ARIZONA WINTER SCHOOL 2022

REPRESENTATION-THEORETIC ASPECT OF AUTOMORPHIC FORMS

PROBLEM SET

This problem set is organized to follow the lectures of Wee Teck Gan and Aaron Pollack. Sections 1-4 deal with automorphic forms and theta correspondences and Sections 5-8 deal with modular forms on exceptional groups. Students will want to have the lecture notes available and references are the references for the lecture notes. In addition, students new to the subject may prefer to look at the following books in order of [3], [2], [1]:

References

[1] D. Bump, J. W. Cogdell, E. de Shalit, D. Gaitsgory, E. Kowalski, and S. S. Kudla. *An introduction to the Langlands program.* Birkhäuser Boston, Inc., Boston, MA, 2003. Lectures presented at the Hebrew University of Jerusalem, Jerusalem, March 12–16, 2001, Edited by Joseph Bernstein and Stephen Gelbart.

[2] J. W. Cogdell, H. H. Kim, and M. R. Murty. *Lectures on automorphic L-functions*, volume 20 of *Fields Institute Monographs.* American Mathematical Society, Providence, RI, 2004.

[3] A. Deitmar. *Automorphic forms.* Universitext. Springer, London, 2013. Translated from the 2010 German original.

1. Automorphic Forms and Theta Correspondence
Gan: Lecture 1 Problems

Problem 1.1. *This exercise concerns the parametrization of unramified representations of $G(F)$ in terms of semisimple classes in the Langlands dual group $G^\vee$.*

(i) Let T be a split torus over a p-adic field F and let $T(\mathcal{O}_F) \subset T(F)$ be the maximal compact subgroup of $T(F)$. A character $\chi : T(F) \to \mathbb{C}^\times$ is unramified if χ is trivial on $T(\mathcal{O}_F)$. The set of unramified characters of is thus

$$\mathrm{Hom}(T(F)/T(\mathcal{O}_F), \mathbb{C}^\times).$$

The dual torus of T is the complex torus defined by

$$T^\vee := X^*(T) \otimes_{\mathbb{Z}} \mathbb{C}^\times,$$

where $X^*(T) = \mathrm{Hom}(T, \mathbb{G}_m)$. Construct a natural bijection

$$\mathrm{Hom}(T(F)/T(\mathcal{O}_F), \mathbb{C}^\times) \cong T^\vee.$$

(ii) Based on (i) and Proposition 1.1 of the lecture notes, deduce that unramified representations of $G = \mathrm{GL}_n(F)$ are paramertrized by semisimple conjugacy classes in $G^\vee = \mathrm{GL}_n(\mathbb{C})$.

Problem 1.2. *This exercise gives you a chance to work with unitary groups in low rank.*

(i) In §1.9 of the lecture notes, we gave as examples the quasi-split unitary groups $\mathrm{U}(V^+)$ with $\dim V^+ = 2$ and 3 and wrote down certain elements as matrices. Let $B \subset \mathrm{U}(V^+)$ be the upper triangular Borel subgroup. Compute the modulus character δ_B as a character of the diagonal torus T.

(ii) At the end of §1.9 of the lecture notes, we introduced the element $u(x, z)$ as a matrix, but one particular entry of the matrix is given as $*$, as it is determined by the others. Determine the entry $*$ explicitly.

(iii) In §1.9 of the lecture notes, we described elements of $\mathrm{U}(V)$ *where* V *is a 3-dimensional Hermitian space. In fact, from Lecture 2 onwards, we will be working with 3-dimensional skew-Hermitian spaces* W. *As what we did for the Hermitian case, write down elements in the Borel subgroup* $B = TU$ *of* $\mathrm{U}(W)$, *with respect to a Witt basis of* W, *i.e. a basis* $\{e, w_0, e^*\}$ *with* $\langle e, e^* \rangle = 1$, $\langle w_0, w_0 \rangle = \delta$ *(with* δ *trace* 0*) and* $\langle e, w_0 \rangle = 0 = \langle e^*, w_0 \rangle$.

Problem 1.3. *To prepare for Lecture 2, you can read the largely self-contained §2.3 which introduces the Heisenberg group* $H(W)$ *associated to a symplectic vector space* W *and then attempt the following problems.*

(i) Let W *be a 3-dimensional skew-Hermitian space as in Problem 1.2(iii) above above, with isometry group* $\mathrm{U}(W)$ *containing the Borel subgroup* $B = TU$. *Write down an isomorphism of* U *with the Heisenberg group associated to a 2-dimensional symplectic space.*

(ii) In §2.3, we introduced the representation

$$\omega_\psi = \mathrm{ind}_{H(X)}^{H(W)} \psi$$

of a Heisenberg group $H(W)$ *on the space* $\mathcal{S}(Y)$ *of Schwarz functions on* Y. *Prove that this representation is indeed irreducible.*

(iii) In the context of §2.3, suppose that $W = W_1 \oplus W_2$ *is the sum of two smaller symplectic spaces, construct a natural surjective group homomorphism*

$$f : H(W_1) \times H(W_2) \longrightarrow H(W).$$

What is the kernel of your homomorphism f*?*

(iv) Let $\omega_{W,\psi}$ *be the irreducible representation of* $H(W)$ *with central character* ψ. *Show that the pullback* $f^*(\omega_{W,\psi})$ *is isomorphic to* $\omega_{W_1,\psi} \otimes \omega_{W_2,\psi}$.

(v) One has a natural embedding

$$f : \mathrm{Sp}(W_1) \times \mathrm{Sp}(W_2) \hookrightarrow \mathrm{Sp}(W).$$

Deduce that f^* *induces a natural isomoprhism of projective representations:*

$$A_{W,\psi} \circ f \cong A_{W_1,\psi} \otimes A_{W_2,\psi}$$

where $A_{W,\psi} : \mathrm{Sp}(W) \to \mathrm{GL}(\mathcal{S})/S^1$ *is as constructed in §2.3.*

Indeed, f *can be lifted to*

$$\tilde{f} : \mathrm{Mp}(W_1) \times \mathrm{Mp}(W_2) \longrightarrow \mathrm{Mp}(W)$$

so that one has an isomoprhism

$$\tilde{f}^*(\omega_{W,\psi}) \cong \omega_{W_1,\psi} \otimes \omega_{W_2,\psi_2}$$

of representations of $\mathrm{Mp}(W_1) \times \mathrm{Mp}(W_2)$.

