PROBLEMS ON FUNDAMENTAL GROUPS OF CURVES ARIZONA WINTER SCHOOL 2020

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Let C be a smooth curve over a number field K with a given base point $b \in C(K)$. The goal of this problem set is to better understand various fundamental groups associated to C:

$$\pi_1(C(\mathbb{C}), b), \quad \pi_1^{\text{ét}}(C_{\bar{K}}, b), \quad \pi_1^{\text{ét}}(C, b), \quad \pi_1^{\mathbb{Q}_p,\text{un}}(C_{\bar{K}}, b), \quad \pi_1^{\text{dR}}(C, b)$$

1. Topological fundamental groups

Problem 1.1. Let X be a topological space. The *path space* $X^{[0,1]}$ of X is the set of all paths in X, i.e., continuous functions $\gamma\colon [0,1] \to X$, equipped with the compact-open topology. (Recall the definition of the compact-open topology: for every compact subset $K \subseteq [0,1]$ and every open subset $U \subseteq X$, let $V(K,U)$ be the set of all paths γ such that $\gamma(K) \subseteq U$. The compact-open topology is the coarsest topology such that all $V(K,U)$ are open.)

There is a natural map

$$X^{[0,1]} \to X \times X, \qquad \gamma \mapsto (\gamma(0), \gamma(1))$$

sending each path to its endpoints. Denote the fiber of a point $(x,y) \in X \times X$ by $P_{x,y}X$, the space of paths from x to y in X. If X is path-connected, prove that all of the fibers $P_{x,y}X$ are homotopy-equivalent via maps that respect path homotopy (i.e., that send homotopic paths to homotopic paths).

A *groupoid* is a category in which all morphisms are isomorphisms. Here's the main example: Let X be a topological space. The *fundamental groupoid* $\Pi_1(X)$ of X is the category whose objects are points of X, and whose morphisms are homotopy classes of paths, i.e., for each $x, y \in X$, we define

$$\mathrm{Hom}_{\Pi_1(X)}(x,y) = \pi_1(X; x, y) = \{\gamma \in X^{[0,1]} : \gamma(0) = x,\ \gamma(1) = y\}/\sim,$$

where $\gamma_1 \sim \gamma_2$ if and only if γ_1 is homotopic to γ_2. The composition operation in this category is induced by concatenation of paths. (If you haven't studied algebraic topology before, it's a good exercise to verify that this is well-defined and associative.) Note that $\pi_1(X,b) = \pi_1(X; b, b) = \mathrm{Aut}_{\Pi_1(X)}(b)$ for all $b \in X$.

Problem 1.2. Let Π be a groupoid. We will refer to objects of Π as points and morphisms as paths, even if Π isn't a fundamental groupoid. There is an obvious notion of path-component for groupoids: two points x and y of Π are in the same path-component if and only if $\mathrm{Hom}_\Pi(x,y)$ is nonempty.

(1) Prove that, if x and y are points of Π in the same path-component, then

$$\mathrm{Aut}_\Pi(x) \cong \mathrm{Aut}_\Pi(y).$$

We call this the *fundamental group* of the path-component of the groupoid.

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(2) Prove that two groupoids Π and Π' are equivalent (as categories) if and only if there is a bijection between path-components of Π and path-components of Π' such that the fundamental groups of corresponding path-components are isomorphic.

Let X be a topological space. A *covering space* of X is a pair (Y, p) where Y is a topological space and $p\colon Y \to X$ is a continuous map such that, for every $x \in X$, there exists an open neighborhood U of x such that $p^{-1}(U)$ can be written as a disjoint union $\bigsqcup_i V_i$, where each V_i is an open subset of Y such that p induces a homeomorphism from V_i to U.

Let $\mathrm{Cov}(X)$ be the category of covering spaces of X: the objects are covering spaces of X, and a morphism $(Y_1, p_1) \to (Y_2, p_2)$ is a continuous map $f\colon Y_1 \to Y_2$ such that $p_2 \circ f = p_1$.

