The geometry of algebra and the algebra of geometry: model categories, infinity categories and spectra

The problems in this problem set are aimed at students who have seen the definitions of the objects under discussion, but may not have worked with them extensively. In the interest of getting students to the point where they can work on the problems, I recomment the following resources.

Category Theory and Model categories Dwyer and Spalinski, "Homotopy Theories and Model Categories," particularly sections 3 and 5. This also includes a quick introduction to category theory in section 2.

Simplicial sets Riehl, "A Liesurely Introduction to Simplicial Sets."

Spectra Schwede, *Symmetric Spectra*. Chapter 1, Sections 1, 4, 5.

These resources are chosen for their ease of reading and introductory nature. I have also tried to make sure that they are relatively short and allow students access to the problems as quickly as possible. They are definitely *not* comprehensive introductions to the subjects at hand.

Pedantic note

All categories are assumed to be locally small. Any category whose nerve we are taking is assumed to be small. Students attempting the problems should be careful to note where set-theoretic issues may arise, and discuss how they should be resolved.

The category **Top** of topological space is actually the category of compactly-generated Hausdorff spaces. (The category of all topological spaces is not cartesian-closed, and is therefore not well-behaved for the purposes of doing homotopy theory.)

1 Simplicial sets and nerves of categories

1.1. There exist two distinct maps $f, g\Delta^0 \longrightarrow \Delta^1$. Show that there exists a simplicial homotopy between these maps in one direction, but not the other. (In other words, there exists a simplicial homotopy from f to g, but not from g to f, or vice versa.)

1.2. Let $\mathcal{I}$ be the category $0 \longrightarrow 1$.

(a) Check that $N\mathcal{I} \cong \Delta^1$.

(b) Let $F, G\colon \mathcal{C} \longrightarrow \mathcal{D}$ be a pair of functors, and suppose that $\alpha\colon F \Longrightarrow G$ is a natural transformation between them. Use the data of α to construct a simplicial homotopy from NF to NG.

(c) Conclude that if $F\colon \mathcal{C} \rightleftarrows \mathcal{D} : G$ is a pair of adjoint functors then $N\mathcal{C}$ is homotopy equivalent to $N\mathcal{D}$.

1.3. Let $\mathcal{O}_n$ be the full subcategory of Δ containing the objects $[0], \ldots, [n]$. Given any simplicial set $X\colon \Delta^{\mathrm{op}} \longrightarrow \mathbf{Set}$ we then have its *n-th truncation* $\widetilde{X}\colon \mathcal{O}_n^{\mathrm{op}} \longrightarrow \mathbf{Set}$ given by precomposition. We define the category $s\mathbf{Set}_n \overset{\text{def}}{=} \mathrm{Func}(\mathcal{O}_n^{\mathrm{op}}, \mathbf{Set})$, with natural transformations as the morphisms.

(a) Check that the n-th truncation is a functor $\tau_n\colon s\mathbf{Set} \longrightarrow s\mathbf{Set}_n$.

(b) Describe the left and right adjoints to the n-th truncation functor.

(c) Look up the definition of a monad/comonad.[1] Describe the monad and comonad constructed on $s\mathbf{Set}$ by these adjunctions. These are called the *n-skeleton* and *n-coskeleton* functors. Which is which?

(d) A simplicial set is said to be *n-skeletal* if its n-skeleton is equal to itself, and *n-coskeletal* if its n-coskeleton is equal to itself. Give an example of a simplicial set which is neither n-skeletal nor n-coskeletal.

(e) Let $\mathcal{C}$ be a category. Show that $N\mathcal{C}$ is 2-coskeletal.

1.4. Show that the functor $N\colon \mathbf{Cat} \longrightarrow s\mathbf{Set}$ is full and faithful. Can you give a condition for identifying objects contained in its essential image?[2]

1.5. For which categories $\mathcal{C}$ is $N\mathcal{C}$ a Kan complex?

1.6. Show that a filtered preorder is contractible.

1.7. Let X be a simplicial set which is k-coskeletal and for which, for all $n < k$, the lift

$$\begin{array}{ccc} \Lambda^n_\ell & \longrightarrow & X \\ \big\downarrow & \nearrow_{\exists} & \\ \Delta^n & & \end{array}$$

exists for all $0 \leq \ell \leq k$. Prove that X is a Kan complex.