2. Automorphic Forms and Theta Correspondence Gan: Lecture 2 Problems

Problem 2.1. *This exercise asks you to reconcile the different ways that $\Theta(\pi)$ have been presented in the lecture notes.*

In §2.5, we have given two descriptions of $\Theta(\pi)$:

- $$\pi \boxtimes \Theta(\pi) = \Omega / \bigcap_{f \in \mathrm{Hom}_{\mathrm{U}(V)}(\Omega,\pi)} \mathrm{Ker}(f).$$
- $\Theta(\pi) = (\Omega \otimes \pi^\vee)_{\mathrm{U}(V)}$.

Show that these are equivalent, and prove the "universal property":

$$\mathrm{Hom}_{\mathrm{U}(V)\times\mathrm{U}(W)}(\Omega, \pi \otimes \sigma) \cong \mathrm{Hom}_{\mathrm{U}(W)}(\Theta(\pi), \sigma)$$

for any smooth representation σ of $\mathrm{U}(W)$.

Problem 2.2. *Deduce Corollary 2.5 (dichotomy) from Theorem 2.4 (conservation relation).*

Problem 2.3. *This problem introduces the Schrodinger model of the Weil representation of a dual pair $\mathrm{U}(V) \times \mathrm{U}(W)$, when one of the spaces is split of even dimension.*

In §2.4, we wrote down some formulas in the Schrodinger model for the Weil representation ω_ψ of a metaplectic group; this model is based on a Witt decomposition of the symplectic space. In §2.5, we considered a dual pair $\mathrm{U}(V) \times \mathrm{U}(W)$ with its splitting

$$\tilde{\iota} : \mathrm{U}(V) \times \mathrm{U}(W) \longrightarrow \mathrm{Mp}(V \otimes_E W)$$

associated to a pair (χ_V, χ_W). One can ask if we can inherit the formulas in the Schrodinger model and write down the action of some elements of $\mathrm{U}(V) \times \mathrm{U}(W)$. For this to be possible, there must be some compatibility between the Witt decomposition we used on $V \otimes W$ and the map $\iota : \mathrm{U}(V) \times \mathrm{U}(W) \to \mathrm{Sp}(V \otimes_E W)$.

More precisely, suppose V is a split Hermitian space and we fix a Witt decomposition $V = X \oplus Y$. Then we inherit a WItt decomposition of $V \otimes W$:

$$V \otimes_E W = (X \otimes_E W) \oplus (Y \otimes_E W).$$

Relative to this Witt decomposition, the Schrodinger model of the Weil representation is realized on $\mathcal{S}(Y \otimes W)$ and one can write down explicit formulas for elements of

$$\mathrm{U}(W) \times P(X) \subset P(X \otimes W),$$

where $P(X)$ is the Siegel parabolic subgroup of $\mathrm{U}(V)$ stabilzing X.

Consider the case when $W = E \cdot w = \langle \delta \rangle$ (with $Tr(\delta) = 0$) is 1-dimensional and $V = Ee \oplus Ee^*$ is the split skew-Hermitian space of dimension 2. From the formulas of the Schrodinger model, deduce (as much as you can) the following actions of $\mathrm{U}(W) \times B(E \cdot e))$ on $\mathcal{S}(Y \otimes V) = S(Ee^* \otimes w)$ (relative to the fixed (χ_V, χ_W)):

- for $g \in \mathrm{U}(W) = E^1$,
$$(g \cdot f)(x) = \chi_V(i(g)) f(g^{-1}x),$$
where $i : E^1 \cong E^\times/F^\times$ is the inverse of the isomorphism $i^{-1} : x \mapsto x/x^c$.
- For $t(a) \in T$, $a \in E^\times$,
$$(t(a) \cdot f)(x) = \chi_W(a) \cdot |a|_E^{1/2} \cdot f(a^c \cdot x).$$
- For $u(z) \in U$, with $z \in E$ and $Tr(z) = 0$,
$$(u(z) \cdot f)(x) = \psi(\delta \cdot z \cdot N(x)) \cdot f(x).$$

To be honest, since we did not explicate the definition of the splitting associated to (χ_V, χ_W), you could not really show the above formulasin full , but you can at least deduce those parts of the formula without χ_V or χ_W. The effects of the choice of (χ_V, χ_W) can be seen from the first two formulas.

Actually, this exercise is setting the scene for Problem 2.4 below, so you may take the formulas above as a given.

Problem 2.4. *This exercise continues from Problem 2.3. It is the first exercise that allows you to work with the Weil representation and to calculate some theta lifts.*

Consider the dual pair $\mathrm{U}(V) \times \mathrm{U}(W)$ *as in Problem 2.3, so that* $W = E \cdot w = \langle \delta \rangle$ *and* V *the split 2-dim. Hermitian space. Fix a pair of splitting characters* (χ_V, χ_W) *and consider the associated Weil representation* $\Omega = \Omega_{\chi_V, \chi_W, \psi}$ *of* $\mathrm{U}(V) \times \mathrm{U}(W)$*. Because* $\mathrm{U}(W)$ *is compact,* Ω *is semisimple as a* $\mathrm{U}(W)$*-module and we can write:*
$$\Omega = \bigoplus_{\mu \in \mathrm{Irr}(\mathrm{U}(W))} \mu \otimes \Theta(\mu).$$
The goal is to understand $\Theta(\mu)$ *as much as possible.*

Using the formulas for the Weil representation $\Omega = \Omega_{\chi_V, \chi_W, \psi}$ *from Problem 2.3 above,*

(i) Compute Ω_U *(the* U*-coinvariants of* Ω*) as a module for* $\mathrm{U}(W) \times T$*.*

(ii) For any nontrivial character ψ' *of* $U \cong F \cdot \delta^{-1}$*, compute* $\Omega_{U,\psi'}$ *as a module for* $\mathrm{U}(W) \times Z(\mathrm{U}(V))$*. (Note that there are two orbits of such nontrivial characters* ψ' *under the conjugation action of* $T(F)$*).*

(iii) Using your answers from (i) and (ii), show that $\Theta(\mu)$ *is nonzero irreducible for any irreducible character* μ *of* $\mathrm{U}(W) = E^1$*. Moreover, show that* $\Theta(\mu)$ *is supercuspidal if and only if* $\mu \neq \chi_V \circ i$ *(see Problem 2.3 for the definition of* i*).*

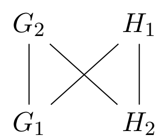
(iv) Show that
$$\Theta(\chi_V \circ i) \hookrightarrow I(\chi_W) := \mathrm{Ind}_{B(E \cdot e)}^{\mathrm{U}(V)} \chi_W.$$
Indeed, your proof should suggest an explicit description of this embedding. More precisely, show that the natural map
$$f \mapsto (h \mapsto (\Omega(h)f)(0))$$
gives a nonzero equivariant map
$$\Omega \longrightarrow (\chi_V \circ i) \otimes I(\chi_W),$$

thus inducing the embedding of $\Theta(\chi_V \circ i)$ *into* $I(\chi_W)$.

