brouwer degree is the quadratic form of Eisenbud-Khimshiashvili-Levine"

Exercises: 1) $f: \mathbb{A}^2_k \to \mathbb{A}^2_k \quad f(x,y) = (4x^3, 2y)$
char $k \neq 2$

Compute $\deg_0 f$

2) f étale at $0 \Rightarrow \deg_0 f = \langle \text{Jac } f \rangle$

with descent data

f étale at $x \Rightarrow \deg_x f = \mathrm{Tr}_{k(x)/k} \langle \text{Jac } f \rangle$

$\mathbb{A}^1$-Milnor #'s

char $k \neq 2$

hypersurface singularity $p \in \{f=0\}$ bifurcates into nodes

$\cong x_1^2 + \dots + x_n^2 = 0$ over k^s

in family $f(x_1, \dots, x_n) + a_1 x_1 + \dots + a_n x_n = t$ parameterized by t, fibers are smooth or have nodes

$k = \mathbb{C}$

For any $(a_1, \dots, a_n)$ sufficiently close to 0

$$\#\text{ nodes in family} = \deg_p^{top} \operatorname{grad} f \quad \text{(Milnor)}$$

$$\| \; \mu_p \quad \text{"Milnor \#"}$$

Over k? char $k \neq 2$

nodes at k-rat'l points:

$$\text{type}(x_1^2 + a x_2^2 = 0) := \langle a \rangle$$

$$\text{type}\left(\sum a_i x_i^2 = 0\right) := \langle 2^n \prod a_i \rangle$$

node at p with $k(p) = L$

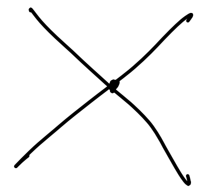
$$\text{type}\left(\sum a_i x_i^2 = 0\right) := \operatorname{Tr}_{L/k} \langle 2^n \prod a_i \rangle$$

Fact: $k \subseteq L$ is separable

Pictures $k=\mathbb{R}$ $\mathbb{R}$-rat'l nodes

$X_1^2+X_2^2=0$

non-split =
non-rat'l
tgt directions



$X_1^2-X_2^2=0$

split =
rat'l tgt directions

Thm (Kass-W)

For generic $(a_1,\dots,a_n)$

$$\sum_{\substack{\text{nodes } p \\ \text{in family}}} \operatorname{type}(p) = \deg_p^{\mathbb{A}^1} \operatorname{grad} f$$
$$\parallel$$
$$\mu_p^{\mathbb{A}^1}$$

"$\mathbb{A}^1$-Milnor #"

Ex $f(x,y)=x^3-y^2$ char $\neq 2,3$

$p=(0,0)\in\{f=0\}$

$\operatorname{grad} f=(3x^2,-2y)$

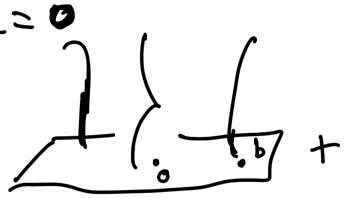
$\deg \operatorname{grad} f = \deg(3x^2)\deg(-2y)$

$$= \begin{array}{cc} & \begin{array}{cc} 1 & x \end{array} \\ \begin{array}{c} 1 \\ x \end{array} & \begin{bmatrix} 0 & \frac{1}{3} \\ \frac{1}{3} & 0 \end{bmatrix} \end{array} \langle -2\rangle = h \qquad \text{rank}=2$$

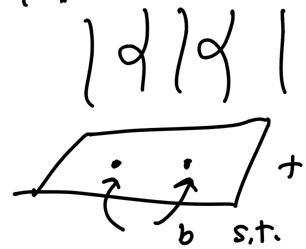
$=\langle 1\rangle+\langle -1\rangle$ So $\mu=2$

family: $y^2 = x^3 + ax + t$

$a = 0$



$a \neq 0$



s.t. Disc $x^3 + ax + t = 0$

$\Leftrightarrow \quad -4a^3 - 27b^2 = 0$

bifurcates into 2 nodes

over $\mathbb{F}_5$: $\langle 1 \rangle = \langle -1 \rangle \Rightarrow$ can't bifurcate into a split & non-split node

over $\mathbb{F}_7$: $\langle 1 \rangle \neq \langle -1 \rangle \Rightarrow$ can't bifurcate into 2 split or 2 non-split nodes

<u>Exercise</u> : compute $\mu^{\mathbb{A}^1}$ for "ADE singularities"

singularity	equation
A_n	$x_1^2 + x_2^{n+1}$
D_n	$x_2(x_1^2 + x_2^{n-2})$
E_6	$x_1^3 + x_2^4$
E_7	$x_1(x_1^2 + x_2^3)$
E_8	$x_1^3 + x_2^5$

which ones are <u>not</u> multiples of $\mathbb{H}$?