1.8. Let $\mathcal{E}$ be a cocomplete category, and let $D\colon \Delta \longrightarrow \mathcal{E}$ be any functor. and

(a) Let $y\colon \Delta \longrightarrow s\mathbf{Set}$ be the Yoneda embedding. Describe y geometrically.

(b) Define $L\colon s\mathbf{Set} \longrightarrow \mathcal{E}$ to be the left Kan extension of D along y. Describe L in terms of coequalizers.

(c) When $\mathcal{E} = \mathbf{Top}$ and $D([n]) = \Delta^n$, describe L.

(d) Prove that L has a right adjoint R, and desribe R in the above situation.

(e) Now consider the case when $\mathcal{E} = \mathbf{Cat}$ and $D([n]) = 0 \longrightarrow \cdots \longrightarrow n$. Describe L and R; what are these usually called?

(f) Now suppose that $\mathcal{E} = s\mathbf{Set}$ and $D([n]) = N(P(n))$, where $P(n)$ is the partial order of subsets of $\{0, \ldots, n\}$. Describe L and R in this case; what are these usually called?

(g) Look up the definition of the homotopy coherent nerve. Can you construct a functor D that produces it?

1.9. Let K, L be two simplicial sets. We define $(K \times L)_n = K_n \times L_n$. Check that $|I \times I| \cong I^2$ explicitly. Prove that $|K \times L| \cong |K| \times |L|$.

1.10. Consider Quillen's Theorem A:

Theorem 1.1 (Quillen's Theorem A)**.** *Let $F\colon \mathcal{C} \longrightarrow \mathcal{D}$ be a functor, and suppose that for all $D \in \mathcal{D}$, F/D is contractible. Then F is a homotopy equivalence.*

(a) Use Theorem A to show that if a category has an initial (resp. terminal) object then it is contractible.

(b) Use Theorem A to show that if $F\colon \mathcal{C} \longrightarrow \mathcal{D}$ has a left (resp. right) adjoint then it is a homotopy equivalence.

(c) Use Theorem A to prove that $B\mathbf{N} \longrightarrow B\mathbb{Z}$ is a weak equivalence.

[1] I recommend Section 5.1 in Riehl's *Category Theory in Context*.

[2] This problem implies that it is completely natural to think of categories as spaces, as long as we are willing to consider simplicial sets as our model of spaces. Thus it is perfectly reasonable to state that "$\mathcal{C}$ is contractible" or "$F\colon \mathcal{C} \longrightarrow \mathcal{D}$ is a homotopy equivalence" in a non-ambiguous manner.

2 Model categories and simplicial categories

2.1. Let Ch_R be the category of bounded below chain complexes over a commutative ring R. (Here we grade chain complexes *homologically*, so differentials go down; you are of course welcome to do this problem with cohomologically graded chain complexes, and look at bounded-above chain complexes instead.) Define a model structure on Ch_R with

cofibrations injections with projective cokernel

fibrations surjections above degree 0

weak equivalences quasi-isomorphisms.

(a) Prove (*without* using the small object argument) that this gives a model structure on Ch_R. Prove that there exists an analogous model structure where the cofibrations are injections. (These are called, respectively, the *projective* and *injective* model structures.)

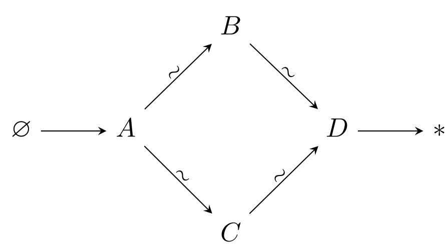
(b) Let M be an R-module. When is M cofibrant? What does the cofibrant replacement of $M[0]$ look like?

(c) Suppose we are given a Quillen pair $F\colon \mathcal{C} \rightleftarrows \mathcal{D} :G$. The *left derived functor of* F is given by composing F with a cofibrant replacement functor. Similarly, the *right derived functor of* G is given by composing G with a fibrant replacement functor. Let M be an R-module. Check that $\cdot \otimes M$ can be thought of a left Quillen functor[3] $\mathrm{Ch}_R \longrightarrow \mathrm{Ch}_{\mathbb{Z}}$ and describe its left derived functor. Can you do a similar construction for $\mathrm{Hom}(M, \cdot)$?