The results of this exercise will be used in the next exercise.

Problem 2.5. *The purpose of this exercise is to indicate a proof of the Howe duality conjecture and Theorem 2.6 for the dual pair* $\mathrm{U}_1 \times \mathrm{U}_1$. *It is very long, but is the highlight of the problem sheet! As mentioned in the notes, the proof makes use of the doubling seesaw argument (among other things). We will introduce some of these notions in turn.*

- *(Seesaw pairs) Suppose a group* E *contains two dual pairs* $G_1 \times H_1$ *and* $G_2 \times H_2$ *(so* G_i *is the centralizer of* H_i *in* E *and vice versa). Suppose that* $G_1 \subset G_2$. *Then it follows that* $H_1 \supset H_2$. *In this situation, we say that the two dual pairs form a seesaw pair, and we often represent this in the following seesaw diagram:*



In this diagram, the diagonal line represents a dual pair, and the vertical line denotes containment, with the group at the bottom contained in the group at the top.

- *(Standard example) Here is the standard example of constructing seesaw pairs in the symplectic groups. Suppose that* $V_1 \oplus V_2$ *is the orthogonal sum of two Hermtian spaces. Set* $\mathcal{W} = (V_1 \oplus V_2) \otimes W$ *(a symplectic space over* F*), and note that*
$$\mathcal{W} = (V_1 \otimes W) \oplus (V_2 \otimes W).$$
This gives the following two dual pairs in $\mathrm{Sp}(\mathcal{W})$:
$$\mathrm{U}(V_1 + V_2) \times \mathrm{U}(W^{\Delta}) \quad \textit{and} \quad (\mathrm{U}(V_1) \times \mathrm{U}(V_2))) \times (\mathrm{U}(W) \times \mathrm{U}(W)).$$
Convince yourself that these form a seesaw pair and draw the relevant seesaw diagram.

- *(Seesaw identity) Suppose one has a seesaw diagram as in the abstract situation above, and let* Ω *be a representation of* E, *which we may restrict to* $G_1 \times H_1$ *and* $G_2 \times H_2$. *Deduce the following seesaw identity: for* $\pi \in \mathrm{Irr}(G_1)$ *and* $\sigma \in \mathrm{Irr}(H_2)$, *one has natural isomorphisms*
$$\mathrm{Hom}_{G_1}(\Theta(\sigma), \pi) \cong \mathrm{Hom}_{G_1 \times H_2}(\Omega, \pi \otimes \sigma) \cong \mathrm{Hom}_{H_2}(\Theta(\pi), \sigma).$$
Note that in the above identity, $\Theta(\sigma)$ *is a representation of* G_2, *whereas* $\Theta(\pi)$ *is a representation of* H_1. *This seesaw identity allows one to transfer a restriction problem from one side of the seesaw to the other.*

- *(Compatible splittings) To apply this seesaw identity to the standard example, there is an extra step, because we need to consider splittings of the dual; pair into the metaplectic group* $\mathrm{Mp}(\mathcal{W})$. *To have a splitting of the metaplectic cover for the dual pair* $\mathrm{U}(V_1) \times \mathrm{U}(W)$, *we need to fix a pair* (χ_{V_1}, χ_W).*; likewise, we need to fix* (χ_{V_2}, χ'_W) *for* $\mathrm{U}(V_2) \times \mathrm{U}(W)$. *Similarly, for the dual pair* $\mathrm{U}(V_1 + V_2) \times \mathrm{U}(W^{\Delta})$, *we may fix* $(\chi_{V_1+V_2}, \chi_{W^{\Delta}})$. *So we see that we have 6 splitting characters to fix here. IF we were* *to choose these randomly, then there is no reason for the resulting splittings to be compatible with each other.*

What does being compatible with each other mean? From the viewpoint of the seesaw diagram, in order to have the seesaw identity, we need to ensure that when the splittings of a group at the top of the diagram is restricted to the subgroup below it, the restriction agrees with the splitting below. This is to ensure that we still have a seesaw situation in $\mathrm{Mp}(\mathcal{W})$.

So for example, we fix the character χ_{W^Δ} *which determines the splitting of* $\mathrm{U}(V_1 + V_2)$. *When restricted to* $\mathrm{U}(V_1) \times \mathrm{U}(V_2)$, *the resulting splitting over* $\mathrm{U}(V_1)$ *and* $\mathrm{U}(V_2)$ *are both associated with* χ_{W^Δ}. *This forces us to take*

$$\chi_{W^\Delta} = \chi_W = \chi'_W,$$

and we shall denote this by χ_W. *Likewise, we fix the character* χ_{V_1} *and* χ_{V_2} *which determines a splitting over* $\mathrm{U}(W) \times \mathrm{U}(W)$. *When restricted to the diagonal* $U(W^\Delta)$, *the resulting splitting of the latter is associated with the character* $\chi_{V_1}\chi_{V_2}$. *This forces us to take*

$$\chi_{V_1+V_2} = \chi_{V_1} \cdot \chi_{V_2}.$$

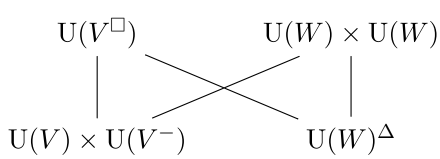
(i) (Doubling seesaw) Now we apply the above to the following concrete situation. We place ourselves in the setting of Theorem 2.6. Hence, let $W = \langle b \cdot \delta \rangle$ *be a 1-dim. skew-Hermitian space, and* $V = \langle a \rangle$ *a 1-dim Hermitian space, so that* $\epsilon(W) = \omega_{E/F}(b)$ *and* $\epsilon(V) = \omega_{E/F}(a)$. *Let* $V^- = \langle -a \rangle$ *and apply the seesaw construction above with* W *as given,*

$$V_1 = V, \quad \text{and} \quad V_2 = V^-$$

so that

$$V^\square := V_1 \oplus V_2 = V \oplus V^-$$

is a split 2-dim. Hermitian space. We thus have the seesaw diagram:



This is called the doubling seesaw, because we have doubled V *(to yield* $V^\square$*), but note that we have introduced a negative sign in the second copy of* V, *so that the doubled-space* $V^\square$ *is split! Indeed, the diagonally embedded* V^Δ *is a maximal isotropic subspace, and one has a Witt decomposition*

$$V^\square = V^\Delta \oplus V^\nabla$$

where $V^\nabla = \{(v, -v) : v \in V\}$.