More generally, one can view $e(V)$ as an integer when $V \to X$ is *relatively oriented*, which we will define to mean that there is a line bundle L and an isomorphism $\mathrm{Hom}(\det TX, \det V) \cong L^{\otimes 2}$. Okonek and Teleman introduced this notation in [OT14]. To see why such a generalization should exist, here are a few general remarks.

The Thom space $\mathrm{Th}(V)$ is locally an r-fold suspension of the base, so $\mathrm{Th}(V)$ can be viewed as a twisted shift of of X itself. Among the manifestations of this viewpoint is the Thom isomorphism between the cohomology of the Thom space and a twisted shift of the cohomology of the base: the Serre Spectral Sequence gives a natural isomorphism

$$H^*(X, \mathcal{O}(V)) \cong \widetilde{H}^{*+r}(\mathrm{Th}(V), \mathbf{Z}),$$

where $\mathcal{O}(V)$ is the local system whose fiber at x is $H^r(V_x, V_x - \{0\})$, called the *orientation sheaf* of V.

Some formal properties of orientation sheaves are as follows. Let $\det V$ denote the highest wedge power of the vector bundle V, so $\det V = \wedge^r V$ because V is rank r. For a linear map $A : \mathbf{R}^r \to \mathbf{R}^r$, the degree of the induced map of spheres $\mathbf{R}^r - \{0\} \to \mathbf{R}^r - \{0\}$ is the sign of the determinant of A. It follows that there is a natural isomorphism

$$\mathcal{O}(V) \cong \mathcal{O}(\det V).$$

Because determinants multiply under tensor product, we obtain a natural isomorphism $\mathcal{O}(V_1 \otimes V_2) \cong \mathcal{O}(V_1) \otimes \mathcal{O}(V_2)$. Because the square of any integer is positive, we obtain a natural isomorphism $\mathcal{O}(V_1) \otimes \mathcal{O}(V_1) \cong \mathbb{Z}$ between the tensor square of the orientation sheaf $\mathcal{O}(V_1)$ and the constant local system. Combining the above, we have a natural isomorphism $\mathcal{O}(V) \cong \mathcal{O}(\det V \otimes L^{\otimes 2})$ for any line bundle $L \to X$. It follows that a relative orientation of V induces an isomorphism $\mathcal{O}(V) \cong \mathcal{O}(TX)$.

A virtual vector bundle is a formal difference of vector bundles, and one can define the Thom spectrum of a virtual vector bundle, with similar properties to a Thom space, for instance there is a Thom isomorphism in this context. Furthermore, for virtual vector bundles Y, W over X, there is a canonical map $\mathrm{Th}(Y) \to \mathrm{Th}(Y \oplus W)$. We therefore have a map

$$\mathrm{Th}(-V) \to \mathrm{Th}(-V \oplus V) \simeq X$$

which induces a map on cohomology $H^0(X, \mathbf{Z}) \to H^0(\mathrm{Th}(-V), \mathbf{Z})$. Composing with the Thom isomorphism $H^{-(-r)}(X, \mathcal{O}(-V)) \cong H^0(\mathrm{Th}(-V), \mathbf{Z})$, we obtain a map

$$H^0(X, \mathbf{Z}) \to H^r(X, \mathcal{O}(-V)). \tag{5}$$

Define the image of 1 in $H^0(X, \mathbf{Z})$ under this map to be the Euler class $e(V)$ in $H^r(X, \mathcal{O}(-V))$.

Poincaré duality is roughly the statement that a compact manifold is self-dual, or more precisely, that the dual of a manifold X is a twisted shift of X: Precisely, the dual of X_+ is $\mathrm{Th}(-TX)$, where X_+ denotes X with a disjoint base point. This induces a Poincaré duality isomorphism $\widetilde{H}^d(X, \mathcal{O}(-TX)) \cong H_0(X, \mathbf{Z}) \cong \mathbf{Z}$. Therefore, if the orientation sheaves $\mathcal{O}(-V)$ and $\mathcal{O}(-TX)$ are isomorphic, which follows from the existence of a relative orientation (we saw this above provided we note that the above also implies $\mathcal{O}(-V) \cong \mathcal{O}(V)$), we may define view $e(V)$ as an integer.