(d) Now modify this to produce a model structure on unbounded chain complexes $\mathrm{Ch}_R^{\geq 0}$.

2.2. Prove that a left Quillen functor preserves weak equivalences between cofibrant objects.

2.3. Let $\mathcal{C}$ be the following category, with $\mathcal{W}$ the marked subcategory:



Find all model structures on $\mathcal{C}$ with $\mathcal{W}$ as the weak equivalences.

2.4. Let $\mathcal{C}$ be a finite preorder closed under all limits and colimits; let $\mathcal{W}$ be a subcategory of $\mathcal{C}$. Prove that there exists a model structure on $\mathcal{C}$ with $\mathcal{W}$ as the weak equivalences if and only if the following two conditions hold:

- $\mathcal{W}$ satisfies 1-of-3: if f and g are composable morphisms such that $gf \in \mathcal{W}$ then $f, g \in \mathcal{W}$.
- Let $\mathcal{W}_c$ be those weak equivalences such that all pushouts are also weak equivalences, and let $\mathcal{W}_f$ be those weak equivalences such that all pullbacks are also weak equivalences. Every morphism in $\mathcal{W}$ factors as a morphism in $\mathcal{W}_c$ followed by a morphism in $\mathcal{W}_f$.

2.5. Show that all model structures on partial orders are Quillen equivalent to a model structure where all morphisms are fibrations and cofibrations and the weak equivalences are isomorphisms. Why doesn't this work for a general category?

2.6. (a) Show that in a model category $\mathcal{C}$, if there exists a zigzag of weak equivalences between two objects, then there exists a zigzag of length at most 4 between these two objects.

[3]How do you need to specify the model structures?

(b) Suppose that $\mathcal{C}$ is a partial order such that $\mathrm{Ho}\,\mathcal{C}$ is essentially large (not equivalent to a small category). Prove that the model structure on $\mathcal{C}$ is not cofibrantly generated.

2.7. Let $\mathcal{C}$ be a category. A *weak factorzation system* is a pair $(\mathcal{L}, \mathcal{R})$ of classes of morphisms in $\mathcal{C}$ such that

- ${}^{\boxslash}\mathcal{R} = \mathcal{L}$
- $\mathcal{L}^{\boxslash} = \mathcal{R}$
- For every morphism $f\colon A \longrightarrow B$ in $\mathcal{C}$ there exists $f_L\colon A \longrightarrow X$ and $f_R\colon X \longrightarrow B$ with $f_L \in \mathcal{L}$ and $f_R \in \mathcal{R}$ such that $f_R f_L = f$.

(a) Show that in a weak factorization system $(\mathcal{L}, \mathcal{R})$, $\mathcal{L}$ is closed under retracts and pushouts and $\mathcal{R}$ is closed under retracts and pullbacks.

(b) Show that a model structure on $\mathcal{C}$ is given by a subcategory $\mathcal{W}$ of weak equivalences satisfying 2-of-3 together with two weak factorization systems $(\mathcal{L}, \mathcal{R})$ and $(\mathcal{L}', \mathcal{R}')$ such that $\mathcal{L} = \mathcal{L}' \cap \mathcal{W}$ and $\mathcal{R}' = \mathcal{R} \cap \mathcal{W}$.

(c) Classify all weak factorization systems on **Set**.

(d) Classify all model structures on **Set**.

2.8. A category $\mathcal{C}$ is a *simplicially enriched category*[4] if there exists a functor

$$\mathbf{Hom}\colon \mathcal{C} \times \mathcal{C}^{\mathrm{op}} \longrightarrow s\mathbf{Set}$$

such that

- $\mathbf{Hom}(A, B)_0 = \mathrm{Hom}(A, B)$ for all $A, B \in \mathcal{C}$.
- $\mathbf{Hom}(A, -)$ has a left adjoint, generally denoted $\cdot \otimes A\colon s\mathbf{Set} \longrightarrow \mathcal{C}$ such that for all $K, L \in s\mathbf{Set}$, $(K \times L) \otimes A \cong K \otimes (L \otimes A)$.
- The functor $\mathbf{Hom}(-, B)$ has a left adjoint $B^{\cdot}\colon s\mathbf{Set} \longrightarrow \mathcal{C}^{\mathrm{op}}$.