(ii) (Doubling seesaw identity) Choose splitting characters χ_V, χ_{V^-} *and* χ_W *as explained above. In fact, we insist further (as we may) that*

$$\chi_V = \chi_{V'} = \chi_W = \gamma \quad \text{(a conjugate-symplectic character of } E^\times).$$

Fix a $\chi \otimes \chi' \in \mathrm{Irr}(\mathrm{U}(V) \times \mathrm{U}(V^-))$ *and the character* $\chi_V|_{E^1}$ *of* $\mathrm{U}(W^\Delta)$*. Write down what the doubling seesaw identity gives.*

(iii) (Duality) We have the two decomposition

$$\Omega_{V,W,\gamma,\psi} = \bigoplus_{\chi} \chi \otimes \Theta_{V,W}(\chi),$$

and

$$\Omega_{V^-,W,\gamma,\psi} = \bigoplus_{\chi'} \chi' \otimes \Theta_{V^-,W}(\chi').$$

Express the Weil representation $\Omega_{V^-,W,\gamma,\psi}$ *in terms of* $\Omega_{V,W,\gamma,\psi}$ *and deduce that*

$$\Theta_{V^-,W}(\chi^{-1}\chi_W|_{E^1}) \cong \overline{\Theta_{V,W}(\chi)} \cdot \chi_V|_{E^1}.$$

Using this, show that the RHS of the seesaw identity can be simplified to:

$$\bigoplus_{\chi,\chi'} \chi \otimes {\chi'}^{-1}\chi_W|_{E^1} \otimes \mathrm{Hom}_{\mathrm{U}(W)}(\Theta_{V,W}(\chi) \otimes \overline{\Theta_{V,W}(\chi')}, \mathbb{C}).$$

as a module for $\mathrm{U}(V) \times \mathrm{U}(V^-) \times \mathrm{U}(W^\Delta)$*. (Note that here and below, we could have replaced* χ_V *and* χ_W *by* γ*, but we have refrained from doing so, in order to make the dependence of* (χ_V, χ_W) *more transparent).*

(iv) (Siegel-Weil) The LHS of the doubling seesaw is

$$\mathrm{Hom}_{\mathrm{U}(V)\times\mathrm{U}(V^-)}(\Theta_{W^\Delta,V^\square}(\chi_V|_{E^1}), \chi \otimes {\chi'}^{-1}\chi_W|_{E^1}).$$

To address this problem, we first need to understand $\Theta_{W^\Delta,V^\square}(\chi_V|_{E^1})$*. This is the local version of the Siegel-Weil formula and is where Problem 2.4 comes in.*

Using your results in Problem 2.4 (iv), show that

$$\Theta_{W^\Delta,V^\square}(\chi_V|_{E^1}) \hookrightarrow \mathrm{Ind}_{B(V^\Delta)}^{\mathrm{U}(V^\square)} \chi_W =: I(\chi_W)$$

(v) (Principal series) Show that the principal series $I(\chi_W)$ *is reducible and in fact is the direct sum of two irreducible summands. How does one distinguish between those two summands? Which of these two summand is* $\Theta_{W^\Delta,V^\square}(\chi_V|_{E^1})$ *equal to? Show that* $\Theta_{W^\Delta,V^\square}(\chi_V|_{E^1})$ *is the unique summand of the induced representation whose* (U,ψ')*-coinvariant is nonzero for* $\psi'(z\delta^{-1}) = \psi(bz)$*.*

This shows that if W *and* W' *are the two 1-dim. skew-Hermitian spaces, then*

$$I(\chi_W) = \Theta_{W,V^\square}(\chi_V|_{E^1}) \oplus \Theta_{W',V^\square}(\chi_V|_{E^1}).$$

(vi) (Mackey theory) The previous part implies that it will be necessary to understand $I(\chi_W)$ *as a module for* $\mathrm{U}(V) \times \mathrm{U}(V^-)$*. This can be approached by Mackey theory, as it involves the restriction of an induced representation.*

Show that $\mathrm{U}(V) \times \mathrm{U}(V^-)$ *acts transitively on the flag variety* $B(V^\Delta)\backslash\mathrm{U}(V^\square)$ *with stabilizer of the identity coset given by* $\mathrm{U}(V)^\Delta$*. From this, deduce that as a* $\mathrm{U}(V) \times \mathrm{U}(V^-)$*-module,* $I(\chi_W)$ *is isomorphic to*

$$C_c^\infty(\mathrm{U}(V)) \otimes (1 \otimes \chi_W|_{E^1}),$$

i.e. a twist of the regular representation. Hence, for $\chi \otimes \chi'^{-1} \in \mathrm{Irr}(\mathrm{U}(V) \times \mathrm{U}(V^-))$*, one has*

$$\mathrm{Hom}_{\mathrm{U}(V)\times\mathrm{U}(V^-)}(I(\chi_W), \chi \otimes \chi'^{-1} \cdot \chi_W|_{E^1}) = \begin{cases} \mathbb{C}, & \text{if } \chi' = \chi; \\ 0, & \text{otherwise.} \end{cases}$$

Moreover, as a $\mathrm{U}(V) \times \mathrm{U}(V^-)$*-module,* $\Theta_{W^\Delta, V^\square}(\chi_V|_{E^1})$ *is a submodule of the above twisted regular representation.*

(vii) (Howe duality) Using (iii) and (vi), show the following:

- *For each* $\chi \in \mathrm{Irr}(\mathrm{U}(V))$*,* $\Theta_{V,W}(\chi)$ *is irreducible or* 0*; in particular it is either* χ *or* 0*.*
- *If* $\chi \neq \chi'$*, then* $\Theta_{V,W}(\chi)$ *and* $\Theta_{V,W}(\chi')$ *are disjoint.*

This is the Howe duality theorem for $\mathrm{U}(V) \times \mathrm{U}(W)$*.*

(viii) (Doubling zeta integral) The remaining issue is to decide for which χ *is* $\Theta_{V,W}(\chi) \neq 0$*. This requires the use of the doubling zeta integral. In this context, the doubling zeta integral is an explicit integral which defines a nonzero element of*

$$\mathrm{Hom}_{\mathrm{U}(V)\times\mathrm{U}(V^-)}(I(\chi_W), \chi \otimes \chi^{-1}\chi_W|_{E^1}) = \mathrm{Hom}_{\mathrm{U}(V)\times\mathrm{U}(V^-)}(C_c^\infty(\mathrm{U}(V)), \chi \otimes \chi^{-1}).$$