This integer can be calculated with some calculus and a section σ of V such that all zeros of σ are isolated. Let p in X be such that $\sigma(p) = 0$. Choosing local coordinates of X near p and a local trivialization of V near p allows us to locally identify σ with a function

$$\sigma : \mathbf{R}^d \to \mathbf{R}^r.$$

When $d = r$, we may take the local degree of this function at the coordinates of p, obtaining an integer $\deg_p \sigma$. However, if we change the local trivialization by a linear function with negative determinant, we will change the sign of $\deg_p \sigma$. Therefore we must choose coordinates and a trivialization which are compatible with a relative orientation in the following sense.

Local coordinates around p give a distinguished local trivialization of TX. Taking the wedge product of a basis of vector fields, we obtain a distinguished section in $\det TX(U)$ for some neighborhood U of p. Similarly, a choice of local trivialization of V gives a distinguished section in $\det V(U)$, by possibly shrinking U. This in turn gives a distinguished section of $\mathrm{Hom}(\det TX, \det V)(U)$

Definition 13. *Local coordinates and a trivialization of V on an open neighborhood U of p are* compatible with the relative orientation *if the distinguished section of $\mathrm{Hom}(\det TX, \det V)(U)$ is the tensor square of a section in $L(U)$.*

When $\deg_p \sigma$ is computed with a choice of local coordinates and trivialization of V which is compatible with a fixed relative orientation the result is independent of the choices, an can be called the *local degree* or *index* of σ at p. The Euler class is the sum of the local terms

$$e(V) = \sum_{p:\sigma(p)=0} \deg_p \sigma$$

This discussion can be transported to $\mathbf{A}^1$-homotopy theory. The first Euler class in the context of $\mathbf{A}^1$-homotopy theory is due to Barge and Morel [BM00], and the technology has benefited from contributions of Morel, Jean Fasel, Aravind Asok, Marc Levine, Frédéric Déglise, Fangzhou Jin, and Adeel A. Khan, and Arpon Raksit. There will be further discussion in Section 4.3. The following point of view is from joint work with Jesse Kass [KW17, Section 4].

4.1. **Orientations.** Let X be a smooth scheme over k of dimension $d = r$. Let $V \to X$ be a rank r algebraic vector bundle.

Definition 14. $V \to X$ *is* oriented *by the data of a line bundle L on X and an isomorphism* $\det V \cong L^{\otimes 2}$.

Some authors use the terminology *weakly oriented* for this concept, reserving the term oriented for isomorphisms $\det V \cong \mathcal{O}$.

Definition 15. $V \to X$ *is* relatively oriented $\mathrm{Hom}(\det TX, \det V)$ *is oriented.*

Example 16. *Let $X = \mathbf{P}^1$. Let $\mathcal{O}(-1)$ denote the tautological bundle on $\mathbf{P}^1$. Then $\det TX \cong TX \cong \mathcal{O}(2)$. Therefore a line bundles $\mathcal{O}(n)$ is relatively orientable if and only if n is even.*

4.2. **Local indices and definition of the Euler class.** Let $V \to X$ be a relatively oriented rank r vector bundle on a smooth scheme over k of dimension $d = r$.

Suppose that σ is a section of V and that p is an isolated zero of σ, meaning that p is point of X such that $\sigma(p) = 0$, and such that there is an open set U containing p such that the only zero of σ in U is p.

We will define the *local index* or *degree* $\deg_p \sigma$ in $\mathrm{GW}(k)$ as follows.

Definition 17. *An étale map $\phi : U \to \mathbf{A}^d$ from a Zariski open neighborhood of p to $\mathbf{A}^d$ will be called* Nisnevich local coordinates around p *if the induced map of residue fields $k(\phi(p)) \to k(p)$ is an isomorphism.*

Nisnevich coordinates are guaranteed to exist if $d = 1$ [BKW18, Proposition 6] or if $k(p)$ is a separable extension of k and $d \geq 1$ [KW17, Lemma 18]. Let ϕ be Nisnevich coordinates around U. Let $\psi : V|_U \to \mathcal{O}_U^r$ be a local trivialization of V.