(a) Prove that $s\mathbf{Set}$ is a simplicially enriched category with $\mathbf{Hom}(K, L)_n = \mathrm{Hom}(K \times \Delta^n, L)$.

(b) More generally, prove that for any simplicially enriched category $\mathcal{C}$, $\mathbf{Hom}(A, B)_n \cong \mathrm{Hom}(\Delta^n \otimes A, B)$.

(c) Prove that **Top** is a simplicially enriched category. What do the n-simplices in $\mathbf{Hom}(X, Y)$ represent?

(d) Prove that for fixed $K \in s\mathbf{Set}$, for any simplicially enriched category $\mathcal{C}$, there exists an adjuction $(K \otimes \cdot) \dashv \cdot^K$.

(e) Prove that for all $K, L \in s\mathbf{Set}$, $B^{K \times L} \cong (B^L)^K$.

2.9. Which objects in the category $s\mathbf{Cat}$ (of simplicial objects in **Cat**) arise from simplicially enriched categories?

2.10. (a) Show that a simplicially enriched category can be encoded as a functor $\Delta^{\mathrm{op}} \longrightarrow \mathbf{Cat}$.

(b) Which functors $\Delta^{\mathrm{op}} \longrightarrow \mathbf{Cat}$ can be obtained from simplicial categories?

2.11. Suppose that $\mathcal{C}$ is a simplicially enriched category and $X \in s\mathcal{C}$. We define

$$|X| = \mathrm{coeq}\left(\coprod_{f\colon [n] \longrightarrow [m]} \Delta^n \otimes X_m \rightrightarrows \coprod_{[n]} \Delta^n \otimes X_n \right).$$

[4]Often, by a horrible abuse of terminology, called a "simplicial category." You should be careful with this, however, since the term "simplicial category" can refer to the category Δ, a simplicially enriched category, or a simplicial object in the category of categories, three **completely distinct concepts**.

(a) Suppose that $\mathcal{C} = s\mathbf{Set}$. Prove that $|X| \cong \operatorname{diag} X$.

(b) Suppose that $\mathcal{C} = \mathbf{Top}$, and suppose that $X\colon \Delta^{\mathrm{op}} \longrightarrow \mathbf{Top}$ has X_n discrete for all n. Thus we can factor $X\colon \Delta^{\mathrm{op}} \xrightarrow{X'} \mathbf{Set} \hookrightarrow \mathbf{Top}$, where the inclusion takes a set to a discrete topological space. Prove that $|X| \cong |X'|$.

(c) Suppose that X is constant. Prove that $|X| \cong X$.

(d) (*) Suppose that $X \longrightarrow Y$ is a map of bisimplicial sets such that $X_n \xrightarrow{\sim} Y_n$ is a weak equivalence for all n. Prove that the map $|X| \longrightarrow |Y|$ is a weak equivalence.[5]

2.12. We will now use the results of the previous problem to prove Quillen's Theorem A.

Let $F\colon \mathcal{C} \longrightarrow \mathcal{D}$ be a functor. We define the category $\mathcal{D}/F$ to have

objects $\bigcup_{D \in \mathcal{D}} \operatorname{ob} D/F$.

morphisms A morphism $(D, f\colon D \longrightarrow F(C)) \longrightarrow (D', f'\colon D' \longrightarrow F(C')$ is a pair of morphisms $g\colon C \longrightarrow C' \in \mathcal{C}$ and $g'\colon D' \longrightarrow D \in \mathcal{D}$ such that the square

$$\begin{array}{ccc} D & \xrightarrow{F(g)} & D' \\ {\scriptstyle f}\downarrow & & \downarrow{\scriptstyle f'} \\ F(C) & \xleftarrow{g'} & F(C') \end{array}$$

commutes.

(a) Let X be the bisimplicial set where $X_{p,q}$ is the set of pairs

$$(D_q \longrightarrow \cdots \longrightarrow D_0 \longrightarrow F(C_0), C_0 \longrightarrow \cdots \longrightarrow C_p).$$

The horizontal and vertical faces are induced from the nerves of $\mathcal{C}$ and $\mathcal{D}$, respectively. Use X to show that $\mathcal{D}/F$ is weakly equivalent to $\mathcal{C}$.