A *universal cover* of X is a connected, simply-connected covering space. It is a standard theorem in algebraic topology that, if X is connected, locally path-connected, and semi-locally simply connected, then X has a universal cover. (For example, this is the case if X is a manifold or the set of $\mathbb{C}$-points of a variety.) Moreover, universal covers are unique (up to non-unique isomorphism).

Problem 1.3. Let X be a path-connected, locally simply-connected space. Fix a base point $b \in X$. Let $p\colon \tilde{X} \to X$ be a universal cover of X. Define

$$\pi_1^{cov}(X) = \mathrm{Aut}(\tilde{X}/X) = \{\text{homeomorphisms } f\colon \tilde{X} \to \tilde{X} \text{ such that } p \circ f = p\},$$

the group of automorphisms of the universal cover. Prove that $\pi_1(X, b) \cong \pi_1^{cov}(X)$.

Problem 1.4. We have a functor from the category of covers of X to the category of sets

$$F_b\colon \mathrm{Cov}(X) \to \mathrm{Set}$$
$$(Y, p) \mapsto p^{-1}(b)$$

sending each cover of X to the fiber of b (and defined in the obvious way on morphisms). Construct a group isomorphism between $\pi_1(X, b)$ and $\mathrm{Aut}(F_b)$, the set of automorphisms of the functor F_b.

Problem 1.5. Let X be a connected, locally path-connected, and locally simply-connected topological space. Let $b \in X$ be a base point. Prove that $\mathrm{Cov}(X)$ is equivalent to the category of left $\pi_1(X, b)$-sets (i.e., sets equipped with a left $\pi_1(X, b)$-action). Show that, under this correspondence, connected covers correspond to sets with transitive $\pi_1(X, b)$-action. Which $\pi_1(X, b)$-sets do Galois covers correspond to? (A cover $Y \to X$ is *Galois* if the natural map $\mathrm{Aut}(Y/X)\backslash Y \to X$ is a homeomorphism.)

Problem 1.6. The above functor F_b restricts to a functor $F_b^{\mathrm{fin}}\colon \mathrm{FinCov}(X) \to \mathrm{FinSet}$ sending finite covering spaces to their (finite) fibers above b. Prove that $\mathrm{Aut}(F_b^{\mathrm{fin}})$ is the profinite completion of $\pi_1(X, b)$. (The *profinite completion* of a group G is the inverse limit of the groups G/N, where N runs through the normal subgroups of finite index in G.)

Problem 1.7. Describe all connected covering spaces of the circle S^1.

Problem 1.8. Compute the fundamental group of a 2-dimensional torus $\mathbb{R}^2/\mathbb{Z}^2$ with one point removed.

Problem 1.9. Let S be a compact Riemann surface of genus g, and let $p_1, \ldots, p_s \in S$ be distinct points. Give a presentation of

$$\pi_1(S \setminus \{p_1, \ldots, p_s\}, b)$$

in terms of generators and relations.

Problem 1.10. Let S be a compact Riemann surface, and let $b \in S$ be a base point. Let J be the Jacobian variety of S, and let $O \in J$ be the identity element. Prove that J is an Eilenberg–MacLane space for the abelianization of the fundamental group of S, i.e., that

$$\pi_i(J, O) \cong \begin{cases} \pi_1(S, b)^{ab} \cong H_1(S) & \text{if } i = 1, \\ 1 & \text{if } i > 1. \end{cases}$$

2. Étale fundamental groups

[Useful references: Milne, *Lectures on Étale Cohomology*; Szamuely, *Galois Groups and Fundamental Groups*.]

Let V be a connected variety over a field K, and fix $b \in V(\bar{K})$. The *étale fundamental group* of V is

$$\pi_1^{\text{ét}}(V, b) = \operatorname{Aut}(F_b^{\text{ét}}),$$

where $F_b^{\text{ét}} \colon \text{FÉt}(V) \to \text{Set}$ is the fiber functor sending each finite étale morphism $p \colon Y \to V$ to the set of $\bar{K}$-points of $p^{-1}(b)$.