Definition 18. *ϕ and ψ are* compatible with the relative orientation *if the distinguished section in $\mathrm{Hom}(\det TX, \det V)(U)$ is the image of the tensor square of a section in $L(U)$ under the isomorphism $\mathrm{Hom}(\det TX, \det V) \cong L^{\times 2}$ coming from the relative orientation.*

Choose Nisnevich local coordinates $\phi : U \to \mathbf{A}^d$. After possibly shrinking U, it is possible to choose a trivialization $\psi : V|_U \to \mathcal{O}_U^r$ which is compatible with the relative orientation (multiplying the first coordinate by some element in $\mathcal{O}_U^*$ will suffice).

$\psi \circ \sigma|_U$ is then an element of $\mathcal{O}(U)^r$. As above, we wish to identify σ with a function $\mathbf{A}^d \to \mathbf{A}^r$, i.e., we wish for each of the r components of $\psi \circ \sigma$ to be in the image of $\phi^* : \mathcal{O}_{\mathbf{A}^d} \to \mathcal{O}_U$. When X is covered by opens of the form $\mathbf{A}^d$, as is the case when X is a Grassmannian, we can choose ϕ to be an isomorphism on local rings, and this is immediate. The general case can be made to work as well, however. We can add an element $G = (g_1, \ldots, g_r)$ of $\mathcal{O}_U^r$ to $\psi \circ \sigma$ so that $G + \psi \circ \sigma = \psi^*(F)$, with $F : \mathbf{A}^d \to \mathbf{A}^r$, and so that each g_i vanishes to a sufficiently high order at p. Then define

$$\deg_p \sigma = \deg_{\phi(p)} F$$

and this is independent of the choice of ϕ, ψ, and G [KW17, Corollary 29].

We then define the Euler Class $e(V, \sigma)$ in $\mathrm{GW}(k)$ of V with respect to the section σ (and the relative orientation):

Definition 19. $e(V, \sigma) = \sum_{p:\sigma(p)=0} \deg_p \sigma$

$e(V, \sigma)$ should be independent of σ in general. It is shown that $e(V, \sigma) = e(V, \sigma')$ when σ and σ' can be connected by a family parametrized by $\mathbf{A}^1$ of sections with only isolated zeros, giving a well-defined Euler class under the hypotheses of [KW17, Corollary 36].

Example 20. *Let 0 denote the origin of the distinguished copy of $\mathbf{A}^1$ in $\mathbf{P}^1$, and let n be an integer. Let $\mathcal{O}(2n \cdot 0) \to \mathbf{P}^1$ denote the line bundle associated to the locally free sheaf of meromorphic functions whose only poles are at 0 and these poles are of order no worse than $2n$. Then the section*

1 *has an isolated zero of order* $2n$ *at* 0. *For any chosen relative orientation,*

$$e(\mathcal{O}(2n \cdot 0)) = \sum_{p:1(p)=0} \deg_p 1 = \deg_0 x^{2n} = n(\langle 1 \rangle + \langle -1 \rangle).$$

4.3. **More perspectives on the Euler class.** Barge and Morel defined an Euler class in the oriented Chow groups of X [BM00]. Namely, let $p : V \to X$ be a rank r vector bundle on a smooth scheme X over k (X of dimension d not necessarily equal to r).

There is a canonical element $\langle 1 \rangle$ in $\widetilde{\mathrm{CH}}^0(X)$. In the Rost–Schmidt complex (4), it is represented by $\langle 1 \rangle$ in $\mathrm{GW}(k(X))$ under the inclusion of the summand corresponding to the codimension 0 scheme X. Let $\sigma : X \to V$ denote the zero section. Since σ is proper, there is a pushforward map $\sigma_* : \widetilde{\mathrm{CH}}^0(X, \omega_{X/k} \otimes \sigma^* p^* \omega_{X/k}) \to \widetilde{\mathrm{CH}}^r(\omega_{V/k} \otimes p^* \omega_{X/k})$. Since $p\sigma = 1$ and $\omega_{V/k} \cong p^* \omega_{X/k} \otimes \det V^*$, this pushforward can be identified with a map

$$\sigma_* : \widetilde{\mathrm{CH}}^0(X) \to \widetilde{\mathrm{CH}}^r(V, \det V^*).$$

The *Euler class* $e(V)$ in $\widetilde{\mathrm{CH}}^r(X, \det V^*)$ is then defined by $e(V) = (p^*)^{-1} \sigma_* \langle 1 \rangle$ [AF16b]. The pushforward followed by the isomorphism $(p^*)^{-1}$ is analogous to the composition (5) defined using Thom spaces.