More precisely, we define

$$Z(s,\chi) : I(s,\chi_W) = \mathrm{Ind}_{B(V^\Delta)}^{\mathrm{U}(V^\square)} \chi_W \cdot |-|_E^s \longrightarrow \mathbb{C}$$

by

$$Z(s,\chi)(f_s) = \int_{\mathrm{U}(V)} f_s(h,1) \cdot \overline{\chi(h)}\, dh.$$

Verify that the integral converges absolutely and defines a nonzero functional

$$Z(,s,\chi) \in \mathrm{Hom}_{\mathrm{U}(V)\times\mathrm{U}(V^-)}(I(s,\chi_W), \chi \otimes \chi^{-1}\chi_W|_{E^1}).$$

Deduce also that $\theta_{V,W}(\chi)$ *is nonzero if and only if* $Z(0,\chi)$ *is nonzero on* $\Theta_{W^\Delta,V^\square}(\chi_V|_{E^1})$*.*

(ix) (Functional equation) There is a standard $\mathrm{U}(V^\square)$*-intertwining operator*

$$M(s) : I(s,\chi_W) \longrightarrow I(-s,\chi_W),$$

whose precise definition need not concern us here. There is a normalization of this intertwining operator (which we will not go into here) with the following properties:

- *one has*

$$M(-s) \circ M(s) = 1.$$

- *At* $s = 0$ *(where* $I(\chi_W)$ *is the sum of two irreducible summands),* $M(s)$ *is holomorphic and* $M(0)$ *acts as* $+1$ *on* $\Theta_{W^\Delta,V^\square}(\chi_V|_{E^1})$ *and as* -1 *on the other summand.*

A basic result in the theory of the doubling zeta integral is that there is a functional equation:

$$\frac{Z(-s,\chi^{-1})(M_s(f_s))}{L_E(\frac{1}{2}-s,\chi_E^{-1}\chi_W)} = \epsilon_E(\frac{1}{2}+s,\chi_E\chi_W^{-1},\psi) \cdot \frac{Z(s,\chi)(f_s)}{L_E(\frac{1}{2}+s,\chi_E\chi_W^{-1})},$$

where we recall that $\chi_E(x) = \chi(x/x^c)$ *for* $x \in E^\times$*. We shall take this as a given.*

(x) (Proof of Theorem 2.6) Using the functional equation and the properties of $M(0)$ *recalled in (ix), as well as the relevant results in earlier parts, prove Theorem 2.6.*

Problem 2.6. *Consider the dual pair* $\mathrm{U}(V) \times \mathrm{U}(W)$ *as in §2.11. Over there, a model for the Weil representation is written down. Understand how the formulas there are deduced from Problem 1.3(v) and Problem 2.1.*

Problem 2.7. *Do the exercise formulated at the end of §2.11.*

3. **Automorphic Forms and Theta Correspondence** **Gan: Lecture 3**

Problem 3.1. *The purpose of this exercise is to do the global analog of Problem 2.4. Hence, we are considering the dual pair* $\mathrm{U}(W) \times \mathrm{U}(V)$ *over a number field* k, *with* $W = E \cdot w = \langle \delta \rangle$ *a 1-dim. skew-Hermitian space (so* $\delta \in E^\times$ *is a trace 0 element) and* $V = Ee \oplus Ee^* = X \oplus Y$ *is a split 2-dim. Hermitian space. As in Problem 2.3, the global Weil representation* Ω *is realized on* $\mathcal{S}(Y_{\mathbb{A}} \otimes W_{\mathbb{A}}) = \mathcal{S}(\mathbb{A}_E e^* \otimes w)$. *The automorphic realization*

$$\theta : \mathcal{S}(Y_{\mathbb{A}} \otimes W_{\mathbb{A}}) \longrightarrow \mathcal{A}(\mathrm{U}(V) \times \mathrm{U}(W))$$

of Ω *is defined by*

$$\theta(\phi)(g,h) = \sum_{v \in V_k} (\Omega(g,h)\phi)(v).$$

For an automorphic character χ *of* $\mathrm{U}(W)$, *its global theta lift* $\Theta(\chi)$ *is spanned by the automorphic forms*

$$\theta(\phi,\chi)(h) = \int_{[\mathrm{U}(W)]} \theta(\phi)(g,h) \cdot \overline{\chi(g)}\, dg$$

as ϕ *ranges over elements of* $\mathcal{S}(Y_{\mathbb{A}} \otimes W_{\mathbb{A}})$.

(i) Recall the Borel subgroup $B = TU$ *of* $\mathrm{U}(V)$ *which is the stabilizer of* $X = E \cdot e$. *For any character* ψ' *of* $U(k)\backslash U(\mathbb{A})$, *compute the* (U, ψ')*-Fourier coefficient*

$$\theta(\phi,\chi)_{U,\psi'}(h) = \int_{[U]} \theta(\phi,\chi)(uh) \cdot \overline{\psi'(u)}\, du.$$

(ii) From your computation in (i), deduce that the global theta lift $\Theta(\chi)$ *is nonzero for any* χ, *and is cuspidal if and only if* $\chi \neq \chi_V \circ i$ *(see Problem 2.3 for the definition of* i*).*

(iii) (Challenging) We would like to express the global theta lift $\theta(\phi, \chi_V \circ i)$ *(which is noncuspidal) as an explicit Eisenstein series. Observe that the map* $\phi \mapsto \theta(\phi, \chi_V \circ i)$ *is an equivariant map*

$$\Omega \twoheadrightarrow (\chi_V \circ i) \otimes \Theta(\chi_V \circ i) \subset (\chi_V \circ i) \otimes \mathcal{A}(\mathrm{U}(V))$$

and that

$$\dim \mathrm{Hom}_{\mathrm{U}(W)\times\mathrm{U}(V)}(\Omega, (\chi_V \circ i) \otimes \Theta(\chi_V \circ i)) = 1.$$

Now we shall produce another element in this 1-dimensional vector space.

