AWS 2020: Selmer varieties and non-abelian descent

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This course will outline the theory of Selmer varieties. We will describe their construction, some known properties, associated reciprocity laws and their application to Diophantine problems. A rough course outline might look like this:

I. Motivational introduction: a review of elliptic curves, descent, and the conjectures of Birch and Swinnerton-Dyer. Brief overview of the Diophantine geometry of curves of higher genus. Arithmetic fundamental groups of elliptic curves.

II. Arithmetic fundamental groups in general. Unipotent completions. Galois actions on fundamental groups. Some preliminary work on the projective line minus three points. De Rham and crystalline fundamental groups. A brief overview of Deligne's long paper on this topic.

III. Non-abelian cohomology. Construction of local and global moduli spaces of principal bundles. p-adic Hodge theory and iterated integrals. Another view of Deligne. Applications to Siegel's theorem.

IV. Finiteness theorems and connections to mixed motives. Grothendieck's section conjecture and effective computation of rational points. Non-abelian reciprocity.

V. Some speculation on effective computation of Selmer varieties. A return to the projective line minus three points. Punctured elliptic curves with complex multiplication and p-adic L-functions.

A project might involve going through some of the existing papers on effective computations of rational or integral points and making precise the equations for Selmer varieties contained therein. If things go well, we can try to compute some Selmer varieties for punctured elliptic curves with CM. This project will be suitable for students with some background in algebraic and arithmetic geometry including some familiarity with elliptic curves, Galois representations, and knowledge of arithmetic cohomologies or a willingness to work with them as a blackbox that gradually lights up. Some knowledge of the algebraic number theory of local and global fields will also be helpful.

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