When V is relatively oriented, we have an isomorphism $\det V^* \cong \omega_{X/k} \otimes (L^*)^{\otimes 2}$, and therefore a pushforward map

$$\widetilde{\mathrm{CH}}^r(X, \det V^*) \to \widetilde{\mathrm{CH}}^{r-d}(\mathrm{Spec}\, k).$$

When $r = d$, we therefore obtain $e(V)$ in $\mathrm{GW}(k)$.

Remark 21. *This should be equal to the Euler class of Section 4.2.*

Morel has an Euler class constructed as the principal obstruction to a nonvanishing section [Mor12, 8.2]. This is known to agree with the Barge–Morel construction at least up to a unit in $\mathrm{GW}(k)$ when V is oriented [AF16a] [Lev17b].

Déglise, Jin and Khan have a construction using 6-functor formalism [DJK18].

There is another construction using Serre Duality of coherent sheaves. I learned about it because Mike Hopkins pulled it out of thin air after a talk I gave. Serre did something similar to Eva Bayer-Fluckiger and A. Raksit independently produced this construction as well. It appears in in work of M. Levine and A. Raksit [LR18] together with a proof that it agrees with the Barge–Morel construction.

4.4. **Lines meeting four lines in space.** with Padma Srinivasan [SW18]. See also the work of Matthias Wendt [Wen18].

5. Arizona Winter School Project Suggestions

There are many beautiful results in enumerative geometry, and it is not so far-fetched to suggest turning any such result into a project, where the classical count over $\mathbf{C}$ becomes an arithmetic count over a field. Some known difficulties with this perspective include the existence of orientations (some vector bundles which occur naturally in enumerative geometry are not relatively oriented), and obtaining enumerative descriptions of local contributions to Euler classes. The purpose of this section is to outline projects which (hopefully) are of a nature to be educational in the Arizona Winter School. Since these projects are intended to be tackled in a period of six days with some success, this list is not intended to be representative of the open problems in the field.

5.1. **Configurations over** k**.** Let $\mathcal{S} \to \mathrm{GW}(1,n)$ denote the tautological bundle on the Grassmannian of lines in $\mathbf{P}^n$. In [Wen18] and [SW18] the Euler class of $\wedge^2\mathcal{S}^* \to \mathrm{Gr}(1,3)$, and more generally, $\oplus_{i=1}^{2n-2}\mathcal{S} \to \mathrm{Gr}(1,n)$ is computed. This corresponds to an arithmetic count of the number of lines meeting $2n-2$ codimension 2 hyperplanes. However, these hyperplanes are all defined over k. One could ask more generally for a count of lines meeting a configuration of $2n-2$ codimension 2 hyperplanes, where the configuration is defined over k, but where the individual codimension 2 hyperplanes are not necessarily defined over k. In other words, we have $2n-2$ hyperplanes defined over $\overline{k}$, which are permuted by the $\mathrm{Gal}(\overline{k}/k)$-action, but where the $\mathrm{Gal}(\overline{k}/k)$-action is potentially non-trivial. An arithmetic count of these lines would correspond to the computation of an Euler class of the vector bundle whose fiber over $[W]$ in $\mathrm{GW}(1,n)$ is $(W^* \wedge W^*)^{2n-2}/S_{2n-2}$, where W denotes a linear subspace of dimension 2 of an $n+1$ dimensional vector space, S_{2n-2} denotes the symmetric group on $2n-2$ objects, and the action of S_{2n-2} on $(W^* \wedge W^*)^{2n-2}$ is by permutation.

Question 22.
1. *Is this vector bundle relatively orientable for certain* n*?*
2. *If so, what is the Euler class?*
3. *Can you give an enumerative interpretation of the local indices?*
4. *If so, what is the resulting theorem?*

Question 23. *Can this be generalized to other counts of subspaces meeting a configuration over* k*? For example, can the count of balanced subspaces of* [Wen18, Section 9.2] *be generalized to where the subspaces are not defined over the ground field? Similarly, if we only require that the two quadrics of* [Wen18, Example 9.4] *are permuted by* $\mathrm{Gal}(\overline{k}/k)$ *instead of being individually defined over* k*, could we obtain a more general result?*