Recall from Problem 2.4(iv) that the map $\phi \mapsto (h \mapsto \Omega(h)\phi(0))$ *defines an equivariant map*

$$j : \Omega \longrightarrow (\chi_V \circ i) \otimes I(\chi_W)$$

whose image isisomorphic to $(\chi_V \circ i) \otimes \Theta(\chi_V \circ i)$. *Now the Eisenstein series is a* $\mathrm{U}(V)$*-equivariant map*

$$E(s,-) : I(s,\chi_W) \longrightarrow \mathcal{A}(\mathrm{U}(V))$$

defined by

$$E(s,f)(h) = \sum_{\gamma \in B\backslash \mathrm{U}(V)} f(\gamma g).$$

This converges only when $Re(s)$ *is sufficiently large (actually* $Re(s) > 1/2$*), but a basic theorem is that it admits a meromorphic continuation to* $\mathbb{C}$ *and that it is holomorphic at* $s = 0$*. Admitting this, we can thus consider* $E(-) := E(0, -)$*.*

Now we have the composite map

$$E \circ j : \Omega \longrightarrow (\chi_V \circ i) \otimes \mathcal{A}(\mathrm{U}(V)).$$

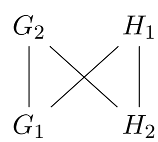
Show that this map is nonzero, so that its image is isomorphic to $(\chi_V \circ i) \otimes \Theta(\chi_V \circ i)$*.*

Note however that the image of $E \circ j$ *is not yet known to be equal to the submodule* $\Theta(\chi_V \circ i) \subset \mathcal{A}(\mathrm{U}(V))$*, but merely isomorphic to it. If we had known that the image is equal to* $\Theta(\chi_V \circ i)$*, then we would have immediately deduce that there is a nonzro constant* $c \in \mathbb{C}^\times$ *such that*

$$\theta(\phi, \chi_V \circ i) = c \cdot E(j(\phi)).$$

Despite this, show nonetheless that the above identity holds (so that the image is indeed equal to $\Theta(\chi_V \circ i)$*) (Hint: you may want to consider the Fourier expansion of both sides). This is the so-called global Siegel-Weil formula.*

Problem 3.2. *The purpose of this exercise is to do global analog of the very long Problem 2.5. We shall use (the global analog of) the notations from Problem 2.5, as well as the same seesaw setup. Before that, let us explain the global seesaw identity in the context of a general seesaw in an ambient group* E*:*



We are now working over a number field k*. Suppose* Ω *is the "Weil representation" of* E *and* Ω *is equipped with an automorphic realization*

$$\theta : \Omega \longrightarrow \mathcal{A}(E).$$

For $\phi \in \Omega$ *and* $f \in \mathcal{A}(G_1)$*, one has the global theta lift:*

$$\theta(\phi, f)(h) = \int_{[G_1]} \theta(\phi)(g,h) \cdot \overline{f(g)}\, dg$$

so that $\theta(\phi, f) \in \mathcal{A}(H_1)$*. Likewise for* $f' \in \mathcal{A}(H_2)$*, one has* $\theta(\phi, f') \in \mathcal{A}(G_2)$ *defined by*

$$\theta(\phi, f') = \int_{[H_2]} \theta(\phi)(g,h) \cdot \overline{f'(h)}\, dh.$$

Now the global seesaw identity is simply:

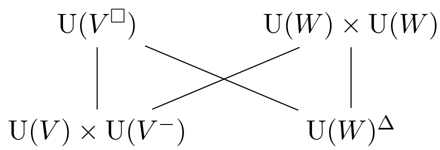
$$\langle \theta(\phi, f), f' \rangle_{H_2} = \langle \theta(\phi, f'), f \rangle_{G_1},$$

where we are using the Petersson inner products on G_1 and H_2 here. Indeed, from definition, it follows that both sides are given by the double integral

$$\int_{[G_1\times H_2]} \theta(\phi)(g,h)\cdot\overline{f(g)}\cdot\overline{f'(h)}\,dg\,dh.$$

Hence the global seesaw identity is simply an application of Fubini's theroem: exchanging the order of integration.

Now we place ourselves in the context of Problem 2.5, with the follwing seesaw:



(i) Taking the automorphic characters $f=\chi\otimes{\chi'}^{-1}\chi_W|_{E^1}$ *on* $\mathrm{U}(V)\times\mathrm{U}(V^-)$ *and* $f'=\chi_V|_{E^1}$ *on* $\mathrm{U}(W^\Delta)$*, write down the resulting global seesaw identity.*

(ii) We now examine the RHS of the seesaw identity (the side of W's). For $\phi\in\Omega_{V^-,W,\psi}$*, show using Problem 2.5(iii) that*

$$\theta(\phi,{\chi'}^{-1}\chi_W|_{E^1})\in\Theta_{V,W,\psi}(\chi')\cdot\chi_V|_{E^1}.$$

(iii) For $\phi_1\in\Omega_{V,W,\psi}$ *and* $\phi_2\in\Omega_{V^-,W,\psi}$*, show that*

$$\int_{[\mathrm{U}(W)]}\theta(\phi_1,\chi)(g)\cdot\theta(\phi_2,{\chi'}^{-1}\chi_W|_{E^1})(g)\cdot\overline{\chi_V(g)}\,dg$$

can be nonzero only if $\chi'=\chi$*. Moreover,* $\theta(\phi_1,\chi)$ *is nonzero if and only if the above integral is nonzero for some* ϕ_2 *(and taking* $\chi'=\chi$*).*

(iv) The LHS of the global seesaw identity (the side of V's) has the form:

$$\int_{[\mathrm{U}(V)\times\mathrm{U}(V^-)]}\theta(\phi_1,\otimes\phi_2,\chi_V|_{E^1})\cdot\overline{\chi(h_1)}\cdot\chi(h_2)\cdot\chi_W(h_2)^{-1}\,dh_1\,dh_2,$$

where

$$\theta(\phi_1,\otimes\phi_2,\chi_V|_{E^1})=\int_{[\mathrm{U}(W^\Delta)]}\theta(\phi_1\otimes\phi_2)(gh)\cdot\overline{\chi_V(g)}\,dg$$

is the global theta lift of $\chi_V|_{E^1}$ *from* $\mathrm{U}(W^\Delta)$ *to* $\mathrm{U}(V^\square)$*. As in Problem 2.5, we now need to explicit the theta lift* $\theta(\phi_1,\otimes\phi_2,\chi_V|_{E^1})$*. This is provided by the global Siegel-Weil formula of Problem 2.7(iii), which expresses* $\theta(\phi_1,\otimes\phi_2,\chi_V|_{E^1})$ *as an Eisenstein series* $E(j(\phi_1\otimes\phi_2))$ *(where we recall that* $j(\phi_1\otimes\phi_2)\in I(\chi_W)$*).*

