AUTOMORPHIC FORMS AND THE THETA CORRESPONDENCE

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1. Lecture 1: The Ramanujan Conjecture

In the first lecture, we shall recall the Ramanujan conjecture for classical modular forms and its reformulation in the language of cuspidal automorphic representations of PGL_2. For more details on this transition and reformulation, please take a look at [8, §4.27 and §4.28]. This reformulation allows one to readily generalize the conjecture to the setting of cuspidal automorphic representations of general connected reductive group G over a number field k. We will then discuss the unitary analog of a construction of Roger Howe and Piatetski-Shapiro [18] which gives a definitive counterexample to the extended Ramanujan conjecture for the unitary group U_3 in three variables. Their original construction gave a counterexample on the group Sp_4, but we will use the same idea to produce a counterexample on U_3 via the method of theta correspondence.

1.1. **The Ramanujan conjecture.** About a century ago, Ramanujan considered the following power series of q

$$\Delta(q) = q \cdot \prod_{n \geq 1} (1 - q^n)^{24}.$$

Expanding this out formally, we have:

$$\Delta(q) = \sum_{n>0} \tau(n) q^n = q - 24q^2 + \ldots$$

Ramanujan conjectured that for all primes p,

$$|\tau(p)| \leq 2 \cdot p^{11/2}.$$

This is the Ramanujan conjecture in question. More generally, for any holomorphic cuspidal Hecke eigenform ϕ of weight k (and level 1) on the upper half plane $\mathfrak{h}$, with Fourier coefficients $\{a_n(\phi)\}_{n \geq 1}$, the Ramanujan-Petersson conjecture asserts that

$$|a_p(\phi)| \leq 2p^{(k-1)/2} \quad \text{for all primes } p.$$

For Hecke eigenforms, the Fourier coefficients $a_p(\phi)$ are also the eigenvalues of the Hecke operator T_p. Hence, the Ramanujan conjecture concerns bounds on cuspidal Hecke eigenvalues.

1.2. **Cuspidal automorphic representations.** The classical theory of modular forms can now be subsumed in a representation theoretic setting. The details of this transition can be found in [8, §3 and §4]. Let us briefly recall this.

Let

- k be a number field with ring of adeles $\mathbb{A} = \prod'_v k_v$, which is a restricted direct product of the local completions k_v for all places v of k.

- G be a connected reductive linear algebraic group over k; for simplicity we may take G to be semisimple;
- fixing any faithful algebraic representation $\rho : G \hookrightarrow \mathrm{GL}_n$ over k, one obtains a system of open compact subgroups $K_v = \rho^{-1}(\mathrm{GL}_n(\mathcal{O}_v)) \subset G(k_v)$, for almost all v, where $\mathcal{O}_v$ is the ring of integers of k_v; for almost all v, K_v is hyperspecial.
- $G(\mathbb{A}) = \prod'_v G(k_v)$ be the adelic group, which is a restricted direct product of $G(k_v)$ relative to the family $\{K_v\}$ of open compact subgroups for almost all v;
- $[G] = G(k)\backslash G(\mathbb{A})$ be the automorphic quotient; the locally compact group $G(\mathbb{A})$ acts on $[G]$ by right translation and there is a $G(\mathbb{A})$-invariant measure (unique up to scaling).

For the theory of classical modular forms, one is taking $k = \mathbb{Q}$ and $G = \mathrm{PGL}_2$. Using the natural identification

$$\mathrm{SL}_2(\mathbb{Z})\backslash\mathfrak{h} \cong \mathrm{PGL}_2(\mathbb{Z})\backslash\mathrm{PGL}_2(\mathbb{R})/\mathrm{O}_2(\mathbb{R}) \cong \mathrm{PGL}_2(\mathbb{Q})\backslash\mathrm{PGL}_2(\mathbb{A})/\mathrm{O}_2(\mathbb{R}) \cdot \prod_p \mathrm{PGL}_2(\mathbb{Z}_p),$$

a classical modular form ϕ on $\mathfrak{h}$ corresponds to a function

$$f : \mathrm{PGL}_2(\mathbb{Q})\backslash\mathrm{PGL}_2(\mathbb{A}) \longrightarrow \mathbb{C}$$

defined by

$$f(g) = (\phi|_k g)(\sqrt{-1}).$$

Replacing ϕ by f allows one to extend the notion of modular forms to the setting of general reductive groups G.

More precisely, an automoprhic form on G is a function

$$f : [G] \longrightarrow \mathbb{C}$$

satisfying some regularity and finiteness properties:

- f is smooth
- f is right K_f-finite (where $K_f = \prod_{v<\infty} K_v$)
- f is of uniform moderate growth
- f is $Z(\mathfrak{g})$-finite.

It is not important for us to know precisely the meaning of the above properties.

Let us denote the space of automorphic forms on G by $\mathcal{A}(G)$. The group $G(\mathbb{A})$ acts on the vector space $\mathcal{A}(G)$ by right translation. An *automorphic representation* π of G is by definition an irreducible subquotient of the $G(\mathbb{A})$-module $\mathcal{A}(G)$. As an irreducible abstract representation of $G(\mathbb{A}) = \prod'_v G(k_v)$, π is of the form:

$$\pi = \otimes'_v \pi_v,$$

a restricted tensor product of irreducible smooth representations π_v of $G(k_v)$. In particular, for almost all v, $\pi_v^{K_v} \neq 0$; we say that π_v is K_v-spherical or K_v-unramified. It is known that $\dim \pi_v^{K_v} = 1$ if π_v is K_v-unramified.

Definition: An automorphic form f on G is called a **cusp form** if, for any parabolic k-subgroup $P = MN$ of G, the N-constant term

$$f_N(g) = \int_{N(k)\backslash N(\mathbb{A})} f(ng)\, dn$$

is zero as a function on $G(\mathbb{A})$.

Let

$$\mathcal{A}_{cusp}(G) \subset \mathcal{A}(G)$$

be the subspace of cusp forms; it is a $G(\mathbb{A})$-submodule (potentially 0). It turns out that, if G is semisimple or has anisotropic center, a cusp form f is rapidly decreasing as a function on $[G]$ (like a Schwarz function) and hence is square-integrable on $[G]$, i.e..

$$\int_{[G]} |f(g)|^2 \, dg < \infty.$$

If we let $\mathcal{A}_2(G)$ denote the $G(\mathbb{A})$-submodule of square-integrable automorphic forms (clearly nonzero), then

$$\mathcal{A}_{cusp}(G) \subset \mathcal{A}_2(G) \subset \mathcal{A}(G).$$

Moreover, each $G(\mathbb{A})$-submodule $\mathcal{A}_{Cusp}(G)$ and $\mathcal{A}_2(G)$ decomposes as a direct sum

$$\mathcal{A}_{cusp}(G) = \bigoplus_{\pi} m_{cusp}(\pi) \cdot \pi \quad \text{and} \quad \mathcal{A}_2(G) = \bigoplus_{\pi} m_2(\pi) \cdot \pi$$

with finite multiplicities (this is not obvious). An irreducible summand π of $\mathcal{A}_{cusp}(G)$ is called a cuspidal automorphic representartion, whereas one of $\mathcal{A}_2(G)$ is called a square-integrable automorphic representation.

1.3. **Classification of unramified representations.** For an abstract irreducible representation $\pi \cong \otimes'_v \pi_v$, we have mentioned that for almost all v, π_v is K_v-unramified, with $K_v \subset G(k_v)$ a so-called hyperspecial maximal compact subgroup. Such K_v-unramified representations can be classified. We recall this classification briefly; the reader can take a look at [8, §4.22-4.26].

For almost all such v, G is quasi-split over k_v and hence possesses a Borel subgroup $B_v = T_v \cdot U_v$ defined over k_v. Hence, T_v is a maximal torus over k_v. For a character $\chi_v : T_v = T_v(k_v) \to \mathbb{C}^\times$, one can form the (normalized) parabolically induced representation

$$I(\chi_v) = \mathrm{Ind}_{B_v}^{G_v} \chi_v,$$

consisting of functions $\phi : G_v \to \mathbb{C}$ satisfying

$$f(utg) = \chi_v(t) \cdot \delta_{B_v}(t)^{1/2} \cdot f(g) \quad \text{for } u \in U_v,\ t \in T_v \text{ and } g \in G_v,$$

where δ_{B_v} is the modulus character defined by

$$\delta_{B_v}(g) = |\det(\mathrm{Ad}(g)|\mathrm{Lie}(U_v))|.$$

and where the action of G_v is by right translation. The representation $I(\chi_v)$ is called a principal series representation.

If χ_v is an unramified character of $T(k_v)$, i.e., χ_v is trivial on $T(k_v) \cap K_v$, we call $I(\chi_v)$ an unramified principal series representation. The following proposition summarizes the classification of K_v-unramified representations:

Proposition 1.1. *(i) An unramified $I(\chi_v)$ is of finite length and contains exactly one irreducible subquotient $\pi(\chi_v)$ which is K_v-unramified. Moreover, $\pi(\chi_v) \cong \pi(\chi'_v)$ if and only if $\chi_v = w \cdot \chi'_v$ for some element w in the Weyl group $W_v = N_{G_v}(T_v)/T_v$.*

(ii) Every K_v-unramified representation of $G(k_v)$ is isomorphic to $\pi(\chi_v)$ for some unramified χ_v.

(iii) Hence the above construction gives a bijection

$$\{K_v\text{-unramified irreps. of } G(k_v)\} \longleftrightarrow \{\text{unramified characters of } T(k_v)\}/(\text{Weyl group action}).$$

The proof of this proposition is through showing the Satake isomorphism; the reader can consult [8, §4.22-23]. There is an elegant way of reformulating the above proposition, using the Langlands dual group $G_v^\vee$ (for split G_v) or the Langlands L-group ${}^LG_v$ in general. This reformulation (for split G_v for simplicity) is a bijection:

$$\{K_v\text{-unramified irreps. of } G(k_v)\} \longleftrightarrow \{\text{semisimple conjugacy classes in } G_v^\vee\}.$$

1.4. **Tempered representations.** A representation π_v is *tempered* if its matrix coefficients (which is a function on G_v) lies in $L^{2+\epsilon}(G_v)$ for all $\epsilon > 0$, i.e. if its matrix coefficients decay sufficiently quickly [8, §4.27]. For unramified representations, we can make do with the following ad-hoc definition.

Definition: A K_v-unramified representation $\pi(\chi_v)$ as above is said to be *tempered* if χ_v is a unitary character, i.e. $|\chi_v| = 1$.

As an example, the trivial representation of G_v is certainly K_v-unramified and is contained in the principal series $I(\delta_{B_v}^{1/2})$. Since $\delta_{B_v}^{1/2}$ is not a unitary character, the trivial representation of G_v is not tempered (unless G_v is compact). From the point of view of matrix coefficients, the trivial representation has constant matrix coefficients which certainly do not decay at all.

1.5. **Representation theoretic formulation of the Ramanujan conjecture.** We can now formulate the Ramanujan conjecture for cuspidal representations of a quasi-split group G.

Conjecture 1.2 (Naive Ramanujan Conjecture)**.** *Let $\pi = \otimes_v' \pi_v$ be a cuspidal representation of a quasi-split G. Then for almost all v, π_v is tempered.*

The transition from Ramanujan's original conjecture to this representation theoretic formulation is not clear at all and was first realized by Satake. This transition is described in [8, §4.28]. The condition that G be quasi-split is there because the conjecture may be easily shown to be false without it. For example, if G is an anisotropic group, so that $[G]$ is compact, then the constant functions, which afford the trivial representation, are certainly cusp forms but the trivial representation is not tempered (as we have remarked above).

1.6. **Counterexample of Howe-PS.** The naive Ramanujan conjecture is expected to be true when $G = \mathrm{GL}_n$, where it has in fact been shown in many cases. However, in the 1977 Corvallis proceedings, Howe and Piatetski-Shapiro [18] constructed an example of a cuspidal automorphic representation of the split group Sp_4. which violates the naive Ramanujan conjecture above. This has led to the following tweak:

Conjecture 1.3 (Revised Ramanujan Conjecture)**.** *Let $\pi = \otimes_v' \pi_v$ be a globally generic cuspidal representation of a quasi-split G. Then for almost all v, π_v is tempered.*

Note that for $G = \mathrm{GL}_n$, all cuspidal representations are known to be globally generic (see [8, §6])

In this series of lectures, we will follow the same basic idea of Howe-PS and construct a similar counterexample for a quasi-split unitary group U_3.

1.7. **ϵ-Hermitian spaces and unitary groups.** Let us first recall some basics about unitary groups and the underlying Hermitian spaces.

We first begin with an arbitrary field F of characteristic 0, and let E be an étale quadratic F-algebra (so E is either a quadratic field extension or $E = F \times F$), with $\mathrm{Aut}(E/F) = \langle c \rangle$ acting on E by $x \mapsto x^c$. With $\epsilon = \pm$, let V be a finite-dimensional ϵ-Hermitian space over E. This means that V is equipped with a nondegenerate E-sesquilinear form $\langle -, - \rangle$, so that

$$\langle v_1, v_2 \rangle^c = \epsilon \cdot \langle v_2, v_1 \rangle \quad \text{and} \quad \langle \lambda v_1, v_2 \rangle = \lambda \cdot \langle v_2, v_2 \rangle$$

for $v_1, v_2 \in V$ and $\lambda \in E$. If $\epsilon = +1$, one gets a Hermitian form; if $\epsilon = -1$, one gets a skew-Hermitian form. Observe that if $\delta \in E^\times$ is a trace 0 element (to F), then multiplication-by-δ takes an ϵ-Hermitian form to a $-\epsilon$-Hermitian form.