As with Galois theory of fields or the topological fundamental group, we have a "fundamental theorem of Galois theory" for schemes:

Theorem (Grothendieck)**.** *The functor $F_b^{\text{ét}}$ induces an equivalence between* FÉt(V) *and the category of finite continuous left $\pi_1^{\text{ét}}(V, b)$-sets. Under this correspondence, connected covers correspond to sets with transitive $\pi_1^{\text{ét}}(V, b)$-action, and Galois covers correspond to quotients of $\pi_1^{\text{ét}}(V, b)$ by a normal open subgroup.*

Problem 2.1. By the Riemann existence theorem, if V is a variety over $\mathbb{C}$, and $M \to V(\mathbb{C})$ is a finite covering map, then there exists a variety Y and an étale cover $Y \to V$ such that $Y(\mathbb{C}) \cong M$ (as complex manifolds), compatibly with the map $M \to V(\mathbb{C})$. Deduce that there is an equivalence of categories

$$\begin{aligned} \text{FÉt}(V) &\to \text{FinCov}(V(\mathbb{C})) \\ Y &\mapsto Y(\mathbb{C}) \end{aligned}$$

between finite étale covers of V and finite covering spaces of $V(\mathbb{C})$. Use this to prove that the étale fundamental group

$$\pi_1^{\text{ét}}(V, b) = \operatorname{Aut}(F_b^{\text{ét}}),$$

where $F_b^{\text{ét}}$ is the étale fiber functor sending each finite étale cover $p \colon Y \to V$ to the fiber $p^{-1}(b)$, is isomorphic to the profinite completion of the topological fundamental group $\pi_1(V(\mathbb{C}), b)$.

On the other hand, when V is a point over a non-algebraically-closed field, the étale fundamental group is the Galois group:

Problem 2.2. Let k be a field. Let k^{sep} be the separable closure of k. Let $b \colon \operatorname{Spec} \bar{k} \to \operatorname{Spec} k$ be induced by an algebraic closure. Prove that

$$\pi_1^{\text{ét}}(\operatorname{Spec} k, b) \cong \operatorname{Gal}(k^{sep}/k).$$

In general, given a geometrically connected variety V over a field k with a point $b \in V(k)$, we have a short exact sequence

$$1 \to \pi_1(V_{\bar{k}}, b) \to \pi_1(V, b) \to \operatorname{Gal}(k^{sep}/k) \to 1.$$

In other words, the *arithmetic fundamental group* $\pi_1(V, b)$ is an extension of the absolute Galois group of k by the *geometric fundamental group* $\pi_1(V_{\bar{k}}, b)$.

Problem 2.3. Let V be a geometrically connected variety over a field k. Suppose $V(k)$ is nonempty. Prove that the above short exact sequence of fundamental groups splits.

Problem 2.4. Compute the étale fundamental group of:

(1) an elliptic curve over $\mathbb{C}$
(2) a nodal cubic curve over $\mathbb{C}$
(3) an elliptic curve over $\mathbb{C}$, with the origin removed
(4) $\mathbb{P}^1_{\bar{\mathbb{Q}}} \setminus \{0, 1, \infty\}$
(5) the multiplicative group $\mathbb{G}_m$ over the field of Laurent series $\mathbb{C}((t))$
(6) $\operatorname{Spec} \mathbb{Z}$

Problem 2.5. Let $f \colon S' \to S$ be a morphism of connected varieties over a field k, and $b' \in S'(\bar{k})$ and $b \in S(\bar{k})$ geometric base points such that $f(b') = b$. The étale property is preserved under base change, so we have a functor $\mathrm{F\acute{E}t}(S) \to \mathrm{F\acute{E}t}(S')$ sending each covering space $Y \to S$ to the cover $Y \times_S S' \to S'$. Since $f(b') = b$, this induces a homomorphism

$$f_* \colon \pi_1^{\text{ét}}(S', b') \to \pi_1^{\text{ét}}(S, b).$$

Prove that:

(1) f_* is surjective if and only if, for every connected finite étale cover $Y \to S$, the base change $Y \times_S S'$ is connected.
(2) f_* is injective if and only if, for every connected finite étale cover $Y' \to S'$, there exists a finite étale cover $Y \to S$, a connected component Y_0 of $Y \times_S S'$, and a morphism $Y_0 \to Y'$ over S'. (For example, this is the case if the map $S' \to S$ itself is étale.)

Problem 2.6. Let X be a smooth complete intersection in $\mathbb{P}^N_{\mathbb{C}}$, and fix $b \in X(\mathbb{C})$. Prove that, if $\dim X \geq 3$, then $\pi_1^{\text{ét}}(X, b)$ is trivial.

3. Tannakian categories

[Useful references: Deligne and Milne, *Tannakian Categories*; Szamuely, *Galois Groups and Fundamental Groups*, ch. 6]

The fundamental groups that we use in the non-abelian Chabauty method are mostly *Tannakian* fundamental groups, which are linear algebraic groups associated to *Tannakian categories.* This can be thought of as a linearization of the theory of fundamental groups.

Definition. A *tensor category* is a category $\mathcal{C}$ together with a functor $\otimes \colon \mathcal{C} \times \mathcal{C} \to \mathcal{C}$ which is (up to functorial isomorphism) associative, commutative, and has an identity object $\mathbb{1}$.

For objects X and Y of $\mathcal{C}$, if the contravariant functor $T \mapsto \operatorname{Hom}(T \otimes X, Y)$ is representable (i.e., isomorphic to the contravariant functor $T \mapsto \operatorname{Hom}(T, A)$ for some object A), then the representing object is called an *internal hom* and denoted $\underline{\mathcal{H}\mathrm{om}}(X, Y)$. The *dual* of an object X is $X^\vee := \underline{\mathcal{H}\mathrm{om}}(X, \mathbb{1})$.

A tensor category is called *rigid* if all internal homs $\underline{\mathcal{H}\mathrm{om}}(X, Y)$ exist; the natural map $\underline{\mathcal{H}\mathrm{om}}(X_1, Y_1) \otimes \underline{\mathcal{H}\mathrm{om}}(X_2, Y_2) \to \underline{\mathcal{H}\mathrm{om}}(X_1 \otimes X_2, Y_1 \otimes Y_2)$ is an isomorphism for all objects X_1, X_2, Y_1, Y_2; and all objects X of $\mathcal{C}$ are *reflexive*, i.e., the natural morphism $X \to X^{\vee\vee}$ to the double dual is an isomorphism.

An *abelian tensor category* is a tensor category $(\mathcal{C}, \otimes)$ such that $\mathcal{C}$ is an abelian category and $\otimes$ is additive in each argument.

A *neutral Tannakian category* over a field k is a rigid abelian tensor category $(\mathcal{C}, \otimes)$ such that $k \cong \mathrm{End}(\mathbb{1})$ and there exists an exact faithful k-linear tensor functor $\omega\colon \mathcal{C} \to \mathrm{Mod}_k$ (the category of k-vector spaces). Such a functor is called a *fiber functor*.

The *Tannakian fundamental group* of $(\mathcal{C}, \otimes, \omega)$ is the functor $\underline{\mathcal{A}\mathrm{ut}}^{\otimes}(\omega)$ of tensor-compatible automorphisms of ω.