5.2. **Bitangents to a smooth plane quartic.** A bitangent to a curve in projective space is a line which is tangent to the curve at 2 points, counted with multiplicity. It is a classical theorem that there are 28 bitangents to a smooth degree 4 curve in $\mathbf{P}^2$ over $\mathbf{C}$. This is closely related to the 27 lines on a complex smooth cubic surface. Namely, let $S \subset \mathbf{P}^3_{\mathbf{C}}$ be a smooth cubic surface, and let p be a point of S not on any line. Projection π_p from p defines a rational map $\pi_p : \mathbf{P}^3_k \dashrightarrow \mathbf{P}^2_k$, and a map $\pi_p^{\mathrm{Bl}} : \mathrm{Bl}_p S \to \mathbf{P}^2$, which expresses S as a degree 2 cover of $\mathbf{P}^2$ branched over a quartic curve C. The images of the 27 lines on S and the exceptional divisor of the blow up give the 28 bitangents of C.

Question 24. *Is there an arithmetic count of the bitangents to a smooth plane quartic?*

One idea along these lines is as follows: perhaps a good local contribution for a bitangent line L is related to the two points of contact with the curve. If these points are defined over $k(L)[\sqrt{D}]$, then $\langle D \rangle$ in $\mathrm{GW}(K(L))$ may be useful. Consider the involution I which appears in the arithmetic count of the lines on a cubic surface described above. Do the images of these fixed points have an independent interpretation as points on the bitangent, for example as points of contact with C?

5.3. **Eisenbud–Khimshaishvili–Levine form is $\mathbf{A}^1$-local degree over extensions of k.** The main result of [KW16] identifies the $\mathbf{A}^1$-local degree $\deg_0 f$ at 0 of a function $f : \mathbf{A}^n_k \to \mathbf{A}^n_k$ with an isolated zero at the origin with the bilinear form ω^{EKL} appearing in the Eisenbud–Levine–Khimshiashvili signature formula

$$\deg_0 f = \omega^{\mathrm{EKL}}.$$

Question 25. *Can this be generalized to equate the local degree $\deg_p f$ with ω^{EKL} when the residue field of p is not k?*

References

[AF16a] Aravind Asok and Jean Fasel, *Comparing euler classes*, 2016. *Preprint to appear in Q.J. Math*, available at `https://arxiv.org/abs/1306.5250`. ↑4.3

[AF16b] ———, *Euler class groups and motivic stable cohomotopy*, Quart. J. Math. **67** (2016), 603–635. *Preprint*, available at `https://arxiv.org/abs/1601.05723`. ↑4.3

[BKW18] Candace Bethea, Jesse Leo Kass, and Kirsten Wickelgren, *An example of wild ramification in an enriched Riemann–Hurwitz formula*, 2018. Preprint, available at `https://arxiv.org/abs/1812.03386`. ↑1, 4.2

[BM00] Jean Barge and Fabien Morel, *Groupe de Chow des cycles orientés et classe d'Euler des fibrés vectoriels*, C. R. Acad. Sci. Paris Sér. I Math. **330** (2000), no. 4, 287–290. MR1753295 ↑2.5, 4, 4.3

[BS95] R. Benedetti and R. Silhol, *Spin and p structures, immersed and embedded surfaces and a result of Segre on real cubic surface*, Topology **34** (1995), 651–678. ↑1

[Cay49] Arthur Cayley, *On the triple tangent planes to a surface of the third order*, Cambridge and Dublin Math. J. **4** (1849), 252–260. ↑1

[DJK18] Frédéric Déglise, Fangzhou Jin, and Adheel Khan, *Fundamental classes in motivic homotopy theory*, 2018. *Preprint*, available at `https://arxiv.org/abs/1805.05920`. ↑4.3

[Dol05] Igor V. Dolgachev, *Luigi Cremona and cubic surfaces*, Luigi Cremona (1830–1903) (Italian), 2005, pp. 55–70. MR2305952 ↑1

[EH16] David Eisenbud and Joe Harris, *3264 and all that—a second course in algebraic geometry*, Cambridge University Press, Cambridge, 2016. MR3617981 ↑1, 1, 1, 2.5

[Fas07] Jean Fasel, *The Chow-Witt ring*, Doc. Math. **12** (2007), 275–312. MR2350291 ↑2.5

[Fas08] ———, *Groupes de Chow-Witt*, Mém. Soc. Math. Fr. (N.S.) **113** (2008), viii+197. MR2542148 ↑2.5