If $(V, \langle -, - \rangle)$ is an ϵ-Hermitian space, let $\mathrm{U}(V)$ be its associated isometry group:

$$\mathrm{U}(V) = \{g \in \mathrm{GL}(V) : \langle gv_1, gv_2 \rangle = \langle v_1, v_2 \rangle \text{ for all } v_1, v_2 \in V\}.$$

Because

$$\mathrm{U}(V, \langle -, - \rangle) = \mathrm{U}(V, \delta \cdot \langle -, - \rangle) \quad \text{for trace 0 elements } \delta \in E^\times,$$

the class of isometry groups one obtains for Hermitian and skew-Hermitian spaces is the same. These isometry groups are called the *unitary groups*: each of them is a connected reductive group with a 1-dimensional anisotropic center $Z \cong E^1$ (the torus defined by the norm 1 elements of $E^\times$).

If $n = dim_E V$, then we say that V is maximally split (or simply split) if V contains a maximal isotropic subspace of dimension $[n/2]$. In that case, $\mathrm{U}(V)$ is quasi-split and thus possesses a Borel subgroup B defined over F. Such a Borel subgroup is obtained as the stabilizer of a maximal flag of isotropic subspaces:

$$0 \subset X_1 \subset \ldots \subset X_{[n/2]}$$

with $\dim X_j = j$.

Let us highlight the special case when $E = F \times F$ is the split quadratic F-algebra. Then V is an $F \times F$-module and hence has the form $V_0 \times V_0^\vee$, for an F-vector space V_0. Up to isomorphism, any Hermitian E-space is isomorphic to the one defined by

$$\langle (v_1, l_1), (v_2, l_2) \rangle = (l_2(v_1), l_1(v_2)) \in E.$$

Then we note that

$$\mathrm{U}(V) \cong \mathrm{GL}(V_0),$$

via the natural action of $\mathrm{GL}(V_0)$ on $V_0 \times V_0^\vee$. We will largely ignore such split cases in the following, as they can be easily handled.

1.8. **Invariants of spaces.** Assume now that F is a local field, E/F a quadratic field extension and fix $\dim V = n$. A Hermitian space V has a natural invariant known as the discriminant:

$$\mathrm{disc}(V) \in F^\times/\mathrm{N}_{E/F}(E^\times).$$

More precisely, if $\{v_1, ..., v_n\}$ is an E-basis and $A = (\langle v_i, v_j\rangle)$ is the matrix of inner products of basis elements, then

$$\mathrm{disc}(V) = (-1)^{n(n-1)/2} \cdot \det(A) \in F^\times/NE^\times.$$

Using the nontrivial quadratic character $\omega_{E/F}$ of $F^\times/N_{E/F}(E^\times)$, it is convenient to encode $\mathrm{disc}(V)$ as a sign:

$$\epsilon(V) = \omega_{E/F}(\mathrm{disc}(V)) = \pm.$$

When F is nonarchimedean, Hermitian spaces are classified by the two invariants $\dim(V)$ and $\epsilon(V)$, so that for each given dimension, there are 2 Hermitian spaces V^+ and V^-. When $F = \mathbb{R}$, however, Hermitian spaces are classified by their signatures (p, q), with $p+q = \dim V$.

Likewise, if W is a skew-Hermitian space, then

$$\mathrm{disc}(W) \in \delta^{\dim W} \cdot F^\times/\mathrm{N}_{E/F}(E^\times)$$

and one sets

$$\epsilon(W) = \omega_{E/F}(\delta^{-\dim W} \cdot \mathrm{disc}(W)) = \pm.$$

Note however that $\epsilon(W)$ depends on the choice of δ.

Assume now that $F = k$ is a number field and E/k is a quadratic field extension. Then a Hermitian space V over E is determined by its localizations $\{V \otimes_k k_v\}$ as v runs over all places of k; in other words, the Hasse principle holds. Note that half the places v will split in E and for these, the local situation is the split case (so the split case cannot be ignored for global considerations). A family of local Hermitian space $\{V_v\}$ (relative to E_v/k_v) arises as the family of localizations of a global Hermitian space relative to E/k if and only if:

- for almost all v, $\epsilon(V_v) = +$;
- $\prod_v \epsilon(V_v) = 1$.

There is an analogous statement for skew-Hermitian spaces which can be formulated, using the observation that multiplication by a nonzero trace 0 element of E switches Hermitian spaces and skew-Hermitian spaces.

1.9. **Examples.** Let us consider some examples in small dimension relative to a quadratic field extension of local fields E/F.

- When $\dim V = 1$, one may identify V with E (by the choice of a basis element), and a Hermitian form is given by $(x, y) \mapsto axy^c$, with $a \in F^\times$; we denote this rank 1 Hermitian space by $\langle a\rangle$. Then $\langle a\rangle \cong \langle b\rangle$ if and only if $a/b \in NE^\times$ and $V^+ = \langle 1\rangle$. In any case, $\mathrm{U}(\langle a\rangle) = E^1 \subset E^\times$ for any $a \in F^\times$.

  We take this occasion to take note of a canonical isomorphism (given by Hilbert's Theorem 90):

  $$E^\times/F^\times \cong E^1$$

  defined by

  $$x \mapsto x/x^c.$$

We will frequently use this isomorphism to identify E^1 as $E^\times/F^\times$.

- When $\dim V = 2$, V^+ is the split Hermitian space $V^+ = Ee_1 \oplus Ee_2$, such that
$$\langle e_1, e_1\rangle = \langle e_2, e_2\rangle = 0 \quad \text{and} \quad \langle e_1, e_2\rangle = 1.$$
This 2-dimensional split Hermitian space is also called a hyperbolic plane and is sometimes denoted by $\mathbb{H}$. The stabilizer of the isotropic line Ee_1 is a Borel subgroup, containing a maximal torus
$$T = \{t(a) = \begin{pmatrix} a & \\ & (a^c)^{-1} \end{pmatrix}, \, a \in E^\times\}$$
and with unipotent radical
$$U = \{u(z) = \begin{pmatrix} 1 & z \\ & 1 \end{pmatrix} : z \in E, \, Tr_{E/F}(z) = 0\}.$$
The other Hermitian space V^- is anisotropic (it has no nonzero isotropic vector). One can describe it in terms of the unique quaternion division F-algebra D. For this, one fixes an F-algebra embedding $E \hookrightarrow D$ (unique up to conjugacy by $D^\times$ by the Skolem-Noether theorem) and regard D as a 2-dimensional E-vector space by left multiplication. One can find $d \in D$ such that d normalizes E and $dxd^{-1} = x^c$ for $x \in E$, and write $D = E \cdot 1 \oplus E \cdot d$. Then one defines a Hermitian form on D by
$$\langle x, y\rangle = \text{projection of } x \cdot y \text{ onto } E \cdot 1.$$
In terms of this model, one can describe the unitary group $\mathrm{U}(V^-)$ as :
$$U(V^-) \cong (E^\times \times D^\times)^1/\nabla F^\times = \{(e,d) : N_E(e) \cdot N_B(b) = 1\}/\nabla F^\times,$$
where $\nabla(F^\times) = \{(t, t^{-1}) : t \in F^\times\}$. The element $(e,d) \in E^\times \times B^\times$ gives rise to the operator
$$x \mapsto e \cdot x \cdot b^{-1}$$
on D.

  If one replaces D by the matrix algebra $M_2(F)$, the above description of V^- and its isometry group $\mathrm{U}(V^-)$ gives rise to a description of $V^+ = \mathbb{H}$ and $\mathrm{U}(V^+)$. This shows that $\mathrm{U}(V^+)$ is intimately connected with GL_2 and $\mathrm{U}(V^-)$ with $D^\times$.

- Consider now the case when $\dim V = 3$. Let $\mathbb{H}$ denote the hyperbolic plane introduced above and recall the 1-dimensional Hermitian space $\langle a\rangle$. The sum $\langle a\rangle \oplus \mathbb{H}$ is then a 3-dimensional Hermitian space with $\epsilon(\langle a\rangle + \mathbb{H}) = \omega_{E/F}(a)$. As a runs over $F^\times/NE^\times$, one obtains the two equivalence classes of 3-dimensional Hermitian spaces (in the nonarchimedean case). Thus, both these spaces are split and since $V^+ \cong a \cdot V^-$ for $a \in F^\times \setminus NE^\times$, we see that $\mathrm{U}(V^+) \cong \mathrm{U}(V^-)$ is a quasisplit group.

  Since this unitary group will play a big role in the Howe-PS construction, let us set up some more notation about it. Let $\langle a\rangle = E \cdot v_0$ and $\mathbb{H} = Ee \oplus Ee^*$ with e and e^* isotropic vectors. Then with respect to the basis $\{e, v_0, e^*\}$ of $\langle a\rangle \oplus \mathbb{H}$, the inner product matrix takes the form

$$\begin{pmatrix} & & 1 \\ & a & \\ 1 & & \end{pmatrix}.$$

The Borel subgroup $B = TU$ stabilizing the isotropic line $E{\cdot}e$ is then upper triangular, with elements of T and U taking the form

$$t(a,b) = \begin{pmatrix} a & & \\ & b & \\ & & (a^c)^{-1} \end{pmatrix} \quad \text{with } a \in E^\times \text{ and } b \in E^1;$$

and

$$u(x,z) = \begin{pmatrix} 1 & 0 & z \\ & 1 & 0 \\ & & 1 \end{pmatrix} \cdot \begin{pmatrix} 1 & -a^c x^c & * \\ & 1 & x \\ & & 1 \end{pmatrix}, \quad \text{with } x, z \in E \text{ and } Tr_{E/F}(z) = 0.$$

1.10. **Basic idea of Howe-PS construction.** We can now give a brief summary of the Howe-PS construction of cuspidal representations of U_3 which violate the naive Ramanujan conjecture.

Let E/k be a quadratic field extension of number fields. Consider a skew-Hermitian space $(W, \langle -,-\rangle)$ of dimension 3 over E. We would like to produce some cusp forms on $\mathrm{U}(W)$ which violates the naive Ramanujan conjecture. These functions on $\mathrm{U}(W)$ will be obtained by restriction (or pullback) of a simpler class of automorphic forms on a larger group containing $\mathrm{U}(W)$. What is this larger group?

By restriction of scalars, we have a 6-dimension space $\mathrm{Res}_{E/k}(W)$ over k and the k-valued form $\mathrm{Tr}_{E/k}(\langle -,-\rangle)$ defines a symplectic form on $\mathrm{Res}_{E/k}(W)$ with associated symplectic group $\mathrm{Sp}(\mathrm{Res}_{E/k}(W))$. This defines an embedding

$$\iota : \mathrm{U}(W) \hookrightarrow \mathrm{Sp}(\mathrm{Res}_{E/k}(W)).$$

It turns out that the simple automorphic forms we need are not really living on $\mathrm{Sp}(\mathrm{Res}_{E/k}(W))$. Rather, the symplectic group $\mathrm{Sp}_{2n}(\mathbb{A})$ has a topological S^1-cover $\mathrm{Mp}_{2n}(\mathbb{A})$ known as the metaplectic group (where S^1 is the unit circle in $\mathbb{C}^\times$):

$$1 \longrightarrow S^1 \longrightarrow \mathrm{Mp}_{2n}(\mathbb{A}) \longrightarrow \mathrm{Sp}_{2n}(\mathbb{A}) \longrightarrow 1$$

and the simpler automorphic forms in question actually live on $\mathrm{Mp}_{2n}(\mathbb{A})$. The need to work with this nonlinear cover accounts for much of the technicality of this subject, but one cannot argue with nature.

These simpler automorphic forms on the metaplectic groups are the theta functions and the automorphic representations they span are called the Weil representations. In order to pull back these theta functions from $\mathrm{Mp}(\mathrm{Res}_{E/k}(W))(\mathbb{A})$ to $\mathrm{Sp}(\mathrm{Res}_{E/k}(W))$, one needs to construct a lifting of ι to

$$\tilde{\iota} : \mathrm{U}(W)(\mathbb{A}) \hookrightarrow \mathrm{Mp}(\mathrm{Res}_{E/k}(W))(\mathbb{A}).$$

This is highly technical but it can be done and Howe-PS then restricted these theta functions to $\mathrm{U}(W)(\mathbb{A})$.

Representation theoretically, if

$$\Omega \subset \mathcal{A}(\mathrm{Mp}(\mathrm{Res}_{E/k}(W)))$$

denotes an automorphic Weil representation, and

$$\tilde{\iota}^* : \mathcal{A}(\mathrm{Mp}(\mathrm{Res}_{E/k}(W))) \to \mathcal{A}(\mathrm{U}(W))$$

denotes the restriction of functions, then one obtains a $\mathrm{U}(W)(\mathbb{A})$-submodule

$$\tilde{\iota}^*(\Omega) \subset \mathcal{A}(\mathrm{U}(W)).$$

Now recall that the center is $\mathrm{U}(W)$ is isomorphic to E^1 as an algebraic group. One can spectrally decompose $\tilde{\iota}^*(\Omega)$ according to central characters.

$$\tilde{\iota}^*(\Omega) = \bigoplus_{\chi} \Omega_\chi$$

as χ runs over the automorphic characters of E^1, or equivalently of $E^\times/k^\times$, where

$$\Omega_\chi = \{f \in \Omega : f(zg) = \chi(z) \cdot f(g) \text{ for all } z \in Z(\mathrm{U}(W)) = \mathbb{A}_E^1 \text{ and } g \in \mathrm{U}(W)(\mathbb{A})\}.$$

What we would like to show is that:

- for each χ, Ω_χ is an irreducible automorphic representation of $\mathrm{U}(W)$ and is cuspidal for many χ.
- for any χ, Ω_χ violates the naive Ramanujan conjecture.