Theorem. *Let $(\mathcal{C}, \otimes)$ be a neutral Tannakian category over k, and let ω be a fiber functor. Then $\underline{\mathcal{A}\mathrm{ut}}^{\otimes}(\omega)$ is represented by an affine group scheme G (which naturally acts on objects of $\mathcal{C}$), and the functor $\mathcal{C} \to \mathrm{Rep}_k(G)$ defined by ω is an equivalence of tensor categories.*

In other words, a neutral Tannakian category can be recovered from its fundamental group as the category of representations of the group (and conversely, an affine group scheme can be recovered from its category of finite-dimensional representations as the Tannakian fundamental group).

Problem 3.1. A real Hodge structure is a finite-dimensional vector space V over $\mathbb{R}$ together with a decomposition $V \otimes \mathbb{C} = \bigoplus_{p,q} V^{p,q}$ such that $V^{p,q}$ and $V^{q,p}$ are complex conjugate subspaces of $V \otimes \mathbb{C}$. Prove that the category of real Hodge structures is a neutral Tannakian category, and that the fundamental group of the fiber functor $(V, (V^{p,q})) \mapsto V$ is isomorphic to $\mathrm{Res}^{\mathbb{C}}_{\mathbb{R}}(\mathbb{G}_m)$.

Problem 3.2. Let X be a manifold. Prove that the category of local systems (i.e., locally constant sheaves) on X is a neutral Tannakian category.

Problem 3.3 (Szamuely, exercise 6.4)**.** Let A be a commutative ring. Let $\mathcal{C}$ be a full subcategory of the category of finitely-generated A-modules containing A and closed under tensor products. Show that $(\mathcal{C}, \otimes)$ is a rigid tensor category if and only if its objects are projective A-modules.

Problem 3.4. Let G be an affine group scheme over a field k.

(1) Prove that G is finite if and only if there is an object X of $\mathrm{Rep}_k(G)$ such that every object of $\mathrm{Rep}_k(G)$ is isomorphic to a subquotient of $X^{\oplus n}$ for some $n \geq 0$.
(2) Prove that G is finite-dimensional (as a k-variety) if and only if there is an object X of $\mathrm{Rep}_k(G)$ such that every object of $\mathrm{Rep}_k(G)$ is isomorphic to a subquotient of a polynomial (with respect to direct sum and tensor product) in X and $X^{\vee}$.

4. Unipotent fundamental groups

[Useful reference: Hain and Matsumoto (2001), "Weighted completion of Galois groups and Galois actions on the fundamental group of $\mathbb{P}^1 - \{0, 1, \infty\}$", Appendix A]

A *unipotent algebraic group* over a field k is an algebraic group isomorphic to a closed algebraic subgroup of the group of upper-triangular matrices with all diagonal entries 1 in $\mathrm{GL}_n(k)$. A *pro-unipotent group scheme* is a group scheme isomorphic to an inverse limit of unipotent algebraic groups.

Let Γ be a topological group, and let k be a topological field of characteristic zero. The *(continuous) k-unipotent completion* of Γ is a pro-unipotent group scheme Γ^{un} over k, together with a continuous group homomorphism $\theta\colon \Gamma \to \Gamma^{\mathrm{un}}(k)$ with the universal property that, for every unipotent algebraic group U/k and every continuous group homomorphism $\varphi\colon \Gamma \to U(k)$, there exists a unique regular homomorphism $\psi\colon \Gamma^{\mathrm{un}} \to U$ such that $\varphi = \psi \circ \theta$.

One can construct unipotent completions as follows:

$$\Gamma^{\mathrm{un}} = \varprojlim_{\rho} U_\rho,$$

where ρ ranges over all Zariski-dense continuous representations $\rho\colon \Gamma \to U_\rho(k)$ from Γ into a unipotent k-group.

Problem 4.1. Verify that the above construction of Γ^{un} satisfies the universal property.