[FK13] Sergey Finashin and Viatcheslav Kharlamov, *Abundance of real lines on real projective hypersurfaces*, Int. Math. Res. Not. IMRN **16** (2013), 3639–3646. MR3090704 ↑1

[Ful98] William Fulton, *Intersection theory*, Second, Ergebnisse der Mathematik und ihrer Grenzgebiete. 3. Folge. A Series of Modern Surveys in Mathematics [Results in Mathematics and Related Areas. 3rd Series. A Series of Modern Surveys in Mathematics], vol. 2, Springer-Verlag, Berlin, 1998. MR1644323 ↑1

[Hoy14] Marc Hoyois, *A quadratic refinement of the Grothendieck-Lefschetz-Verdier trace formula*, Algebr. Geom. Topol. **14** (2014), no. 6, 3603–3658. MR3302973 ↑1

[HS12] Asaf Horev and Jake P. Solomon, *The open Gromov–Witten–Welschinger theory of blowups of the projective plane*, 2012. *Preprint*, available at `https://arxiv.org/abs/1210.4034`. ↑1

[KW16] Jesse Kass and Kirsten Wickelgren, *The class of Eisenbud–Khimshiashvili–Levine is the local $\mathbb{A}^1$-brouwer degree*, 2016. Accepted for publication in Duke Mathematical Journal, available at `https://arxiv.org/abs/1608.05669`. ↑1, 5.3

[KW17] ______, *An arithmetic count of the lines on a smooth cubic surface*, 2017. *Preprint*, available at `https://arxiv.org/abs/1708.01175`. ↑1, 1, 1, 1, 4, 4.2, 4.2, 4.2

[Lev17a] Marc Levine, *The intrinsic stable normal cone*, 2017. *Preprint*, available at `https://arxiv.org/abs/1703.03056`. ↑1

[Lev17b] ______, *Toward a new enumerative geometry*, 2017. *Preprint*, available at `https://arxiv.org/abs/1703.03049`. ↑1, 4.3

[Lev18a] ______, *Motivic Euler characteristics and Witt-valued characteristic classes*, 2018. *Preprint*, available at `https://arxiv.org/abs/1806.10108`. ↑1

[Lev18b] ______, *Toward an algebraic theory of Welschinger invariants*, 2018. *Preprint*, available at `https://arxiv.org/abs/1808.02238`. ↑1

[LR18] Marc Levine and Arpon Raksit, *Motivic Gauss–Bonnet formulas*, 2018. *Preprint*, available at `https://arxiv.org/abs/1808.08385`. ↑4.3

[Mor12] Fabien Morel, $\mathbb{A}^1$*-algebraic topology over a field*, Lecture Notes in Mathematics, vol. 2052, Springer, Heidelberg, 2012. MR2934577 ↑2.5, 4.3

[MV99] F. Morel and V. Voevodsky, $\mathbb{A}^1$*-homotopy theory of schemes*, Inst. Hautes Études Sci. Publ. Math. **90** (1999), 45–143 (2001). MR1813224 (2002f:14029) ↑1

[OT14] Christian Okonek and Andrei Teleman, *Intrinsic signs and lower bounds in real algebraic geometry*, J. Reine Angew. Math. **688** (2014), 219–241. MR3176620 ↑1, 4

[Sch58] L. Schläfli, *An attempt to determine the twenty-seven lines upon a surface of the third order, and to divide such surfaces into species in reference to the reality of the lines upon the surface*, Quart. J. Pure Appl. Math **2** (1858). ↑1

[Seg42] B. Segre, *The Non-singular Cubic Surfaces*, Oxford University Press, Oxford, 1942. MR0008171 ↑1

[SW18] Padmavathi Srinivasan and Kirsten Wickelgren, *An arithmetic count of the lines meeting four lines in* $P\hat{3}$, arXiv:1810.03503 [math] (October 2018), available at `1810.03503`. ↑1, 4.4, 5.1

[Wen18] Matthias Wendt, *Oriented Schubert calculus in Chow–Witt rings of Grassmannians*, 2018. *Preprint* available at `https://arxiv.org/abs/1808.07296`. ↑1, 4.4, 5.1, 23

[WW19] Kirsten Wickelgren and Ben Williams, *Unstable motivic homotopy theory*, 2019. *Preprint*, available at `https://arxiv.org/abs/1902.08857`. ↑2.2, 2.3

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