One can view the map $\chi \mapsto \Omega_\chi$ as a lifting of automorphic representations from U_1 to U_3. This lifting is an instance of the theta correspondence, which we will discuss in the next two lectures.

2. **Lecture 2: Local Theta Correspondence**

The next two lectures will be devoted to a discussion of the theory of theta correspondence, so as to understand the construction of Howe-PS in its proper context. Two possible references for this are the survey papers of D. Prasad [27] and S. Gelbart [16]. In particular, the latter is concerned with theta correspondence for unitary groups. In this second lecture, we shall focus on the local theta correspondence, for which a basic reference is the book [26] of Moeglin-Vigneras-Waldspurger. Hence, we will be working over a local field F (of characteristic 0), and for simplicity, we shall assume F is nonarchimedean.

2.1. **Basic idea.** From the last lecture, we saw that the Howe-PS construction basically gives a map

$$\{\text{Automorphic characters of } \mathrm{U}_1 = E^1\} \to \{\text{Automorphic representations of } \mathrm{U}_3\}$$

sending χ to Ω_χ.

Now given any two groups G and H, one may ask more generally: what are some ways of constructing such a lifting from $\mathrm{Irr}(G)$ to $\mathrm{Irr}(H)$? Here, $\mathrm{Irr}(G)$ denotes the set of equivalence classes of irreducible representations of G.

A standard procedure is as follows. Suppose for simplicity that G and H are finite groups and Ω is a (finite-dim) representation of $G \times H$. Then one may decompose Ω into irreducible $G \times H$-summands:

$$\Omega = \bigoplus_{\pi \in \mathrm{Irr}(G)} \bigoplus_{\sigma \in \mathrm{Irr}(H)} m(\pi, \sigma) \cdot \pi \otimes \sigma.$$

One can rewrite this as:

$$\Omega = \bigoplus_{\pi \in \mathrm{Irr}(G)} \pi \otimes V(\pi)$$

where

$$V(\pi) = \bigoplus_{\sigma \in \mathrm{Irr}(H)} m(\pi, \sigma)\sigma.$$

This gives a map

$$\mathrm{Irr}(G) \longrightarrow R(H) \quad (\text{Grothedieck group of } \mathrm{Irr}(H))$$

sending π to $V(\pi)$. Since we are interested in getting irreducible representations of H as outputs, we ask: for what Ω is $V(\pi)$ is irreducible or zero for any π? If we can find such an Ω, then the map $\pi \mapsto V(\pi)$ would be a map

$$\mathrm{Irr}(G) \longrightarrow \mathrm{Irr}(H) \cup \{0\}.$$

An example of an Ω that has this property is certainly the trivial representation. However the map so obtained is not very interesting. On the other hand, if $\dim \Omega$ is too big, then $\dim V(\pi)$ will have to be big for many π's as well, so that $V(\pi)$ tends to be reducible. Thus, a simple heuristic is that Ω cannot be too big nor too small.

In practice, one can try an Ω arising in the following way. Suppose there is an (almost injective) group homomorphism

$$\iota : G \times H \longrightarrow E \quad \text{for some group } E.$$

One can take Ω to be an irreducible representation of E of smallest possible dimension > 1 and then pull it back to $G \times H$.

The theory of theta correspondence, which was systematically developed by Howe, arises in this way.

2.2. **Reductive dual pairs.** Let F be a field of characteristic 0, and let E be an étale quadratic F-algebra, with $\mathrm{Aut}(E/F) = \langle c \rangle$. Let V be a finite-dimensional Hermitian space over E and W a skew-Hermitian space. Then $V \otimes_E W$ is naturally a skew-Hermitian space over E. By restriction of scalars, we may regard $V \otimes_E W$ as an F-vector space which is equipped with a natural symplectic form $\mathrm{Tr}(\langle -, - \rangle_V \otimes \langle -, - \rangle_W)$. Then one has a natural map of isometry groups

$$\iota : \mathrm{U}(V) \times \mathrm{U}(W) \longrightarrow \mathrm{Sp}(V \otimes_E W)$$

which is injective on each factor $\mathrm{U}(V)$ and $\mathrm{U}(W)$. The images of $\mathrm{U}(V)$ and $\mathrm{U}(W)$ are in fact mutual commutants of each other in the symplectic group, and such a pair of groups is called a *reductive dual pair*.

Howe has given a complete classification of all such dual pairs in the symplectic group. A further example (perhaps easier than the one above) is obtained as follows. If V is a symmetric bilinear space (or a quadratic space) and W a symplectic space over F, then $V \otimes_F W$ inherits a natural symplectic form (by tensor product) and one has

$$\mathrm{O}(V) \times \mathrm{Sp}(W) \longrightarrow \mathrm{Sp}(V \otimes W).$$

2.3. **Metaplectic groups and Heisenberg-Weil representations.** Assume now that F is a local field. The symplectic group $\mathrm{Sp}(V \otimes_E W)$ has a nonlinear S^1-cover $\mathrm{Mp}(V \otimes_E W)$ known as the metaplectic group:

$$1 \longrightarrow S^1 \longrightarrow \mathrm{Mp}(V \otimes_E W) \longrightarrow \mathrm{Sp}(V \otimes_E W) \longrightarrow 1$$

The construction of this central extension is a lecture course in itself, but since its construction is such a classic result, we feel obliged to give a sketch.

Let us work in the context of an arbitrary symplectic vector space W over F (in place of the cumbersome notation $V \otimes_E W$). Let

$$H(W) = W \oplus F$$

and equip it with the group law

$$(w_1, t_1) \cdot (w_2, t_2) = (w_1 + w_2, t_1 + t_2 + \frac{1}{2}\langle w_1, w_2 \rangle).$$

This is the so-called Heisenberg group. It is a 2-step nilpotent group with center $Z = F$ and commutator $[H(W), H(W)] = Z$. The definition of this group law is motivated by the Heisenberg commutator relations from quantum mechanics, hence the name.

The irreducible (smooth) representations of $H(W)$ can be classified and come in 2 types:

- the 1-dimensional representations: these factor through

$$H(W)/[H(W), H(W)] = H(W)/Z \cong W$$

and hence are given by characters of W. Observe that these are precisely the representations trivial on the center Z.

- the other irreducible representations have nontrivial central character. For a fixed $\psi : Z = F \longrightarrow \mathbb{C}^\times$, the Stone-von Neumann theorem asserts that $H(W)$ has a unique irreducible representation ω_ψ with central character ψ. Moreover, the representation ω_ψ is unitary.

One can give an explicit construction of ω_ψ. Let

$$W = X \oplus Y$$

be a Witt decomposition of W so that X and Y are maximal isotropic subspaces. Then $H(X) = X + F$ is an abelian subgroup of $H(W)$ and one can extend ψ to a character of $H(Y)$ by setting $\psi(x, 0) = 1$ (i.e. extending trivially to X). Then

$$\omega_\psi \cong \mathrm{ind}_{H(X)}^{H(W)} \psi \quad \text{(compact induction)}.$$

This induced representation is realized on $\mathcal{S}(Y) := C_c^\infty(Y)$, and the action of $h \in H(W)$ can be easily written down:

$$\begin{cases} (\omega_\psi(0,z)f)(y') = \psi(z) \cdot f(y'), \text{ for } z \in F; \\ (\omega_\psi(y,0)f)(y') = f(y+y'), \text{ for } y \in Y \\ (\omega_\psi(x,0)f)(y') = \psi(\langle y', x\rangle) \cdot f(y'), \text{ for } x \in X. \end{cases}$$

This action preserves the natural inner product on $\mathcal{S}(Y)$.

The symplectic group $\mathrm{Sp}(W)$ acts on $H(W)$ as group automorphisms via:

$$g \cdot (w,t) = (g \cdot w, t).$$

Observe that the action on Z is trivial. Hence the representation ${}^g\omega_\psi = \omega_\psi \circ g^{-1}$ is irreducible and has the same nontrivial central character as ω_ψ. By the Stone-von Neumann theorem, these two representations are isomorphic, i..e there exists an invertible operator $A_\psi(g)$ on the underlying vector space $\mathcal{S}$ of ω_ψ such that

$$A_\psi(g) \circ {}^g\omega_\psi(h) = \omega_\psi \circ A_\psi(g) \quad \text{for all } h \in H(W).$$

By Schur's lemma, the operator $A_\psi(g)$ is well-defined up to $\mathbb{C}^\times$. By the unitarity of ω_ψ, we can insist that $A_\psi(g)$ is unitary and hence it is well-defined up to the unit circle $S^1 \subset \mathbb{C}^\times$. Hence we have a map

$$A_\psi : \mathrm{Sp}(W) \longrightarrow \mathrm{GL}(\mathcal{S})/S^1.$$

When one pulls back the short exact sequence

$$1 \longrightarrow S^1 \longrightarrow \mathrm{GL}(\mathcal{S}) \longrightarrow \mathrm{GL}(\mathcal{S})/S^1 \longrightarrow 1$$

by the map A_ψ, one obtains the desired metaplectic group $\mathrm{Mp}_\psi(W)$:

$$\begin{tikzcd} 1 \arrow[r] & S^1 \arrow[r] \arrow[d, equal] & \mathrm{Mp}_\psi(W) \arrow[r] \arrow[d, "\tilde{A}_\psi"] & \mathrm{Sp}(W) \arrow[r] \arrow[d, "A_\psi"] & 1 \\ 1 \arrow[r] & S^1 \arrow[r] & \mathrm{GL}(\mathcal{S}) \arrow[r] & \mathrm{GL}(\mathcal{S})/S^1 \arrow[r] & 1 \end{tikzcd}$$

Hence, this construction produces not just the group $\mathrm{Mp}_\psi(W)$ but also a natural representation

$$\tilde{A}_\psi : \mathrm{Mp}_\psi(W) \longrightarrow \mathrm{GL}(\mathcal{S}).$$

Thus, we have a representation ω_ψ of $\mathrm{Mp}_\psi(W) \ltimes H(W)$. In other words, the irreducible representation ω_ψ of $H(W)$ extends to the semidirect product $H(W) \rtimes \mathrm{Mp}_\psi(W)$ (with $\mathrm{Mp}_\psi(W)$ acting on $H(W)$ via its projection to $\mathrm{Sp}(W)$). We call this a Heisenberg-Weil representation; its restriction to $\mathrm{Mp}_\psi(W)$ is simply called a Weil representation.

It turns out that the isomorphism class of the extension defining $\mathrm{Mp}_\psi(W)$ is independent of ψ; so we shall write $\mathrm{Mp}(W)$ henceforth, suppressing ψ. The Weil representation ω_ψ of $\mathrm{Mp}(W)$ is, in some sense, the smallest infinite-dimensional representation of the metaplectic group. We only make two further remarks here:

- ω_ψ is a genuine representation of $\mathrm{Mp}(W)$, in the sense that $\omega_\psi(z) = z$ for all $z \in S^1$, so that ω_ψ does not factor to a smaller cover of $\mathrm{Sp}(W)$.
- ω_ψ is not an irreducible representation of $\mathrm{Mp}(W)$, but rather a direct sum of two irreducible representations (the even and odd Weil representaitons):

$$\omega_\psi = \omega_\psi^+ \oplus \omega_\psi^-.$$

  Indeed, with ω_ψ realized on $\mathcal{S}(Y)$ as above, ω_ψ^+ is realized on the subspace of even functions (i.e. $f(-x) = f(x)$) whereas ω_ψ^- is realized on the subspace of odd functions.

2.4. **Schrodinger model.** One can ask if it is possible to write down some formulas for the action of elements of $\mathrm{Mp}(W)$, for example on the model $\mathcal{S}(Y)$ of ω_ψ. Let $P(X)$ be the maximal parabolic subgroup of $\mathrm{Sp}(W)$ stabilizing the maximal isotropic subspace X; this is the so-called Siegel parabolic subgroup. Its Levi decomposition has the form $P(X) = M(X) \cdot N(X)$, with $M(X) \cong \mathrm{GL}(X)$ and

$$N(X) = \{n(B) : B \in \mathit{Sym}^2 X \subset \mathrm{Hom}(Y, X)\},$$

where

$$n(B) = \begin{pmatrix} 1 & B \\ 0 & 1 \end{pmatrix} \quad \text{(relative to } W = X \oplus Y.\text{)}$$

The metaplectic cover splits canonically over $N(X)$ and noncanonically over $M(X)$. Then one can write down formulas for the action of elements lying over $g \in \mathrm{GL}(X)$ and $n(B) \in N(X)$:

$$\begin{cases} (\omega_\psi(g)f)(y) = |\det_X(g)|^{\dim W/2} \cdot f(g^{-1} \cdot y) \\ (\omega_\psi(n(B))f)(y) = \psi(\frac{1}{2} \cdot \langle By, y\rangle) \cdot f(y). \end{cases}$$

To describe the action of $\mathrm{Mp}(W)$ (at least projectively), one needs to give the action of an extra Weyl group element

$$w = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$$

which together with $P(X)$ generates $\mathrm{Sp}(W)$. The action of this w is given by Fourier transform (up to scalars).