(b) Consider the diagram

$$\begin{array}{ccccc} \mathcal{C} & \longleftarrow & \mathcal{D}/F & \longrightarrow & \mathcal{D}^{\mathrm{op}} \\ {\scriptstyle F}\downarrow & & \downarrow & & \| \\ \mathcal{D} & \longleftarrow & \mathcal{D}/1_{\mathcal{D}} & \longrightarrow & \mathcal{D}^{\mathrm{op}} \end{array}$$

where the middle vertical functor is induced by F. Use this and the result of part (a) to prove Quillen's Theorem A.

3 Spectra

For these problems, we use the category of symmetric spectra. Analogous results exist in most models of spectra.

3.1. Let $f\colon A \longrightarrow X$ be an inclusion of spaces. Prove that the sequence

$$A \longrightarrow X \longrightarrow X \cup_A CA$$

[5] This is a special case of a much more general phenomenon; however, to explore it in more detail we would need to discuss Reedy model category structures, which would take us far afield.

is a homotopy cofiber sequence by showing that any map $X \longrightarrow Y$ such that the precomposition $A \longrightarrow Y$ is null-homotopic factors through $X \cup_A CA$. Show that

$$X \longrightarrow X \cup_A CA \longrightarrow \Sigma A$$

is also a homotopy cofiber sequence. Use this to produce a long exact sequence in the cohomology represented by a spectrum E.

3.2. Show that the category of spectra is simplicial using the definition

$$\mathbf{Hom}(X,Y)_n = \mathrm{Hom}(X \wedge \Delta[n]_+, Y),$$

where for any pointed simplicial set K and spectrum E, $(E \wedge K)_n = E_n \wedge K$.

3.3. (a) Show that the category of chain complexes has a symmetric monoidal structure with unit given by $\mathbf{Z}[0]$ defined by

$$(C_\bullet \otimes D_\bullet)_n = \oplus_{i+j=n} C_i \otimes D_j.$$

Show that this product distributes over the direct sum of chain complexes, in the sense that

$$(C \oplus D) \otimes E \cong (C \otimes E) \oplus (D \otimes E).$$

(b) Show that $\wedge$ gives a symmetric monoidal structure to the category of pointed spaces and distributes over $\vee$.

(c) Consider the category of spectra as defined in Bousfield–Friedlander. We wish to construct a symmetric monoidal structure on this category with the smash product $\wedge$ as the symmetric monoidal structure. We can try to define such a smash product structure analogously to the tensor product on chain complexes by defining

$$(X \wedge Y)_n = \bigvee_{i+j=n} X_i \wedge Y_j.$$

Explain why this does *not* give a symmetric monoidal structure on spectra with $\mathbb{S}$ as the unit.

3.4. Let G be a finite group, considered as a category (i.e. the category has one object, G is the morphisms, and composition is multiplication). For any category $\mathcal{C}$, a *G-representation* in $\mathcal{C}$ is a functor $G \longrightarrow \mathcal{C}$.

(a) Let $\alpha\colon H \longrightarrow G$ be a homomorphism of groups. There is an induced functor

$$\mathrm{Res}^G_H\colon \mathrm{Rep}(G,\mathcal{C}) \longrightarrow \mathrm{Rep}(H,\mathcal{C}).$$

Show that this functor has a left adjoint; this is denoted Ind^G_H.

(b) Suppose that $\mathcal{C}$ is a symmetric monoidal category such that $\otimes$ commutes with coproducts. A *symmetric sequence* is a sequence

$$X_0, X_1, X_2, \ldots$$

such that $X_i \in \mathrm{Rep}(\Sigma_i, \mathcal{C})$. Show that the category of symmetric sequences in $\mathcal{C}$ has a symmetric monoidal structure with

$$(X \otimes Y)_n = \coprod_{i+j=n} \mathrm{Ind}^{\Sigma_n}_{\Sigma_i \times \Sigma_j}(X_i \otimes Y_j).$$

(c) How does this definition relate to the smash product of symmetric spectra? (Hint: consider $\mathbb{S}$-modules.)