Problem 4.2. Here are two ways of constructing the $\mathbb{Q}_p$-unipotent fundamental group $\pi_1^{\mathbb{Q}_p,un}(X,b)$ of a smooth variety $X/\bar{\mathbb{Q}}$ with base point $b \in X(\bar{\mathbb{Q}})$:

(1) take the $\mathbb{Q}_p$-unipotent completion of the (geometric) étale fundamental group of X;
(2) take the Tannakian fundamental group of the category of unipotent $\mathbb{Q}_p$-smooth sheaves (i.e., locally constant p-adic sheaves which have a filtration whose associated graded pieces are direct sums of the trivial $\mathbb{Q}_p$-sheaf) on X.

Prove that the resulting pro-unipotent group schemes are isomorphic. (Hint: prove that (unipotent) local systems correspond to (unipotent) representations of the fundamental group.)

The following two exercises show the difference between unipotent groups in characteristic zero and in positive characteristic:

Problem 4.3. Let k be a field of characteristic zero. Let U be a unipotent algebraic group over k. Prove that U is connected.

Problem 4.4. Let k be a field of characteristic $p > 0$. Let G be a finite p-group. Prove that every finite-dimensional k-representation of G is unipotent.

Problem 4.5. Let k be a field of characteristic zero. Let Γ be a group such that the abelianization Γ^{ab} is finite. Prove that the k-unipotent completion of Γ is trivial.

Problem 4.6. Let $f\colon Y \to X$ be a dominant map of smooth connected curves over $\bar{\mathbb{Q}}$, and let $y \in Y$ and $x \in X$ be (geometric) base points such that $f(y) = x$. Prove that the induced map

$$\pi_1^{\mathbb{Q}_p,un}(Y,y) \to \pi_1^{\mathbb{Q}_p,un}(X,x)$$

is surjective.

Problem 4.7. Let U be a unipotent algebraic group, and let $U^{ab} = U/[U,U]$ be its abelianization. Let $G \subseteq U$ be a closed subgroup such that G surjects onto U^{ab} via the natural map. Prove that $G = U$. Use this to show that a group with trivial abelianization has trivial unipotent completion.

Problem 4.8. Give an example of a smooth projective variety whose étale fundamental group is nontrivial, but whose $\mathbb{Q}_p$-unipotent fundamental group is trivial.

Problem 4.9. Let $X = \mathbb{P}^1_{\bar{\mathbb{Q}}} \setminus \{0,1,\infty\}$. Let U be the $\mathbb{Q}_p$-unipotent fundamental group of X. Let $U_n = U/U^{n+1}$, where the subscript denotes the lower central series of U, defined inductively by $U^1 = U$ and $U^{n+1} = [U,U^n]$ for $n \geq 1$.

Give an asymptotic formula for the dimension of U_n (as a variety over $\mathbb{Q}_p$) as n grows. (Easier version: just get the leading term. Harder version, if you like combinatorics: get an exact formula.)

Problem 4.10. Let U_n be as in the previous problem. Give a faithful representation of U_2 and U_3 as matrix groups.

Problem 4.11. Let C be a smooth projective curve of genus $g \geq 2$ over a number field K, and let $b \in C(K)$ be arbitrary. Let $U = \pi_1^{\mathbb{Q}_p,un}(C_{\bar{K}}, b)$ be the $\mathbb{Q}_p$-unipotent fundamental group of C, and let $U_n = U/U^{n+1}$, as before. Give an asymptotic formula for the dimension of U_n.

Problem 4.12. Let U_n be as in the previous problem, and let $W_n = U_n/[[U_n, U_n], [U_n, U_n]]$ be the quotient of U_n by the third level of the derived series. Give an asymptotic formula for the dimension of W_n. (This quotient is sometimes more convenient for computations; for example, Coates and Kim use this in their 2010 work on curves with CM Jacobian.)