The above gives the Schrodinger model of the Weil representation (which is related to the Schrodinger description of quantum mechanics). This concludes our brief sketch of the construction of the metaplectic group and its Weil representations.

2.5. **Weil representations of unitary groups.** Let us return to the setting of our unitary dual pair

$$\iota : \mathrm{U}(V) \times \mathrm{U}(W) \longrightarrow \mathrm{Sp}(V \otimes_E W).$$

By the above, the metaplectic group $\mathrm{Mp}(V \otimes_E W)$ has a distinguished representation ω_ψ depending on a nontrivial additive character ψ of F. If the embedding ι can be lifted to a homomorphism

$$\tilde{\iota} : \mathrm{U}(V) \times \mathrm{U}(W) \longrightarrow \mathrm{Mp}(V \otimes_E W),$$

then we obtain a representation $\omega_\psi \circ \tilde{\iota}$ of $\mathrm{U}(V) \times \mathrm{U}(W)$.

Such splittings have been constructed and classified by S. Kudla [21]. They are not unique but can be specified by picking two characters χ_V and χ_W of $E^\times$ such that

$$\chi_V|_{F^\times} = \omega_{E/F}^{\dim V} \quad \text{and} \quad \chi_W|_{F^\times} = \omega_{E/F}^{\dim W}.$$

One way of doing this is, for example, fixing a character γ of $E^\times$ such that $\gamma|_{F^\times} = \omega_{E/F}$ (i.e. a conjugate symplectic character) and then taking

$$\chi_V = \gamma^{\dim V} \quad \text{and} \quad \chi_W = \gamma^{\dim W}.$$

In any case, if we fix this pair (χ_V, χ_W) of splitting characters, then Kudla provides a splitting

$$\tilde{\iota}_{\chi_V,\chi_W,\psi} : \mathrm{U}(V) \times \mathrm{U}(W) \longrightarrow \mathrm{Mp}(V \otimes_E W)$$

of ι. In fact, the choice of χ_V gives rise to a splitting

$$\iota_{V,W,\chi_V,\psi} : \mathrm{U}(W) \longrightarrow \mathrm{Mp}(\mathbb{V} \otimes_E W)$$

over $\mathrm{U}(W)$, whereas the choice of χ_W gives a splitting

$$\iota_{V,W,\chi_W,\psi} : \mathrm{U}(V) \longrightarrow \mathrm{Mp}(V \otimes_E W).$$

Hence, the splitting over the two members of the dual pair can be constructed somewhat independently of each other. This is a manifestation of a basic property of the metaplectic cover: if two elements of $\mathrm{Sp}(V \otimes_E W)$ commute, then any lifts of them in $\mathrm{Mp}(V \otimes_E W)$ also commute with each other.

With a splitting fixed, we may pull back the Weil representation ω_ψ and obtain a representation

$$\Omega_{V,W,\chi_V,\chi_W,\psi} := \tilde{\iota}^*_{\chi_V,\chi_W,\psi}(\omega_\psi) \quad \text{of } \mathrm{U}(V) \times \mathrm{U}(W).$$

We call this $\Omega_{\chi_V,\chi_W,\psi}$ a Weil representation of $\mathrm{U}(V) \times \mathrm{U}(W)$; we will often suppress the subscript for ease of notation.

While we have not described the splitting $\tilde{\iota}_{\chi_V,\chi_W,\psi}$ explicitly, we highlight a few basic properties:

- (Change of (χ_V, χ_W)) One can easily describe the effect of changing (χ_V, χ_W) on $\Omega_{\chi_V,\chi_W,\psi}$. More precisely, if (χ'_v, χ'_W) is another pair of splitting characters, then
$$\Omega_{V,W,\chi'_V,\chi'_W,\psi} \cong \Omega_{V,W,\chi_V,\,\chi_W,\psi} \otimes (\chi'_V/\chi_V \circ i \circ \det_W) \otimes (\chi'_W/\chi_W \circ i \circ \det_V),$$

where $i : E^1 \to E^\times/F^\times$, taking note that χ'_V/χ_V and χ'_W/χ_W are characters of $E^\times/F^\times$.

- (Scaling) Given $a \in F^\times$, one can scale the additive character ψ to obtain $\psi_a(x) = \psi(ax)$. One can also scale the Hermitian or skew-Hermitian forms on V and W, otaining V^a and W^a. If we identify $\mathrm{U}(V)$ and $\mathrm{U}(V^a)$ as the same subgroup of $\mathrm{GL}(V)$, then one has:

$$\Omega_{V,W,\chi_V,\chi_W,\psi_a} \cong \Omega_{V^a,W,\chi_V,\chi_W,\psi} \cong \Omega_{V,W^a,\chi_V,\chi_W,\psi}.$$

- (Duality) Recalling the Weil representation is unitarizable, so that its dual is isomorphic to its complex conjugate, we note:

$$\overline{\Omega_{V,W,\chi_V,\chi_W,\psi}} \cong \Omega_{V,W,\chi_V^{-1},\chi_W^{-1},\overline{\psi}}.$$

- (Center) If we take

$$\chi_V = \gamma^{\dim V} \quad \text{and} \quad \chi_W = \gamma^{\dim W}$$

for a conjugate symplectic character γ, and identify $Z_{\mathrm{U}(V)}$ and $Z_{\mathrm{U}(W)}$ with E^1 (and hence with each other), $\tilde{\iota}_{\chi_V,\chi_W,\psi}$ agrees on $Z_{\mathrm{U}(V)}$ and $Z_{\mathrm{U}(W)}$.

2.6. **Local theta lifts.** In the theory of local theta correspondence, one studies how the representation $\Omega_{V,W,\chi_V,\chi_W,\psi}$ decomposes into irreducible pieces. For this, one would of course need to know a lot more about the representation $\Omega_{V,W,\chi_V,\chi_W,\psi}$ than what has been described above. For example, one may demand if there are formulas for the group action? Ultimately, such formulas would have been derived from those in the Schrodinger model we described earlier and an explicit knowledge of the splitting $\tilde{\iota}_{\chi_V,\chi_W,\psi}$. We will come to this in the particular case of interest later on in this lecture. At the moment, let us just formulate some questions one may ask and describe the answers to some of these questions.

We will write $\mathrm{Irr}(\mathrm{U}(W))$ for the set of equivalence classes of irreducible smooth representations of $\mathrm{U}(W)$. Unlike the case of finite (or compact) groups, the representation Ω is infinite-dimensional and is not necessarily semisimple as a $\mathrm{U}(V) \times \mathrm{U}(W)$-module. So when one talks about the decomposition of Ω into irreducible constituents, one can understand this in potentially two ways:

- for which $\pi \otimes \sigma \in \mathrm{Irr}(\mathrm{U}(V)) \times \mathrm{Irr}(\mathrm{U}(W))$ is $\pi \otimes \sigma$ a subrepresentation of Ω?
- for which $\pi \otimes \sigma \in \mathrm{Irr}(\mathrm{U}(V)) \times \mathrm{Irr}(\mathrm{U}(W))$ is $\pi \otimes \sigma$ a quotient of Ω?

It turns out that it is more fruitful to consider the second question above. Hence, Ω determines a subset of $\mathrm{Irr}(\mathrm{U}(V) \times \mathrm{Irr}(\mathrm{U}(W))$:

$$\Sigma_\Omega = \{(\pi, \sigma) : \pi \otimes \sigma \text{ is a quotient of } \Omega\}$$

and thus a correspondence between $\mathrm{Irr}(\mathrm{U}(V)) \times \mathrm{Irr}(\mathrm{U}(W))$. This is the correspondence in "theta correspondence". If $\pi \otimes \sigma \in \Sigma_\Omega$, we say that σ is a local theta lift of π and vice versa.

One can reformulate the above definition in a slightly different way, which is more convenient for the question of determining all possible theta lifts of a given π. For $\pi \in \mathrm{Irr}(\mathrm{U}(V))$, one considers the maximal π-isotypic quotient of Ω:

$$\Omega / \bigcap_{f \in \mathrm{Hom}_{\mathrm{U}(V)}(\Omega,\pi)} \mathrm{Ker}(f),$$

which is a $\mathrm{U}(V) \times \mathrm{U}(W)$-quotient of Ω expressible in the form

$$\pi \otimes \Theta(\pi),$$

for some smooth representation $\Theta(\pi)$ of $\mathrm{U}(W)$ (possibly zero, and possibly infinite length a priori). We call $\Theta(\pi)$ the *big theta lift* of π. An alternative way to define $\Theta(\pi)$ is:

$$\Theta(\pi) = (\Omega \otimes \pi^{\vee})_{\mathrm{U}(V)},$$

the maximal $\mathrm{U}(V)$-invariant quotient of $\Omega \otimes \pi^{\vee}$. In any case, it follows from definition that there is a natural $\mathrm{U}(V) \times \mathrm{U}(W)$-equivariant map

$$\Omega \twoheadrightarrow \pi \otimes \Theta(\pi),$$

which satisfies the "universal property" that for any smooth representation σ of $\mathrm{U}(W)$,

$$\mathrm{Hom}_{\mathrm{U}(V)\times\mathrm{U}(W)}(\Omega, \pi \otimes \sigma) \cong \mathrm{Hom}_{\mathrm{U}(W)}(\Theta(\pi), \sigma) \quad \text{(functorially)}.$$

The local theta lifts of π are then the irreducible quotients of $\Theta(\pi)$.

2.7. **Howe duality conjecture.** The goal of local theta correspondence is to determine the representation $\Theta(\pi)$ or rather its irreducible quotients. Recall that our hope is that $\Theta(\pi)$ is close to irreducible or at least not too big. To this end, we first note the following basic result of Howe and Kudka:

Proposition 2.1 (Finiteness)**.** *For any $\pi \in \mathrm{Irr}(\mathrm{U}(V))$, $\Theta(\pi)$ is of finite length. In particular, if $\Theta(\pi)$ is nonzero, it has (finitely many) irreducible quotients and we may consider its maximal semisimple quotient (its cosocle) $\theta(\pi)$. Moreover, for any $\pi \in \mathrm{Irr}(\mathrm{U}(V))$ and $\sigma \in \mathrm{Irr}(\mathrm{U}(W))$,*

$$\dim \mathrm{Hom}_{\mathrm{U}(V)\times\mathrm{U}(W)}(\Omega, \pi \otimes \sigma) < \infty.$$

We call $\theta(\pi)$ the *small theta lift* of π. The local theta lifts of π are precisely the irreducible summands of $\theta(\pi)$.

We can now formulate a fundamental theorem [32, 15]:

Theorem 2.2 (Howe duality theorem)**.** *(i) If $\Theta(\pi)$ is nonzero, then it has a unique irreducible quotient. In other words, $\theta(\pi)$ is irreducible or 0.*

(ii) If $\theta(\pi) \cong \theta(\pi') \neq 0$, then $\pi \cong \pi'$.

Hence, one has an map

$$\theta_{\chi_V,\chi_W,\psi} : \mathrm{Irr}(\mathrm{U}(V)) \longrightarrow \mathrm{Irr}(\mathrm{U}(W)) \cup \{0\}$$

which is injective when restricted to the subset of $\mathrm{Irr}(\mathrm{U}(V))$ which is not sent to 0. Moreover,

$$\dim \mathrm{Hom}_{\mathrm{U}(V)\times\mathrm{U}(W)}(\Omega, \pi \otimes \sigma) \leq 1$$

for any $\pi \in \mathrm{Irr}(\mathrm{U}(V))$ and $\sigma \in \mathrm{Irr}(\mathrm{U}(W))$.

Another way to formulate the theorem is to note that the subset/correspondence Σ_Ω is the graph of a bijective function between $pr_V(\Sigma_\Omega) \subset \mathrm{Irr}(\mathrm{U}(V))$ and $pr_W(\Sigma_\Omega) \subset \mathrm{Irr}(\mathrm{U}(W))$, where pr_V refers to the projection to $\mathrm{Irr}(\mathrm{U}(V))$. Thus, we see that local theta correspondence is an instance of the Basic Idea highlighted at the beginning of the lecture.

2.8. **Questions.** After the Howe duality theorem above, we can ask the following questions:

(a) (Nonvanishing) For a given $\pi \in \mathrm{Irr}(\mathrm{U}(V))$, decide if $\theta_{\chi_V,\chi_W,\psi}(\pi)$ is nonzero.

(b) (Identity) Describe the map $\theta_{\chi_V,\chi_W,\psi}$ explicitly. In other words, If $\theta_{\chi_V,\chi_W,\psi}(\pi)$ is nonzero, can one describe it in terms of π in another way?

Nowadays, both these questions have rather complete answers, but it is too much to describe these answers for this course. Instead, we will highlight some relevant answers for our application.

2.9. **Rallis' tower.** For the nonvanishing question, Rallis observed that it is fruitful to consider theta correspondence in a family. Let W_0 be an anisotropic skew-Hermitian space over E (for example a 1-dimensional one), and for $r \geq 0$, let

$$W_r = W_0 \oplus \mathbb{H}^r$$

where $\mathbb{H}$ is the hyperbolic plane. The collection $\{W_r \mid r \geq 0\}$ is called a Witt tower of spaces. Observe that:

- $\dim W_r \mod 2$ is independent of r;
- $\mathrm{disc}(W_r)$ or equivalently the sign $\epsilon(W_r)$ is independent of r.