3.5. An *Ω-spectrum* is a spectrum each of whose structure maps induces a weak equivalence $X_n \longrightarrow \Omega X_{n+1}$. Look up the axioms of a generalized cohomology theory. Let E be an Ω-spectruc. Prove that the sequence of functors $h^n\colon Y \longmapsto [Y, E_n]$ is a generalized cohomology theory.[6]

[6]Brown's Representability Theorem gives the converse to this statement: every generalized cohomology theory is represented by a spectrum. However, the categories of cohomology theories and of spectra are *not* equivalent: the category of spectra is richer, and thus usually a more fruitful category to work with.

3.6. Let $X\colon \mathbf{FinSet}_* \longrightarrow \mathbf{Top}$ be a functor from the category of finite pointed sets to $\mathbf{Top}$. Write $S^1 \cong \Delta^1/\partial\Delta^1$, and define $S^n = S^1 \wedge \cdots \wedge S^1$. Note that this is a simplicial set which factors through the forgetful functor $\mathbf{FinSet}_* \longrightarrow \mathbf{Set}$; we can thus think of S^n as a functor $\Delta^{\mathrm{op}} \longrightarrow \mathbf{FinSet}_*$. Define

$$Y_n = |X \circ S^n|.$$

Note that Σ_n acts on Y_n by permuting the S^1-coordinates of S^n. Prove that Y is a symmetric spectrum.

3.7. Let R be a commutative ring spectrum. Let M be a right R-module spectrum and N a left R-module spectrum. We define

$$M \wedge_R N = \mathrm{coeq}(M \wedge R \wedge N \rightrightarrows M \wedge N).$$

There is a spectral sequence

$$E^2_{s,t} \cong \mathrm{Tor}^{\pi_* R}_{s,t}(\pi_* M, \pi_* N) \Longrightarrow \pi_*(M \wedge_R N).$$

Now let k be a field and R a k-module. Prove that $HR \wedge_{Hk} HR \cong H(R \otimes_k R)$, but $HR \wedge HR \ncong H(R \otimes_{\mathbb{Z}} R)$.

4 Replacing algebra with geometry

4.1. Check that the definition of group homology of G with coefficients in a left G-module M agrees with the homology of the bar construction $B(*, G, M)$ (considered as a simplicial set).

4.2. Let $\mathcal{V}$ be a closed symmetric monoidal category. A *category $\mathcal{C}$ enriched in $\mathcal{V}$* is

- a collection of objects $\mathrm{ob}\,\mathcal{C}$,
- for every $A, B \in \mathrm{ob}\,\mathcal{C}$, an object $\mathcal{C}(A,B) \in \mathcal{V}$,
- a *composition law*

  $$\circ\colon \mathcal{C}(B,C) \otimes \mathcal{C}(A,B) \longrightarrow \mathcal{C}(A,C)$$

  for every triple $A, B, C \in \mathrm{ob}\,\mathcal{C}$, and
- a morphim $I \longrightarrow \mathcal{C}(A,A)$ called the *identity* for every $A \in \mathrm{ob}\,\mathcal{C}$. Here, I is the unit of the monoidal structure in $\mathcal{V}$.

(a) Write down associativity and unitality laws for composition.

(b) Check that an ordinary category is a category enriched over $\mathbf{Set}$, where $\otimes = \times$.

(c) Check that an abelian category is a category enriched over AbGp, with $\otimes$ being the tensor product of abelian groups. Why 't it work if we take the monoidal structure $(\mathrm{AbGp}, \oplus)$ instead of $(\mathrm{AbGp}, \otimes)$?

(d) Check that a simplicially enriched category is a category enriched over $s\mathbf{Set}$.

4.3. Let $X_\bullet$ be a simplcial abelian group. Define

$$C_n = \bigcap_{i=1}^{n} \ker d_i.$$

Prove that $X_\bullet \longmapsto C_\bullet$ gives a functor $s\mathbf{Set} \longrightarrow \mathrm{Ch}_{\mathbb{Z}}$. Construct an inverse functor and show that this gives an equivalence of categories.

4.4. Let R be a ring, and consider its Hochschild complex. Write down a simplicial set which has the Hochschild complex as its homology. Now use this simplicial set to produce an analogous spectrum whose geometric realization we can call $THH(R)$. (You'll need to replace R with HR.) Explain why $\pi_* THH(R) \ncong HH_*(R)$.

4.5. Use the simplicial structure on the category of spectra to explain why we can define $THH(R) \stackrel{\mathrm{def}}{=} S^1 \otimes HR$.