(v) On replacing $\theta(\phi_1,\otimes\phi_2,\chi_V|_{E^1})$ *by the Eisenstein series* $E(j(\phi_1\otimes\phi_2))$*, the integral in (iv) becomes (a special value of) the gloibal doubling zeta integral:*

$$Z(s,\chi)(f)=\int_{[\mathrm{U}(V)\times\mathrm{U}(V^-)]}E(s,f)(h_1,h_2)\cdot\overline{\chi(h_1)}^{-1}\cdot\chi(h_2)\cdot\overline{\chi_W(h_2)}^{-1}\,dh_1\,dh_2,$$

for $f \in I(s, \chi_W)$. The theory of this doubling zeta integral is discussed in Ellen Eischen's lectures. As discussed there, this doubling zeta integral represents the L-value $L(1/2 + s, \chi_E \cdot \chi_W^{-1})$. More precisely, for $f_S \in I(s, \chi_W)$, we have:

$$Z(s,\chi)(f_s) = \frac{L_E(\frac{1}{2}+s, \chi_E \cdot \chi_W^{-1})}{L(1+2s, \omega_{E/k})} \cdot \prod_v \frac{Z_v(s,\chi_v)(f_{s,v}) \cdot L(1+2s, \omega_{E_v/k_v})}{L_{E_v}(\frac{1}{2}+s, \chi_{E_v} \cdot \chi_{W,v}^{-1})},$$

where the product over v is finite (as almost all terns are equal to 1). Using this identity and the last assertion in Problem 2.5 (viii), prove Theorem 3.3 in the lecture notes.

Problem 3.3. *Do the guided exercise in §3.12 of the lecture notes. This is the global analog of Problem 2.6.*

4. Automorphic Forms and Theta Correspondence Gan: Lecture 4 Problems

Problem 4.1. *For an A-parameter ψ considered in Lecture 4, i.e. Saito-Kurokawa type for PGSp_4, Howe-PS type fop U_3 and the short and long root type for G_2, compute the global component group S_ψ and the quadratic character ϵ_ψ.*

Problem 4.2. *Do the same for the original Howe-PS A-parameter on PGSp_4 defined as follows:*

$$\psi = \rho \times \mathrm{id} : L_k \times \mathrm{SL}_2(\mathbb{C}) \longrightarrow \mathrm{O}_2(\mathbb{C}) \times \mathrm{SL}_2(\mathbb{C}) \longrightarrow \mathrm{Sp}_4(\mathbb{C}),$$

where the second arrow is defined by the natural map associated to the tensor product of a 2-dim. quadratic space and a 2-dim symplectic space (which yields a 4-dim symplectic space). In addition, note that a homomorphism

$$\rho : L_k \longrightarrow \mathrm{O}_2(\mathbb{C})$$

determines:

- *by composition with the determinant map on $\mathrm{O}_2(\mathbb{C})$ a quadratic étale k-algebra E*
- *an automorphic character χ_ρ of $E^1 \cong E^\times/k^\times$.*

5. Classical modular forms and G_2
Pollack: Lecture 1 Problems

Problem 5.1. *Do exercise 3.1.1 in the notes.*

Problem 5.2. *The point of this exercise is to directly find the Fourier expansion of modular forms $\varphi \in A_\ell(\Gamma)$, in the notation of section 3.1.2 of the notes, without using as a crutch the Fourier expansion of classical modular forms.*

Suppose $r \in \mathbb{R}$. Say that a function $F : \mathrm{SL}_2(\mathbb{R}) \to \mathbb{C}$ is a moderate growth generalized Whittaker function *of type r, ℓ if:*

1. *F is of moderate growth*
2. *$F(\left(\begin{smallmatrix} 1 & x \\ & 1 \end{smallmatrix}\right) g) = e^{irx}F(g)$ for all $x \in \mathbb{R}$.*
3. *$F(gk_\theta) = e^{-i\ell\theta}F(g)$ for all $k_\theta = \left(\begin{smallmatrix} \cos(\theta) & -\sin(\theta) \\ \sin(\theta) & \cos(\theta) \end{smallmatrix}\right) \in \mathrm{SO}(2)$.*
4. *$X_- F(g) \equiv 0$.*

Prove that:

1. *If $r < 0$, then $F = 0$*
2. *If $r > 0$, then F is proportional to the function W_n defined at the end of section 3.1.2.*

Deduce the Fourier expansion of modular forms φ in $A_\ell(\Gamma)$.

Problem 5.3. *Prove that $\mathfrak{g}_2$ is a simple Lie algebra, i.e., verify Propositions 2.1.1 and 2.1.2 in the notes.*

Problem 5.4. *Understand the structure of $\mathfrak{g}_2$ as a Lie algebra: Do exercises 2.2.1, 2.2.2, and 2.2.3 in the notes.*

6. The Fourier expansion on G_2
Pollack: Lecture 2 Problems

Problem 6.1. *Let $P = MN$ be the Heisenberg parabolic of G_2, with $Z = [N, N]$ its center. (See the end of section 2.2 of the notes). Suppose φ is an automorphic form on $G_2(\mathbb{A})$, and φ_Z its constant term along Z. Prove that if $\varphi_Z \equiv 0$, then $\varphi \equiv 0$.*

Using cuspidal holomorphic Siegel modular forms on Sp_4, show that the analogous statement with G_2 replaced by Sp_4 is false. (For the Sp_4 analogue, let Z be the root space of the highest root in Sp_4.)

Recall that if $f(u, v) = au^3 + bu^2v + cuv^2 + dv^3$ and $f'(u, v) = a'u^3 + b'u^2v + c'uv^2 + d'v^3$, then the symplectic product

$$\langle f, f' \rangle = ad' - 3bc' + 3cb' - da'.$$

If $m \in \mathrm{GL}_2$, define $m \cdot f(u, v) = \det(m)^{-1}f((u, v)m)$.

Problem 6.2. *Prove that the following conditions are equivalent on a real binary cubic form f:*

1. *$\beta_f(m) := \langle f, m \cdot (u - iv)^3 \rangle \neq 0$ for all $m \in \mathrm{GL}_2(\mathbb{R})$. (This quantity appears in the Fourier expansion of modular forms on G_2.)*
2. *$f(z, 1) \neq 0$ for all z in the upper half-plane.*
3. *f splits into three linear factors over $\mathbb{R}$.*

One says that f is positive semi-definite *if it satisfies the equivalent conditions above. Modular forms on G_2 have Fourier coefficients corresponding to these positive semi-definite f.*

Problem 6.3. *Suppose $F(Z) = \sum_{T\geq 0} a_F(T)e^{2\pi i \mathrm{tr}\,(TZ)}$ is a Siegel modular form of weight ℓ for Sp_{2n}. It is well-known that if F is cuspidal then the Fourier coefficients $a_F(T)$ satisfy $a_F(T) \neq 0$ implies $\det(T) \neq 0$. The purpose of this problem is to prove this fact.*

(1) Let φ_F be the automorphic function $\varphi_F : \mathrm{Sp}_{2n}(\mathbb{R}) \to \mathbb{C}$ corresponding to F. Define φ_F.
(2) Prove that $\varphi_F(g)$ has a Fourier expansion of the form $\varphi_F(g) = \sum_{T\geq 0} a_F(T)W_T(g)$, with functions $W_F(g)$ that satisfy $W_T(m) = \det(Y)^{\ell/2}e^{-2\pi \mathrm{tr}\,(TY)}$. Here $m = \begin{pmatrix} r & \\ & {}^t r^{-1}\end{pmatrix}$ with $r \in \mathrm{GL}_n(\mathbb{R})$ and $Y = r\,{}^t r$.
(3) Observe that the functions $W_T(m)$ above are only bounded when $\det(T) \neq 0$.
(4) Suppose now that F is a cuspidal Siegel modular form. Deduce, using the fact the cusp form φ_F is bounded, that $a_F(T) \neq 0$ implies $\det(T) \neq 0$.