Hence, in the nonarchimedean case, there are precisely two Witt towers of skew-Hermitian spaces with a fixed dimension modulo 2, and any given skew-Hermitian space W is a member of a unique Witt tower.

One can then consider a family of theta correspondences associated to the tower of reductive dual pairs $(\mathrm{U}(V), \mathrm{U}(W_r))$ with respect to a fixed pair of splitting characters (χ_V, χ_W) (note that we can fix χ_{W_r} independently of r, since the parity of $\dim W_r$ is independent of r). Kudla showed:

Proposition 2.3. *(i) For $\pi \in \mathrm{Irr}(\mathrm{U}(V))$, there is a smallest $r_0 = r_0(\pi)$ such that $\Theta_{V,W_{r_0},\psi}(\pi) \neq 0$. Moreover, $r_0 \leq \dim V$.*

(ii) For any $r > r_0$, $\Theta_{V,W_r,\psi}(\pi) \neq 0$ (tower property).

(iii) Suppose that π is supercuspidal. Then $\Theta_{V,W_{r_0},\psi}(\pi)$ is irreducible supercuspidal. For $r > r_0$, $\Theta_{V,W_r,\psi}(\pi)$ is irreducible but not supercuspidal.

Some remarks:

- We call this smallest $r_0(\pi)$ from the proposition above the *first occurrence index* of π in the Witt tower (W_r) (relative to our fixed (χ_V, χ_W, ψ)).
- The fact that $r_0 \leq \dim V$ means that when W is sufficiently large, more precisely when W has an isotropic subspace of dimension $\geq \dim V$, the map $\theta_{V,W,\psi}$ is nonzero on the whole of $\mathrm{Irr}(\mathrm{U}(V))$. When $r \geq \dim V$, we say that the theta lifting is in the *stable range*.
- The nonvanishing problem (a) highlighted above is reduced to the question of determining the first occurrence indices for the two Witt towers.

In fact, our job is halved because the two first occurrence indices (for a given $\dim W_\bullet \mod 2$) are not independent of each other. Rather, we have the following theorem of Sun-Zhu [30]:

Theorem 2.4 (Conservation relation)**.** *Consider the two towers (W_v) and (W'_r) of skew-Hermitian spaces with fixed $\dim W_\bullet \mod 2$, and let r_0 and r'_0 be the respective first occurrence indices of a fixed $\pi \in \mathrm{Irr}(\mathrm{U}(V))$ (relative to a fixed data (χ_V, χ_W, ψ)). Then*

$$\dim W_{r_0} + \dim W'_{r'_0} = 2 \dim V + 2.$$

In particular r_0 and r'_0 determine each other.

The conservation relation above implies the following dichotomy theorem (which you should try to prove):

Corollary 2.5. *Suppose that W and W' belong to the two different Witt towers of skew-Hermitian spaces (with $\dim W \mod 2$ fixed), and $\dim W + \dim W' = 2\dim V$. Then for any $\pi \in \mathrm{Irr}(\mathrm{U}(V))$, exactly one of the two theta lifts $\Theta_{V,W,\psi}(\pi)$ and $\Theta_{V,W',\psi}(\pi)$ is nonzero.*

2.10. $\mathrm{U}_1 \times \mathrm{U}_1$**.** Let us illustrate the dichotomy theorem in the base case where $\dim V = \dim W_0 = 1$. Let W_0 and W'_0 be the two skew-Hermitian spaces of dimension 1. For any $\chi \in \mathrm{Irr}(\mathrm{U}(V)) = \mathrm{Irr}(E^1)$, the dichotomy theorem implies that exactly one of the theta lifts $\theta_{V,W_0,\psi}(\chi)$ or $\theta_{V,W'_0,\psi}(\chi)$ is nonzero. Now here is an interesting question: which of these lifts is nonzero? This turns out to be a highly nontrivial and beautiful result of Moen and Rogawski [29, Prop. 3.4] and Harris-Kudla-Sweet [17]:

Theorem 2.6. *The theta lift $\theta_{V,W_0,\psi}(\chi)$ (with respect to splitting characters (χ_V, χ_W)) is nonzero if and only if*

$$\epsilon(V) \cdot \epsilon(W_0) = \epsilon_E(1/2, \chi_E \cdot \chi_W^{-1}, \psi(Tr_{E/F}(\delta -))).$$

Here, recall that δ is a nonzero trace zero element of E and the definition of the sign $\epsilon(W_0)$ depends on δ. Moreover, χ_E is the character of $E^\times/F^\times$ defined by $\chi_E(x) = \chi(x/x^c)$ and the local epsilon factor on the right is that defined in Tate's thesis.

This theorem shows that the question of nonvanishing of theta lifting has deep arithmetic connections. One way to prove this theorem is via the doubling seesaw argument, which relates the theta lifting to the doubling zeta integral of Piatetski-Shapiro and Rallis, a theory that produces the standard L-function and epsilon factor. This is the approach of [17]. We have no time to go into this here, but would like to mention that this doubling zeta integral plays an important role in Ellen Eischen's lectures.

After addressing nonvanishing, the next issue is that of identity: what is $\theta_{V,W_0,\psi}(\chi)$ if it is nonzero? Suppose we pick $\chi_V = \chi_W$ (as is allowed here), then the splitting over the two U_1's agree (on identifying them with E^1) and so the theta lifting is the identity map $\Theta(\chi) = \chi$ on its domain (i.e. outside its zero locus). With our knowledge of how the Weil representation changes when we change (χ_V, χ_W), this allows one to figure out the general case:

$$\Theta_{\chi_V,\chi_W,\psi}(\chi) = \chi \cdot (\chi_W^{-1}\chi_V \circ i) \quad \text{on its domain.}$$

2.11. **Application to Howe-PS setting.** We shall now specialize to the particular case we are interested in. Set $V = \langle 1 \rangle$ to be the 1-dimensional Hermitian space with form $(x, y) \mapsto xy^c$ so that

$$\mathrm{U}(V) = E^1 \cong E^\times / F^\times.$$

We consider the theta correspondence for $\mathrm{U}(V)$ with the two odd-dimensional Witt towers. (W_r) and (W'_r), with $\dim W_r = \dim W'_r = 2r + 1$. Because $\mathrm{U}(V)$ is compact, one in fact has a direct sum decomposition:

$$\Omega = \bigoplus_\chi \chi \otimes \Theta(\chi).$$

as χ runs over the characters of E^1. Now let us record some consequences of the general results discussed above:

- For any χ, $\Theta_{V,W_r,\psi}(\chi)$ and $\Theta_{V,W',\psi}(\chi)$ are irreducible or 0. This is because, with $\mathrm{U}(V)$ being compact, any χ is supercuspidal.
- For any χ and $r > 0$, $\theta_{V,W_r,\psi}(\chi)$ and $\theta_{V,W'_r,\psi}(\chi)$ are both nonzero; this is because we are already in the stable range when $r > 0$.
- What if $r = 0$ (i.e. the dual pair $\mathrm{U}_1 \times \mathrm{U}_1$)? This is the situation addressed by the dichotomy theorem: exactly one of $\theta_{V,W_0,\psi}(\chi)$ and $\theta_{V,W'_0,\psi}(\chi)$ is nonzero. Exactly which one is nonzero is highly non-obvious but we shall come to this later.
- Suppose without loss of generality that $\theta_{V,W'_0,\psi}(\chi) = 0$. Then $\Theta_{V,W_1,\psi}(\chi)$ is supercuspidal.

I will leave it as an exercise for the reader to deduce the above assertions from the results discussed above. Instead, I will describe the proof of some of those results in the special case of $\mathrm{U}_1 \times \mathrm{U}_3$. This is where we do the "dirty work", which will be formulated as a series of guided exercises below.

2.12. **Guided exercise.** The Weil representation Ω for $\mathrm{U}(V) \times \mathrm{U}(W_1) = \mathrm{U}_1 \times \mathrm{U}_3$ can be given a realization as follows. Let ω be the Heisenberg-Weil representation for

$$(\mathrm{U}(V) \times \mathrm{U}(W_0)) \ltimes H(V \otimes_E W_0) = (\mathrm{U}_1 \times \mathrm{U}_1) \ltimes H(V \otimes_E W_0)$$

where we recall that $H(V \otimes_E W_0)$ is the Heisenberg group associated to the 2-dimensional symplectic space $V \otimes_E W_0$ over F. It is in fact not easy to give a concrete model for this representation (even starting with a Schridinger model for Mp_2). Then the Weil representation Ω for $\mathrm{U}(V) \times \mathrm{U}(W_1)$ can be realized on

$$S(Ee^* \otimes V) \otimes \omega$$

which can be thought of as the space of Schwarz functions on the 1-dimensional E-vector space $Ee^* \otimes V$ valued in ω. One can give explicit formulas for the action of $\mathrm{U}(V) \times B$, where $B = TU$ is the Borel subgroup stabilizing the isotropic line $E \cdot e$ as follows.

(a) for $h \in \mathrm{U}(V) = E^1$,

$$(h \cdot f)(x) = \omega(h) \left(f(h^{-1} \cdot x) \right), \quad \text{with } x \in E \text{ and } f \in \Omega.$$

(b) for an element

$$t(a,b) = \begin{pmatrix} a & & \\ & b & \\ & & (a^c)^{-1} \end{pmatrix} \in T \quad \text{with } a \in E^\times \text{ and } b \in E^1,$$

one has:

$$(t(a,b)f)(x) = \chi_V(a) \cdot |a|^{1/2} \cdot \omega(b)\left(f(a^c \cdot x)\right),$$

where we regard $b \in E^1$ as an element of $\mathrm{U}(W_0)$.

(c) for an element

$$u(0,z) = \begin{pmatrix} 1 & 0 & z \\ & 1 & 0 \\ & & 1 \end{pmatrix} \in U, \quad \text{with } z \in F,$$

one has:

$$(u(0,z)f)(x) = \psi(zN(x)) \cdot f(x).$$

(d) for an element

$$u(y,0) = \begin{pmatrix} 1 & * & * \\ & 1 & y \\ & & 1 \end{pmatrix},$$

one has:

$$(u(y,0)f)(x) = \omega(h(xy,0))(f(x)),$$

where $h(xy) = (xy, 0) \in H(V \otimes_E W_0)$ is regarded as an element in the Heisenberg group. (The 2 asterisks in $u(y,0)$ are determined by y; work out what they should be).

Now recall that we have:

$$\Omega = \bigoplus_{\chi \in \mathrm{Irr}(E^1)} \chi \otimes \Theta(\chi),$$

and our goal is now to understand $\Theta(\chi)$ as much as possible.

Given the above information, here is the guided exercise:

<u>Exercise</u>:

(i) Let $Z = \{u(0,z) : z \in F\} \subset U$. This is the center of U. Compute the coinvariant space

$$\Omega_Z = \Omega / \langle z \cdot f - f : z \in Z, f \in \Omega \rangle,$$

as a representation of $\mathrm{U}(V) \times B/Z = E^1 \times T \cdot U/Z$. Indeed, show that the natural projection $\Omega \to \Omega_Z$ is given by the evaluation-at-0 map

$$ev_0 : \mathcal{S}(E) \otimes \omega \longrightarrow \omega.$$

(ii) From the answer in (i), deduce the following:

(a) For any $\chi \in \mathrm{Irr}(\mathrm{U}(V))$, $\Theta(\chi)_Z = \Theta(\chi)_U$.

(b) Suppose that χ has nonzero theta lift $\theta_0(\chi)$ to $\mathrm{U}(W_0)$ with respect to ω, $\Theta(\chi)$ is nonzero and non-supercuspidal. Indeed, one has a nonzero $T = E^\times \times \mathrm{U}(W_0)$-equivariant map

$$\Theta(\chi)_N \to \chi_V |-|^{1/2} \otimes \theta_0(\chi),$$

so that by Frobenius reciprocity, there is a nonzero equivariant map

$$\Theta(\chi) \to \mathrm{Ind}_B^{\mathrm{U}(W)} \left(\chi_V |-|^{-1/2} \otimes \theta_0(\chi) \right) \quad \text{(normalized induction)}$$

taking note that $\delta_B^{1/2}(t(a,b)) = |a|_E$. Hence we see that $\Theta(\chi)$ contains a constituent of the latter principal series representation, which is nontempered (since $|-|^{1/2}$ is not unitary).

(c) Suppose that χ has zero theta lift to $\mathrm{U}(W_0)$. From (i), deduce that $\Theta(\chi)$ is supercuspidal (i.e. $\Theta(\chi)_U = 0$).

(iii) Now compute the twisted coinvariant space

$$\Omega_{Z,\psi} = \Omega / \langle z \cdot f - \psi(z) \cdot f : z \in Z, f \in \Omega \rangle$$

as a representation of $\mathrm{U}(V) \times T_\psi$, where

$$T_\psi = \{ t(a,b) : a, b \in E^1 \} \subset T$$

is the stabilizer of (Z, ψ) in T.

(iv) Deduce from the answer in (iii) that for any $\chi \in \mathrm{Irr}(\mathrm{U}(V))$, $\Theta(\chi) \neq 0$.