5. De Rham fundamental groups

[Useful references: Kim's 3rd AWS lectures notes; Hain's lecture notes from the 2005 AWS, titled "Lectures on the Hodge–de Rham theory of the fundamental group of $\mathbb{P}^1 \setminus \{0, 1, \infty\}$"]

Let X be a smooth variety over a field K of characteristic zero. Let $\mathrm{Un}(X)$ be the category of unipotent vector bundles on X, i.e., (algebraic) vector bundles $\mathcal{V}$ equipped with a flat connection

$$\nabla_{\mathcal{V}} \colon \mathcal{V} \to \Omega_{X/K} \otimes \mathcal{V}$$

that admit a filtration $\mathcal{V} = \mathcal{V}_n \supset \cdots \supset \mathcal{V}_1 \supset \mathcal{V}_0 = 0$ by subbundles stable under the connection, such that each $\mathcal{V}_{i+1}/\mathcal{V}_i$ is a trivial bundle with connection. (The morphisms are maps of sheaves preserving the connection.)

Problem 5.1. Verify that $\mathrm{Un}(X)$ is a neutral Tannakian category. Verify that, given any $b \in X(K)$, the functor $F_b^{\mathrm{dR}} \colon \mathrm{Un}(X) \to \mathrm{Mod}_K$ sending each $(\mathcal{V}, \nabla_{\mathcal{V}})$ to the fiber of $\mathcal{V}$ above b is a fiber functor.

The *de Rham fundamental group* of X is the Tannakian fundamental group

$$\pi_1^{\mathrm{dR}}(X, b) = \underline{\mathcal{A}\mathrm{ut}}^{\otimes}(F_b^{\mathrm{dR}}).$$

Problem 5.2. Compute the de Rham fundamental group of the multiplicative group $\mathbb{G}_m$.

Problem 5.3. Let $\mathrm{Un}_n(X)$ be the full Tannakian subcategory of $\mathrm{Un}(X)$ generated by bundles admitting a unipotent filtration of length $\leq n$. Prove that the Tannakian fundamental group of $\mathrm{Un}_n(X)$, with fiber functor given by the restriction of F_b^{dR} to $\mathrm{Un}_n(X)$, is isomorphic to the quotient of $\pi_1^{\mathrm{dR}}(X, b)$ by the $(n+1)$-st level of its lower central series.

The de Rham fundamental group over $\mathbb{Q}_p$ can be related to p-adic iterated integrals on differential forms. Here's the Archimedean analogue: Let M be a (real or complex) manifold. Let $F = \mathbb{R}$ or $\mathbb{C}$. (A good reference for the below are Hain's 2005 AWS notes, which are also the source of the below exercises. If you want a more thorough understanding of iterated integrals and their relation to fundamental groups, do more of the exercises from those notes.)

Definition. A *homotopy functional* on M is a map $f \colon M^{[0,1]} \to F$ such that the value of f at a path $\gamma \in M^{[0,1]}$ depends only on the homotopy class of γ.

Definition. Let $w_1, \ldots, w_r$ be smooth 1-forms on M with values in F. Let $\gamma \colon [0, 1] \to M$ be a path. Define

$$\int_\gamma w_1 \ldots w_r = \int_{0 \leq t_1 \leq \cdots \leq t_n \leq 1} f_1(t_1) \ldots f_r(t_r)\, dt_1 \ldots dt_r,$$

where $\gamma^* w_j = f_j(t)\,dt$. (Note that this can be viewed as integration over a simplex $\Delta^r = \{(t_1, \dots, t_r) \in \mathbb{R}^r : 0 \le t_1 \le \cdots \le t_r \le 1\}$.) An *iterated integral* is a linear combination of functions $M^{[0,1]} \to F$ of the above form.

Let's look at some key combinatorial identities for iterated integrals that shed light on their highly nonabelian algebraic structure.