Problem 6.4. *The Fourier expansion of modular forms on G_2 of weight ℓ is in terms of functions $W_f(g) : G_2 \to \mathbf{V}_\ell$ that can be expressed as*

$$W_f(m) = \det(m)^\ell |\det(m)| \sum_{-\ell\leq v\leq \ell} \left(\frac{|\beta_f(m)|}{\beta_f(m)}\right)^v K_v(|\beta_f(m)|)\frac{x^{\ell+v}y^{\ell-v}}{(\ell+v)!(\ell-v)!}.$$

Here $\beta_f(m)$ was defined above, $m \in \mathrm{GL}_2(\mathbb{R})$ is in the Levi of the Heisenberg parabolic, and $\{x^{\ell+v}y^{\ell-v}\}_v$ is a certain basis of $\mathbf{V}_\ell$.

(1) Recall that a binary cubic is said to be non-degenerate if its discriminant is nonzero. Prove that if $f \geq 0$ but f is degenerate, then the function $W_f(m)$ is unbounded.
(2) Deduce, as in the previous exercise, that if φ is a cusp form, the Fourier coefficients satisfy $a_\varphi(f) \neq 0$ implies $\mathrm{disc}(f) \neq 0$.

7. Examples of modular forms on G_2
Pollack: Lecture 3 Problems

Problem 7.1. *Suppose φ is a level one modular form of weight ℓ, and let $a_\varphi(f)$ be the Fourier coefficient of φ corresponding to the integral binary cubic f. Prove that $a_\varphi(\gamma \cdot f) = \det(\gamma)^\ell a_\varphi(f)$ for all $\gamma \in \mathrm{GL}_2(\mathbb{Z})$.*

Problem 7.2 (This question requires a lot of work.)**.** *Let $\{x^{2\ell}, \dots, y^{2\ell}\}$ be the weight basis of $V_\ell = Sym^{2\ell}(\mathbb{C}^2)$ corresponding to the $\mathfrak{sl}_2$-triple defined in section 4.1.1 of "Modular forms on G_2 and their Standard L-function". Consider the function $f_\ell : G_2(\mathbb{R}) \to \mathbf{V}_\ell$ defined as:*

(1) $f_\ell(nmg) = \det(m)^\ell |\det(m)| f_\ell(g)$ for all $n \in N(\mathbb{R})$ and $m \in M(\mathbb{R}) \simeq \mathrm{GL}_2(\mathbb{R})$. Here $P = MN$, so that N is the unipotent radical of the Heisenberg parabolic and M is its Levi subgroup.
(2) $f_\ell(gk) = k^{-1} \cdot f_\ell(g)$ for all $k \in K_{G_2} = (\mathrm{SU}(2) \times \mathrm{SU}(2))/\mu_2$.
(3) $f_\ell(1) = x^\ell y^\ell$.

Prove that f_ℓ is annihilated by the differential operator D_ℓ that defines modular forms of weight ℓ. Deduce that absolutely convergent degenerate Eisenstein series define modular forms.

Problem 7.3 (This is an open question)**.** *Prove that the absolutely convergent degenerate Eisenstein series have nonzero non-degenerate Fourier coefficients. To do this, it is necessary and sufficient to verify that the following integral is not identically* 0*:*

$$\int_{N(\mathbb{R})} f_\ell(wng)e^{-i\langle f,\overline{n}\rangle}\,dn$$

where

(1) *$N(\mathbb{R})$ is the Heisenberg unipotent radical*
(2) *w is the Weyl group element that conjugates P to its opposite*
(3) *f is a non-degenerate binary cubic that is positive semi-definite*
(4) *$\overline{n}$ denotes the image of n in N/Z, which can be identified with the space of real binary cubics.*

If you can't prove it in general (which again, is open), can you get a computer to evaluate the integral in special cases of ℓ and $g = 1$?

8. **Beyond G_2**
Pollack: Lecture 4 Problems

Problem 8.1 (Positive definite Fourier coefficients for F_4)**.** *Let $C = \mathbb{R}$ and $J = H_3(C) = H_3(\mathbb{R})$ be the symmetric 3×3 matrices. Recall that $W_J = \mathbb{R} + J + J^\vee + \mathbb{R} = \mathbb{R} + J + J + \mathbb{R}$ where we identify J with $J^\vee$ via the trace pairing. The quartic form on W_J is defined as $q((a,b,c,d)) = (ad-(b,c))^2 + 4a\det(c) + 4d\det(b) - 4\mathrm{tr}\,(b^\#, c^\#)$. Here $\# : J \to J$ is the quadratic polynomial map satisfying $X^\# = \det(X)X^{-1}$ when X is invertible. Let $\mathcal{H}_J = \{X + iY : X, Y \in J, Y > 0\}$ where $Y > 0$ means that Y is positive definite. Note that $\mathcal{H}_J$ is the symmetric space for $\mathrm{Sp}_6(\mathbb{R})$. Suppose $w = (a,b,c,d) \in W_J$. Write $w \geq 0$, and say w is positive semi-definite, if $p_w(Z) = aN(Z) + (b, Z^\#) + (c, Z) + d$ is nonzero on $\mathcal{H}_J$.*

(1) *Prove that $w = (1,0,0,1)$ has $q(w) > 0$ and w is not positive semi-definite. (In fact, one can show that if $q(w) > 0$, then w is not positive semi-definite.)*
(2) *Prove that $w = (-1,0,1,0)$ has $q(w) < 0$ and $w \geq 0$.*
(3) *Construct a $w \in W_J$ with $q(w) < 0$ but w not positive semidefinite.*

Problem 8.2. *Check that the vector space with bracket $\mathfrak{g}(J)$ defined in section 6.1 of the notes is a Lie algebra. (This is proved in Proposition 6.1.1 of the notes, but following the proof will require understanding material in sections 4 and 5 of the notes.)*