What one sees from this guided exercise is that to understand the theta lifts $\Theta(\pi)$ (for example to detect its nonvanishing or supercuspidality), it is useful to consider various twisted coinvairant spaces $\Omega_{N,\psi}$ where $N \subset \mathrm{U}(W)$ is an abelian subgroup and ψ is a (possibly trivial) character of N. Such twisted coinvariant spaces (or twisted Jacquet modules) are local analogs of the Fourier coefficients of modular forms. They are readily computable from the concrete model of the Weil representation analogous to the one above. It also shows that the Weil representations have an inductive structure with respect to the Rallis tower. In this guided exercise, we see that we are basically reduced to the following two problems:

- the irreducibility of $\Theta(\chi)$ (as given by the Howe duality theorem);
- the understanding of the $\mathrm{U}_1 \times \mathrm{U}_1$ theta correspondence (which we discussed earlier).

2.13. **Split case.** Note that the case when $E = F \times F$ is also necessary for global applications. In this case, the dual pair is $\mathrm{GL}_m \times \mathrm{GL}_n$. The Weil representation is, up tp twists by 1-dimensional characters, the natural action of $\mathrm{GL}_m(F) \times \mathrm{GL}_n(F)$ on the space $\mathcal{S}(M_{m\times n}(F))$ of Schwarz functions on the space of $m \times n$ matrices. The study of this local theta correspondence is essentially completed in the paper [24] of A. Minguez. Hence we shall say no more about this case in this paper.

2.14. **Remarks.** We have given a discussion of the theory of classical theta correspondence which is based on reductive dual pairs in the symplectic group. But this idea is clearly more robust. One may ask:

- Can one classify all reductive dual pairs $G \times H$ in any simple Lie group E, as opposed to just for $E = \mathrm{Sp}_{2n}$?
- Is there an understanding of the smallest infinite-dimensional representation Ω of any such E?
- If so, when one pulls back Ω to $G \times H$, does one obtain a transfer or lifting of representations analogous to those described in this lecture?

These questions started to be explored in the mid-1980's. Reductive dual pairs have been classified on the level of Lie algebras by Rubenthaler. The construction and classification of so-called minimal representations of a simple Lie group E was begun by Kostant, Vogan, Kazhdan, Savin, Torasso and others; see [13]. Finally the study of the resulting theta correspondence began in the 1990's but the theory is not as systematic as the classical case. It is only recently that one has somewhat complete results in several families of examples. In the project for this course, you will work with a particular instance of this exceptional theta correspondence.

3. **Lecture 3: Global Theta Correspondence**

In this third lecture, we will discuss the global theta correspondence. We will see that almost all of the considerations and constructions of the previous lecture make sense in the global setting, once they are appropriately construed. We will work over a number field k with associated local field k_v for each place v of k and with adele ring $\mathbb{A} = \prod_v' k_v$. We fix a quadratic field extension E/k and consider a pair of Hermitian space V and skew-Hermitian space W relative to E/k.

3.1. **Basic idea.** Let us return to the basic idea of Lecture 2: in the local setting, the Weil representation allows one to define the local theta lifting

$$\theta : \mathrm{Irr}(\mathrm{U}(V)) \longrightarrow \mathrm{Irr}(\mathrm{U}(W)) \cup \{0\}.$$

If $\mathrm{U}(V) \times \mathrm{U}(W)$ is in the stable range (with V smaller), one even has

$$\theta : \mathrm{Irr}(\mathrm{U}(V)) \longrightarrow \mathrm{Irr}(\mathrm{U}(W)).$$

In the global setting, one might imagine that by taking (restricted) tensor product taken over all places v of a number field k, one gets

$$\theta : \mathrm{Irr}(\mathrm{U}(V)(\mathbb{A})) \longrightarrow \mathrm{Irr}(\mathrm{U}(W)(\mathbb{A})).$$

This is the case, but we are interested not in the lifting of abstract irreducible representations, but rather of cuspidal automorphic representations. Cuspidal automorphic representations are representations which are realized in the space of cuspidal automoprhic forms (i.e. functions on $[G]$). So what we need is a map

$$\{\text{Cusp forms on } \mathrm{U}(V)\} \longrightarrow \{\text{Automorphic forms on } \mathrm{U}(W)\}.$$

We are thus interested in procedures which allow one to lift functions on a (measure) space X to functions on another space Y. A standard such procedure is via a kernel function K, i.e. a function $K : X \times Y \to \mathbb{C}$. Given such a function K, one gets a linear map

$$T_K : \mathcal{C}(X) \longrightarrow \mathcal{C}(Y)$$

defined by

$$T_K(f)(y) = \int_X K(x, y) \cdot f(x)\, dx,$$

assuming convergence is not an issue.

Let's apply this simple idea to our setting. Recall from Lecture 2 that we have

$$\tilde{\iota} : \mathrm{U}(V) \times \mathrm{U}(W) \longrightarrow \mathrm{Mp}(V \otimes_E W),$$

We shall see that the Weil representation Ω is automorphic on $\mathrm{Mp}(V \otimes_E W)$, i.e. there is an equivariant map

$$\theta : \Omega \longrightarrow \mathcal{A}(\mathrm{Mp}(V \otimes_E W)).$$

So for any $\phi \in \Omega$, we have an automorphic form $\theta(\phi)$ on $\mathrm{Mp}(V \otimes_E W)$: these are the theta functions. Pulling back $\theta(\phi)$ by $\tilde{\iota}$, we may regard $\theta(\phi)$ as a function on $[\mathrm{U}(V)] \times [\mathrm{U}(W)]$. We can thus use $\theta(\phi)$ as a kernel function to transfer functions on $[\mathrm{U}(V)]$ to functions on $[\mathrm{U}(W)]$. In other words, each $\theta(\phi)$ gives a linear map

$$\theta_\phi : \{ \text{ Cusp forms on } \mathrm{U}(V)\} \longrightarrow \{\text{Automoprhic forms on } \mathrm{U}(W)\}.$$

As we consider all these θ_ϕ together, we have a map

$$\{\text{Cuspidal automorphic representations of } \mathrm{U}(V)\} \to \{\text{Automorphic representations of } \mathrm{U}(W)\}$$

3.2. **Adelic metaplectic groups.** We shall now give more precise formulation of the above basic idea. Fix a non-trivial additive character $\psi = \prod_v \psi_v$ of $F\backslash\mathbb{A}$. Suppose that W is a symplectic vector space over k. Then for each v, we have seen the metaplectic group

$$1 \longrightarrow S^1 \longrightarrow \mathrm{Mp}(W_v) \longrightarrow \mathrm{Sp}(W_v) \longrightarrow 1$$

For almost all v, it is known that the covering splits uniquely over the hyperspecial maximal compact subgroup K_v, so that we may regard K_v as an open compact subgroup opf $\mathrm{Mp}(W_v)$. Then one can form the restricted direct product:

$$\prod_v{}' \mathrm{Mp}(W_v) \quad \text{(with respect to the family } \{K_v\}).$$

This contains as a central subgroup $\bigoplus_v S^1$. If we quotient out the restricted direct product above by the central subgroup

$$Z = \{(z_v) \in \bigoplus_v S^1 : \prod_v z_v = 1\}$$

we get the adelic metatplectic group

$$1 \longrightarrow S^1 \longrightarrow \mathrm{Mp}(W)(\mathbb{A}) \longrightarrow \mathrm{Sp}(W) \longrightarrow 1.$$

Note that though we use the notation $\mathrm{Mp}(W)(\mathbb{A})$, $\mathrm{Mp}(W)$ is not an algebraic group and we are not taking the group of adelic points of an algebraic group.

An important property of this adelic metaplectic over is that it splits (canonically) over the group $\mathrm{Sp}(W)(k)$ of k-rational points, so that one can canonically regard $\mathrm{Sp}(W)(k)$ as a subgroup of $\mathrm{Mp}(W)(\mathbb{A})$. As a result one can consider the automorphic quotient

$$[\mathrm{Mp}(W)] = \mathrm{Sp}(W)(k)\backslash\mathrm{Mp}(W)(\mathbb{A})$$

and introduce the space of genuine automorphic forms on $\mathrm{Mp}(W)(\mathbb{A})$: these are the automorphic functons

$$f : [\mathrm{Mp}(W)] \longrightarrow \mathbb{C}$$

such that for all $z \in S^1$,

$$f(zg) = z \cdot f(g).$$

3.3. **Global Weil representations.** We may consider the global Weil representation

$$\omega_\psi := \bigotimes_v{}' \omega_{\psi_v} \quad \text{of} \quad \prod_v{}' \mathrm{Mp}(W_v).$$

This factors to a representation ω_ψ of $\mathrm{Mp}(W)(\mathbb{A})$ (for clearly Z acts trivially). If $W = X \oplus Y$ is a Witt decomposition, we have seen that for each v, ω_{ψ_v} can be realized on $\mathcal{S}(Y_v)$. Hence, ω_ψ can be realized on

$$\bigotimes_v{}' \mathcal{S}(Y_v) = \mathcal{S}(Y_{\mathbb{A}}).$$

In other words, ω_ψ is realized on a very concrete space.

3.4. **Theta functions.** It turns out that one has a $\mathrm{Mp}(W)(\mathbb{A})$-equivariant map

$$\theta : \mathcal{S}(Y_{\mathbb{A}}) \longrightarrow \mathcal{A}(\mathrm{Mp}(W))$$

defined by averaging over the k-rational points of Y:

$$\theta(f)(g) = \sum_{y \in Y_k} (\omega_\psi(g) \cdot f)(y).$$

The fact that $\theta(f)$ is left-invariant under $\mathrm{Sp}(W)(k)$ is a consequence of the Poisson summation formula. The functions $\theta(f)$ are called theta functions.

The map θ is non-injective. More precisely, since

$$\omega_{\psi_v} = \omega^+_{\psi_v} \oplus \omega^-_{\psi_v} \quad \text{for each } v,$$

we see that as an abstract representation,

$$\omega_\psi = \bigoplus_S \omega_{\psi,S}$$

where

$$\omega_{\psi,S} = (\bigotimes_{v \in S} \omega^-_{\psi_v}) \otimes (\bigotimes_{v \notin S} \omega^+_{\psi_v})$$

for finite subsets S of places of k. Then

$$\mathrm{Ker}(\theta) = \bigoplus_{|S| \text{ odd}} \omega_{\psi,S},$$

and we have

$$\theta : \bigoplus_{|S| \text{ even}} \omega_{\psi,S} \hookrightarrow \mathcal{A}(\mathrm{Mp}(W)).$$

3.5. **Pulling back.** Now let us return to our reductive dual pair $\mathrm{U}(V) \times \mathrm{U}(W)$ over k. Recall that we have

$$\iota : \mathrm{U}(V)(\mathbb{A}) \times \mathrm{U}(W)(\mathbb{A}) \longrightarrow \mathrm{Sp}(V \otimes_E W)(\mathbb{A}).$$

If we fix a pair of automorphic characters (χ_V, χ_W) of $E^\times$ with

$$\chi_V|_{\mathbb{A}^\times} = \omega_{E/k}^{\dim_E V} \quad \text{and} \quad \chi_W|_{\mathbb{A}^\times} = \omega_{E/k}^{\dim_E W},$$

then as in the local case, one obtains an associated lifting

$$\tilde{\iota}_{\chi_V,\chi_W,\psi} : \mathrm{U}(V)(\mathbb{A}) \times \mathrm{U}(W)(\mathbb{A}) \longrightarrow \mathrm{Mp}(V \otimes W)(\mathbb{A}).$$

This lifting $\tilde{\iota}$ sends the group $\mathrm{U}(V)(k) \times \mathrm{U}(W)(k)$ into $\mathrm{Sp}(V \otimes W)(k) \subset \mathrm{Mp}(V \otimes W)(\mathbb{A})$. Hence, the pullback of a function in $\mathcal{A}(\mathrm{Mp}(V \otimes W))$ by $\tilde{\iota}$ gives a function on $[\mathrm{U}(V)] \times [\mathrm{U}(W)]$, Thus we have

$$\theta : \Omega \longrightarrow \mathcal{A}(\mathrm{Mp}(V \otimes W)) \longrightarrow \mathcal{A}(\mathrm{U}(V) \times \mathrm{U}(W)).$$

3.6. **Global theta liftings.** Now for $f \in \mathcal{A}_{cusp}(\mathrm{U}(V))$ and $\varphi \in \Omega$, we set:

$$\theta(\varphi, f)(g) = \int_{[\mathrm{U}(V)]} \theta(\varphi)(g,h) \cdot \overline{f(h)}\, dh.$$

The cuspidality of f implies that the integral above converges absolutely (because of the rapid decay of f). Then $\theta(\varphi, f)$ is an automorphic form on $\mathrm{U}(W)$.

Suppose that $\pi \subset \mathcal{A}_{cusp}(\mathrm{U}(V))$ is an irreducible cuspidal automorphic representation. Let

$$\Theta(\pi) = \langle \theta(\varphi, f) : f \in \pi,\ \varphi \in \Omega \rangle \subset \mathcal{A}(\mathrm{U}(W)).$$

This is a $\mathrm{U}(W)(\mathbb{A})$-submodule of $\mathcal{A}(\mathrm{U}(W))$ and we call it the **global theta lift** of π.

3.7. **Questions.** The main questions concerning global theta lifting are analogs of those in the local case:

- Is $\Theta(\pi)$ nonzero?
- Is $\Theta(\pi)$ contained in the space of cusp forms?

In addition, we can ask:

- For $\pi = \otimes'_v \pi_v$, how is the global $\Theta(\pi)$ related to the local theta liftings $\theta(\pi_v)$ for all v?

We shall begin by addressing this issue of local-global compatibility.