Problem 5.4 (Hain 2005, exercise 5)**.** Show that

$$\Delta^r = \bigcup_{j=0}^{r} \{(t_1, \dots, t_r) \in \mathbb{R}^r : 0 \le t_1 \le \cdots \le t_j \le 1/2 \le t_{j+1} \le \cdots \le t_r\},$$

and that there is a natural identification of $\Delta^j \times \Delta^{r-j}$ with

$$\{(t_1, \dots, t_r) \in \mathbb{R}^r : 0 \le t_1 \le \cdots \le t_j \le 1/2 \le t_{j+1} \le \cdots \le t_r\}.$$

Problem 5.5 (Hain 2005, exercise 6)**.** Let r and s be nonnegative integers. A *shuffle of type* (r, s) is a permutation $\sigma \in S_{r+s}$ such that

$$\sigma^{-1}(1) < \sigma^{-1}(2) < \cdots < \sigma^{-1}(r)$$

and

$$\sigma^{-1}(r+1) < \sigma^{-1}(r+2) < \cdots < \sigma^{-1}(r+s).$$

Let $\mathrm{Sh}(r, s)$ be the set of shuffles of type (r, s). Prove that

$$\Delta^r \times \Delta^s = \bigcup_{\sigma \in \mathrm{Sh}(r,s)} \{(t_1, t_2, \dots, t_{r+s}) : 0 \le t_{\sigma(1)} \le t_{\sigma(2)} \le \cdots \le t_{\sigma(r+s)} \le 1\}.$$

Problem 5.6. Deduce the following identities for iterated integrals given smooth 1-forms $w_1, \dots, w_{r+s}$ and paths $\alpha, \beta \in M^{[0,1]}$ on a manifold M:

- **Coproduct:** If $\alpha(1) = \beta(0)$, then
$$\int_{\alpha\beta} w_1 \dots w_r = \sum_{j=0}^{r} \int_\alpha w_1 \dots w_j \int_\beta w_{j+1} \dots w_r.$$
- **Product:**
$$\int_\alpha w_1 \dots w_r \int_\alpha w_{r+1} \dots w_{r+s} = \sum_{\sigma \in \mathrm{Sh}(r,s)} \int_\alpha w_{\sigma(1)} \dots w_{\sigma(r+s)}.$$
- **Antipode:**
$$\int_{\alpha^{-1}} w_1 w_2 \dots w_r = (-1)^r \int_\alpha w_r w_{r-1} \dots w_1.$$

Problem 5.7. Let M be a connected F-manifold, where F is $\mathbb{R}$ or $\mathbb{C}$, and fix $x \in M$. Let $A = H^0(\mathrm{Ch}(P_{x,x}M; F))$ be the set of iterated integrals $P_{x,x}M \to F$ that are homotopy functionals. Equip A with the comultiplication given by

$$\int w_1 \dots w_r \mapsto \sum_{j=0}^{r} \int w_1 \dots w_j \otimes \int w_{j+1} \dots w_r,$$

the augmentation given by evaluation at the constant loop at x, and the antipode given by inverting paths; verify that this makes A into a Hopf algebra.

Let $\mathbb{Z}\pi_1(M, x)$ be the group algebra of the (topological) fundamental group of M. Prove that integration defines a Hopf algebra homomorphism from A into the Hopf algebra of continuous functionals $\mathrm{Hom}_F(\mathbb{Z}\pi_1(M, x), F)$ (with coproduct and antipode induced by the operations on the group algebra).

Problem 5.8. Prove that, with notation as above, if an element $f \in A$ is in the subalgebra $L_n A \subseteq A$ spanned by $\int w_1 \dots w_r$ with $r \leq n$, then the kernel of f (considered as a functional on $\mathbb{Z}\pi_1(M, x)$) contains J^{m+1}, where J is the augmentation ideal of $\mathbb{Z}\pi_1(M, x)$.

In fact, these give isomorphisms, demonstrating that iterated integrals contain a large amount of information about the fundamental group:

Theorem (Chen)**.** *The integration map $A \to \mathrm{Hom}_F(\mathbb{Z}\pi_1(M, x), F)$ is a Hopf algebra isomorphism, and for each $n \geq 0$ induces an isomorphism*

$$L_n A \cong \mathrm{Hom}_F(\mathbb{Z}\pi_1(M, x)/J^{n+1}, F).$$