3.8. **Compatibility with local theta lifts.** How is the representation $\Theta(\pi)$ related to the abstract irreducible representation $\Theta^{abs}(\pi) := \otimes_v \theta(\pi_v)$? We have:

Proposition 3.1. *Suppose that $\Theta(\pi)$ is non-zero and is contained in the space $\mathcal{A}_2(\mathrm{U}(W))$ of square-integrable automorphic forms on $\mathrm{U}(W)$. Then $\Theta(\pi) \cong \Theta^{abs}(\pi)$.*

Proof. Since $\Theta(\pi) \subset \mathcal{A}_2(\mathrm{U}(W))$, it is semisimple. Let σ be an irreducible summand of $\Theta(\pi)$. Then consider the linear map

$$\Omega \otimes \pi^\vee \otimes \sigma^\vee \longrightarrow \mathbb{C}$$

defined by:

$$\varphi \otimes \overline{f_1} \otimes \overline{f_2} \mapsto \int_{[\mathrm{U}(W)]} \theta(\varphi, f_1)(g) \cdot \overline{f_2(g)}\, dg.$$

This map is non-zero and $\mathrm{U}(V) \times \mathrm{U}(W)$-invariant. Thus it gives rise to a non-zero equivariant map

$$\Omega \longrightarrow \pi \otimes \sigma,$$

and thus for all v, a non-zero $\mathrm{U}(V_v) \times \mathrm{U}(W_v)$-equivariant map

$$\Omega_v \longrightarrow \pi_v \otimes \sigma_v.$$

In other words, we must have

$$\sigma_v \cong \theta(\pi_v).$$

Hence, $\Theta(\pi)$ must be an isotypic sum of $\Theta^{abs}(\pi)$. Moreover, the multiplicity-one statement in the Howe duality theorem implies that

$$\dim \mathrm{Hom}_{\mathrm{U}(V)(\mathbb{A}) \times \mathrm{U}(W)(\mathbb{A})}(\Omega, \pi \otimes \Theta^{abs}(\pi)) = 1.$$

Thus $\Theta(\pi)$ is in fact irreducible and isomorphic to $\Theta^{abs}(\pi)$. $\square$

3.9. **Cuspidality and Nonvanishing.** As in the local case, it is useful to consider a Rallis tower of theta lifitngs, corresponding to a Witt tower $W_r = W_0 \oplus \mathbb{H}^r$ of skew-Hermitian spaces, with W_0 anisotropic. One has the analogous results:

Proposition 3.2. *Let π be a cuspidal automorphic representation of $\mathrm{U}(V)$, and consider its global theta lift $\Theta_{V,W_r,\psi}(\pi)$ on $\mathrm{U}(W_r)$ (relative to a fixed pair (χ_V, χ_W)). Then one has:*

(i) There is a smallest $r_0 = r_0(\pi) \leq \dim V$ such that $\Theta_{V,W_{r_0},\psi}(\pi) \neq 0$. Moreover, $\Theta_{V,W_{r_0},\psi}(\pi)$ is contained in the space of cusp forms.

(ii) For all $r > r_0$, $\Theta_{V,W_r,\psi}(\pi)$ is nonzero and noncuspidal.

(iii) For all $r \geq \dim V$, $\Theta_{V,W_{r_0},\psi}(\pi) \subset \mathcal{A}_2(\mathrm{U}(W))$.

As in the local case, we call $r_0 = r_0(\pi)$ the first occurrence of π in the relevant Witt tower, and we call the range where $r \geq \dim V$ the stable range.

3.10. **The case of** $\mathrm{U}_1 \times \mathrm{U}_1$**.** In parallel with the local setting, we may consider the theta lift for the basic case of $\mathrm{U}(V) \times \mathrm{U}(W)$ with $\dim V = \dim W = 1$. We have seen that the nonvanishing of local theta lifts is controlled by a local root number. Here is the global theorem:

Theorem 3.3. *Let χ be an automorphic character of $\mathrm{U}(V) = E^1$. Its global theta lift $\Theta_{\chi_V,\chi_W,\psi}(\pi)$ on $\mathrm{U}(W)$ is nonzero if and only if :*

(i) *for each place v, the local theta lift $\theta_{\chi_{V,v},\chi_{W,v},\psi_v}(\chi)$ is nonzero;*
(ii) *the global L-value*

$$L(1/2, \chi_E \chi_W^{-1}) \neq 0.$$

Note that under condition (i), our local theorem for $\mathrm{U}_1 \times \mathrm{U}_1$ implies that

$$\epsilon(1/2, \chi_{E,v}\chi_{W,v}^{-1}, \psi_v(\mathrm{Tr}(\delta -))) = \epsilon(V_v) \cdot \epsilon(W_v) \quad \text{for all } v;$$

where δ is a trace 0 element of E. On taking product over all places v, we see that

$$\epsilon(1/2, \chi_E \chi_W^{-1}) = 1$$

since $\prod_v \epsilon(V_v) = 1 = \prod_v \epsilon(W_v)$. Hence, there is a chance that the global L-value in (ii) is nonzero!

The proof of this Theorem is a global analog of the local theorem in the $\mathrm{U}_1 \times \mathrm{U}_1$ case, via the doubling seesaw and doubling zeta integral. One then gets the Rallis inner product formula, which relates the Petersson inner product of two global theta lifts and the special L-value. This result thus gives an interpretation for the nonvanishing of the central L-value.

3.11. **The case of** $\mathrm{U}_1 \times \mathrm{U}_3$**.** Let us now specialize to the case of interest, where $\dim V = 1$ and $\dim W = \dim W_1 = 3$. Let χ be an automorphic character of E^1. Then χ is cuspidal since E^1 is anisotropic. We can thus apply the above general results to conclude:

Corollary 3.4. $\Theta_{V,W_1}(\chi)$ *is nonzero and square-integrable. It is cuspidal if and only if the global theta lift of χ to $\mathrm{U}(W_0) = \mathrm{U}_1$ is zero.*

Using all the results we have seen, we can now construct a counterexample to the naive Ramanujan conjecture:

- Consider the global theta correspondence for $\mathrm{U}(V) \times \mathrm{U}(W_1)$ with $\dim V = 1$ and $\dim W_1 = 3$, and take $\chi_V = \chi_W$ and $\chi = 1$ (the trivial character of $\mathrm{U}(V)(\mathbb{A})$). The global theta lift $\Theta_{V,W,\psi}(1)$ is a nonzero irreducible square-integrable automorphic representation.
- If 1 has zero global theta lift $\theta_0(1) \cong 1$ to the lower step $\mathrm{U}(W_0) = \mathrm{U}_1$ of the Rallis tower, then $\Theta(1)$ is cuspidal irreducible. Moreover, for almost all v, $\Theta(1)_v \cong \theta(1_v)$ is an unramified representation belonging to the principal series
$$\mathrm{Ind}_{B_v}^{\mathrm{U}(W_v)} \left(\chi_{V,v}| - |_v^{-1/2} \otimes 1_v\right)$$
and thus is nontempered. Hence $\Theta(1)$ would be a counterexample to the naive Ramanujan conjecture.
- On the other hand, if 1 has nonzero theta lift to $\mathrm{U}(W_0)$, then we may select two places v_1 and v_2 and replace W_{v_1} and W_{v_2} by the other local skew-Hermitian space. In other words, we can find a global skew-Hermitian space W' such that
$$W'_{v_1} \neq W_{v_1} \quad \text{and} \quad W'_{v_2} \neq W_{v_2}$$
but
$$W'_v \cong W_v \quad \text{for all } v \neq v_1, v_2.$$
Such a global skew-Hermitian space exists, by our classification of global Hermitian spaces.

  Now consider the global theta lift $\Theta_{V,W'}(1)$ on $\mathrm{U}(W')$ and observe that
$$\theta_{V_{v_1},W'_{v_1}}(1) = 0 = \theta_{V_{v_2},W'_{v_2}}(1)$$
because of dichotomy. Hence we can repeat the above argument in replacing W by W'.

3.12. **Guided exercise.** As in the local case, it will be instructive to carry out some of the dirty work in proving some of the above results, at least in our setting of $\mathrm{U}_1 \times \mathrm{U}_3$. The guided exercise below is the global analog of the local guided exercise in Lecture 2, following the various notation there.

To do the exercise, it is necessary to write down the theta function $\theta(\phi)$ (for $\phi \in \Omega$) as explicitly as possible. Recall that Ω is realized on
$$\mathcal{S}(\mathbb{A}_E e^* \otimes v_0) \otimes \omega$$
where ω is the global Heisenbegr-Weil representation of
$$(\mathrm{U}(V) \times \mathrm{U}(W_0)) \ltimes H(V \otimes_E W_0).$$
We have an automorphic functional
$$\theta_0 : \omega \longrightarrow \mathcal{A}(\mathrm{Mp}_2) \to \mathbb{C}$$
where the last arrow is the evaluation at 1. The automorphic realization
$$\theta : \Omega = \mathcal{S}(\mathbb{A}_E) \otimes \omega \longrightarrow \mathcal{A}(\mathrm{U}(V) \times \mathrm{U}(W))$$

is then given by

$$\theta(f)(g) = \sum_{x \in E} \theta_0 \left(\Omega(g)(f)(x)\right).$$

Now for the exercise:

Exercise:

- For a cusp form f on $\mathrm{U}(V)$ and $\phi \in \Omega$, compute:

$$\theta(\phi, f)_Z(g) := \int_{[Z]} \theta(\phi, f)(zg)\, dz$$

where $Z = \{u(0, z) : z \in k\}$ is the center of the unipotent radical U of B, as well as

$$\theta(\phi, f)_U(g) = \int_{[U]} \theta(\phi, f)(ug)\, du.$$

- Likewise, compute:

$$\theta(\phi, f)_{Z,\psi_{1/2}}(g) := \int_{[Z]} \overline{\psi(z/2)} \cdot \theta(\phi, f)(zg)\, dz$$

- From these computations, deduce as much of Cor. 3.4 as possible.

4. **Lecture 4: Arthur's Conjecture**

In this final lecture, we will discuss an influential conjecture of Arthur [2] which explains and classifies all possible failures of the naive Ramanujan conjecture. We will then illustrate with some examples, including the Howe-PS example discussed earlier, the Saito-Kurokawa example for PGSp_4 and the case of the exceptional group G_2. The latter will be relevant for our project.

4.1. **A basic hypothesis.** In the formulation of Arthur's conjecture, one needs to make a (serious) assumption:

(Basic Hypothesis): There is a topological group L_k (depending only on the number field k) satisfying the following properties:

- the identity component L_k^0 of L_k is compact and the group of components L_k/L_k^0 is isomorphic to the Weil group W_k of k;
- for each place v, there is a natural conjugacy class of embeddings $L_{k_v} \hookrightarrow L_k$, where L_{k_v} is the Weil group if k_v is archimedean, and the Weil-Deligne group $W_{k_v} \times \mathrm{SU}_2(\mathbb{C})$ if k_v is non-archimedean.
- there is a natural bijection between the set of isomorphism classes of irreducible representations of L_k of dimension n and the set of cuspidal representations of $GL_n(\mathbb{A})$.

This assumption is basically the main conjecture in the Langlands program for GL_n. We can view it as a classification of the cuspidal representaitons of GL_n in terms of irreducible n-dimensional Galois representations.

4.2. **Arthur's conjecuture.** Arthur's conjecture is a classification of the constituents of $\mathcal{A}_2(G)$, i.e.a classification of the square-integrable automorphic representations of G. This classification proceeds in two steps. The first step is approximately the classification of these constituents up to near equivalence (this is not entirely true, but for the groups discussed here, it is expected to be so). Here, we say that two representations $\pi_1 = \otimes_v' \pi_{1,v}$ and $\pi_2 = \otimes_v' \pi_{2,v}$ of $G(\mathbb{A})$ are *nearly equivalent* if for almost all places v, $\pi_{1,v}$ and $\pi_{2,v}$ are isomorphic: this is an equivalence relation.

More precisely, Arthur speculated that there is a decomposition

$$\mathcal{A}_2(G) = \bigoplus_{\psi} \mathcal{A}_{2,\psi},$$

where each $\mathcal{A}_{2,\psi}$ is a near equivalence class, and the direct sum runs over equivalence classes of discrete A-parameters ψ. For a split group G, an A-parameter is an admissible map

$$\psi : L_k \times SL_2(\mathbb{C}) \longrightarrow G^{\vee}$$

where $G^{\vee}$ is the complex Langlands dual group of G. One property of being admissible is that $\psi(L_k)$ should be bounded in $G^{\vee}$. An A-parameter is discrete if the centralizer group

$$S_{\psi} := Z_{G^{\vee}}(\psi)/Z(G^{\vee})$$

is finite.

The second step of the classification is to describe the decomposition of each $\mathcal{A}_{2,\psi}$. This has a local-global structure. That is, for each place v, there will be a finite set of unitary representations of $G(k_v)$ (depending on the A-parameter ψ). If we pick an element π_v from each of these finite sets at each place v, we may form the (restricted) tensor product $\pi := \otimes_v \pi_v$, which is a representation of $G(\mathbb{A})$. Then $\mathcal{A}_{2,\psi}$ is the sum of such representations, with some multipllcities. Let us now be more precise.

4.3. **Local A-packets.** The global A-parameter ψ gives rise (by restriction) to a local A-parameter

$$\psi_v : L_{k_v} \times SL_2(\mathbb{C}) \longrightarrow G^\vee$$

for each place v of k. Set

$$S_{\psi_v} = \pi_0 \left(Z_{G^\vee}(\psi_v)/Z(G^\vee) \right).$$

This is the local component group of ψ_v. To each irreducible representation η_v of S_{ψ_v}, Arthur speculated that one can attach a unitarizable finite length (possibly reducible, possibly zero) representation π_{η_v} of $G(k_v)$. The set

$$A_{\psi_v} = \{\pi_{\eta_v} \ : \ \eta_v \in \mathrm{Irr}(S_{\psi_v})\}$$

is called a local A-packet. Among other things, it is required that

- for almost all v, π_{η_v} is irreducible and unramified if η_v is the trivial character 1_v. In fact, for almost all v, π_{η_v} is the unramified representation whose Satake parameter is:

$$s_{\psi_v} = \psi_v \left(Fr_v \times \begin{pmatrix} q_v^{1/2} & \\ & q_v^{-1/2} \end{pmatrix} \right),$$

  where Fr_v is a Frobenius element at v and q_v is the number of elements of the residue field at v.

4.4. **Global A-packets.** With the local packets A_{ψ_v} at hand, we may define the global A-packet by:

$$A_\psi = \{\pi = \otimes_v \pi_{\eta_v} : \pi_{\eta_v} \in A_{\psi_v}\, \eta_v = 1_v \text{ for almost all } v\}.$$

It is a set of nearly equivalent representations of $G(\mathbb{A})$, indexed by the irreducible representations of the compact group

$$\mathcal{S}_{\psi,\mathbb{A}} := \prod_v S_{\psi,v}.$$

If $\eta = \bigotimes_v \eta_v$ is an irreducible character of $\mathcal{S}_{\psi,\mathbb{A}}$, then we may set

$$\pi_\eta = \bigotimes_v \pi_{\eta_v}.$$

This is possible because for almost all v, $\eta_v = \mathbf{1_v}$ and $\pi_{\mathbf{1_v}}$ is required to be unramified by the above.

4.5. **Multiplicity formula.** The space $\mathcal{A}_{2,\psi}$ will be the sum of the elements of A_ψ with some multiplicities. More precisely, note that there is a natural map

$$S_\psi \longrightarrow S_{\psi,\mathbb{A}}.$$

Arthur attached to ψ a quadratic character ϵ_ψ of S_ψ (whose definition is given below). Now if η is an irreducible character of $S_{\psi,\mathbb{A}}$, we set

$$m_\eta = \frac{1}{\# S_\psi} \cdot \left(\sum_{s \in S_\psi} \epsilon_\psi(s) \cdot \eta(s) \right).$$

Then Arthur conjectures that

$$\mathcal{A}_\psi \cong \bigoplus_\eta m_\eta \pi_\eta.$$

4.6. **The character ϵ_ψ.** The definition of the quadratic character ϵ_ψ is quite subtle. For a discrete ψ, one considers the adjoint action of

$$S_\psi \times L_k \times \mathrm{SL}_2(\mathbb{C}) \quad \text{on } \mathrm{Lie}(G^\vee) \text{ via } \psi$$

and decomposes it into the direct sum of irreducible summands, each of which has the form

$$\eta \otimes \rho \otimes S_r$$

where S_r denotes the r-dimensional irreducible representation of $\mathrm{SL}_2(\mathbb{C})$. Note that this is an orthogonal representation, since the adjoint representation has a nondegenerate invariant symmetric bilinear form (e.g. the Killing form if G is semisimple).

We consider only those irreducible components $\eta \otimes \rho \otimes S_k$ satisfying the following properties:

- $\eta \otimes \rho \otimes S_k$ is orthogonal;
- k is even, so that S_k is a symplectic representation.
- ρ is symplectic, and
$$\epsilon(1/2, \rho) = -1.$$

The conditions above imply that η is orthogonal, so that $\det(\eta)$ is a quadratic character of S_ψ. Let $\mathcal{T}$ br the set of irreducible summands satisfying these conditions. Then we set

$$\epsilon_\psi(s) = \prod_{\tau \in \mathcal{T}} \det(\eta)(s) \quad \text{for } s \in S_\psi.$$

As an example, suppose that ψ is trivial on $\mathrm{SL}_2(\mathbb{C})$. Then the set $\mathcal{T}$ is empty (since the only S_k which occurs above is the trivial representation S_1).

4.7. **Tempered and non-tempered parameters.** An A-parameter ψ is called *tempered* if ψ is trivial when restricted to $SL_2(\mathbb{C})$. In this case, the representations in A_ψ are conjectured to be tempered (this corresponds to the boundedness condition on $\psi(L_k)$). A non-tempered A-parameter is one for which $\psi(\mathrm{SL}_2(\mathbb{C}))$ is not trivial. In this case, for almost all v, the unramified representation π_{1_v} (which has Satake parameter s_{ψ_v}) is nontempered.

Thus, according to Arthur's conjecture, the cuspidal representations in

$$\bigoplus_{\text{nontempered } \psi} \mathcal{A}_\psi$$

are precisely those which violate the naive Ramanujan conjecture. On the other hand, the cuspidal representations in $\mathcal{A}_\psi$ for tempered ψ are all tempered and the representation $\pi_1 = \otimes_v \pi_{1_v}$ should be globally generic. In this sense, Arthur's conjecture provides an explanation and classification of nontemepred cusp forms.

Though the group L_k is conjectural, we really only need its irreducible representations in formulating Arthur's conjecture for classical groups. As such, under our (basic hypothesis), we can replace all occurrences of "an irreducible n-dimensional representation of L_k by "an irreducible cuspidal representation of GL_n". Then we may view Arthur's conjecture as a description of $\mathcal{A}_2(G)$ in terms of cuspidal representations of GL's. Understood in this way, when G is a quasi-split classical group, Arhtur's conjecture has been verified in the works of Arthur [3] and Mok [25].

4.8. **Connection with theta correspondence.** Here is a natural question one can ask concerning theta correspondence and Arthur's conjecture. We have seen in Lecture 3 that when $\mathrm{U}(V) \times \mathrm{U}(W)$ is a dual pair in the stable range (with V the smaller space), then for any cuspidal representation π of $\mathrm{U}(V)$, its global theta lift $\Theta(\pi)$ on $\mathrm{U}(W)$ is a nonzero irreducible summand of $\mathcal{A}_2(\mathrm{U}(W))$. Since square-integrable automorphic representations have A-parameters, it is natural to ask how the A-parameters of $\Theta(\pi)$ and π are related.

If we view A-parameters as representing near equivalence classes, answering this question is about identifying the local theta lifts of unramified representations and then detecting the automorphy of the family of unramified local theta lifts. This line of reasoning leads to:

Conjecture 4.1 (Adam's conjecuture). *If $\pi \subset \mathcal{A}_{cusp}(\mathrm{U}(V))$ has A-parameter Ψ (thought of as an $\dim V$-dimensional representation of $L_E \times \mathrm{SL}_2(\mathbb{C})$), then under the global theta lift with respect to (χ_V, χ_W, ψ), the global theta lift of π (which is a summand in $\mathcal{A}_2(\mathrm{U}(W))$) has A-parameter*

$$\chi_V \cdot (\chi_W^{-1} \Psi \oplus S_{\dim W - \dim V}).$$

Given Arthur's conjecture, this is largely a local unramified issue. What is subtle about Adam's conjecture is its local analog (which we don't discuss here)

4.9. **Examples.** We shall illustrate Arthur's conjecture with several families of examples in low rank in the rest of the lecture. For the groups SO_5, U_3 and G_2, we shall write down a family of nontempered A-paremeters. For each such A-parameter ψ, we will examine the consequences of Arthur's conjecture. This will involve determinning:

- the local and global component groups associated to ψ;
- the quadratic character ϵ_ψ;
- the size of the local A-packets and the structure of the submodule $\mathcal{A}_\psi$.

We will then see if the description of these A-parameters provide some clues to how the A-packets and $\mathcal{A}_\psi$ may be constructed.

4.10. **Saito-Kurokawa example.** Let $G = \mathrm{SO}_5 = \mathrm{PGSp}_4$, so that its Langlands dual group is $G^\vee = \mathrm{Sp}_4(\mathbb{C})$. We have the subgroup

$$SL_2(\mathbb{C}) \times SL_2(\mathbb{C}) \subset Sp_4(\mathbb{C}) = G^\vee.$$

These two commuting SL_2's play symmetrical roles, as they correspond to a pair of orthogonal long roots in the C_2 root system.

We will consider A-parameters of the form:

$$\psi = \rho \times \mathrm{Id} : L_k \times \mathrm{SL}_2(\mathbb{C}) \longrightarrow \mathrm{SL}_2(\mathbb{C}) \times \mathrm{SL}_2(\mathbb{C}) \subset \hat{G}.$$

Such an A-parameter ψ is specficied by giving an (admissible) homomorphism

$$\rho : L_k \longrightarrow \mathrm{SL}_2(\mathbb{C}).$$

Note also that

$$Z_{G^\vee}(\psi) = Z_{\mathrm{SL}_2}(\rho) \times Z_{\mathrm{SL}_2}$$

Hence, the A-parameter ψ is discrete if and only if $Z_{\mathrm{SL}_2}(\rho)$ is finite, or equivalently if the 2-dimensional representation of L_k afford by ρ is irreducible. By our (basic hypothesis), to give such a ρ is the same as giving a cuspidal representation $\tau = \tau_\rho$ of GL_2 with trivial central character, i.e. a cuspidal representation of PGL_2.

A discrete A-parameter of $G = \mathrm{SO}_5$ of the above form is called a Saito-Kurokawa A-parameter. We have just seen that such A-parameters are parametrized by cuspidal representations of PGL_2.

Given a parameter $\psi = \psi_\tau$, let us compute the various quantities that appear in Arthur's conjecture. As we saw above

$$S_\psi = (Z_{\mathrm{SL}_2}(\rho_\tau) \times Z_{\mathrm{SL}_2})/Z_{\mathrm{Sp}_4} = (\mu_2 \times \mu_2)/\Delta\mu_2 = \mu_2$$

Likewise the local component groups S_{ψ_τ} are given by

$$S_{\psi_{\tau,v}} = \begin{cases} \mu_2, & \text{if } \rho_{\tau_v} \text{ is irreducible;} \\ 1, & \text{if } \rho_{\tau_v} \text{ is reducible.} \end{cases}$$

The condition that $\rho_{\tau,v}$ be irreducible is equivalent to τ_v being a discrete series representation of $\mathrm{PGL}_2(F_v)$.

4.10.1. **Local Arthur packets.** Now Arthur's conjecture predicts that for each place v, the local A-packets $A_{\psi_{\tau,v}}$ has the form:

$$A_{\psi_{\tau,v}} = \begin{cases} \{\pi^+_{\tau_v}\}, & \text{if } \tau_{\mathrm{v}} \text{ is not discrete series} \\ \{\pi^+_{\tau_v}, \pi^-_{\tau_v}\}, & \text{if } \tau_{\mathrm{v}} \text{ is discrete series.} \end{cases}$$

Here, $\pi^+_{\tau_v}$ is indexed by the trivial character of $S_{\psi_{\tau,v}}$.

Of course, we know what π^+_v has to be for almost all v: it is irreducible unramified with Satake parameter $s_{\psi_{\tau,v}}$. This unramifed representation π^+_v is the unramfied constituent of the induced representation

$$I_P(\tau_v, 1/2) = \mathrm{Ind}_P^G | - |_v^{1/2} \otimes \tau_v.$$

where $P = MN$ is the Siegel parabolic subgroup of SO_5 with Levi factor $\mathrm{GL}_1 \times \mathrm{SO}_3 = \mathrm{GL}_1 \times \mathrm{PGL}_2$. From this, we see that the representations in the global A-packet are nearly equivalent to the constitutents of $Ind_{P(\mathbb{A})}^{G(\mathbb{A})}|-|^{1/2}\otimes\tau$. Moreover, their local components are nontempered for almost all v.

4.10.2. **Global A-packets.** Let S_τ be the set of places v where τ_v is discrete series, so that the global A-packet has $2^{\#S_\tau}$ elements.

To compute the multiplicity m_η of $\pi_\eta \in A_{\psi_\tau}$ we need to know the quadratic character ϵ_{ψ_τ} of S_{ψ_τ}. By a short computation (which you should do), ϵ_{ψ_τ} is the non-trivial character of $S_{\psi_\tau}\cong\mu_2$ if and only if $\epsilon(1/2,\tau)=-1$. Here $\epsilon(s,\tau)$ is the global ϵ-factor of τ.

Now if $\pi=\otimes_v\pi_{\tau_v}^{\epsilon_v}\in A_{\psi_\tau}$, then the multiplicity associated to π by Arthur's conjecture is:

$$m(\pi)=\begin{cases}1, \text{ if } \epsilon_\pi:=\prod_v\epsilon_v=\epsilon(1/2,\tau);\\ 0, \text{ if } \epsilon_\pi=-\epsilon(1/2,\tau).\end{cases}$$

Thus, we should have:

$$\mathcal{A}_{\psi_\tau}\cong\bigoplus_{\pi\in A_{\psi_\tau}:\epsilon_\pi=\epsilon(1/2,\tau)}\pi.$$

4.10.3. **Construction.** How can one construct the A-packets A_{ψ_τ} and the space $\mathcal{A}_{\psi_\tau}$? It seems that we need a lifting to go from τ to these square-integrable representations of PGSp_4. Since the theta correspondence is 1-to-1, one cannot hope to use theta correspondence to go from τ to $\mathcal{A}_{\psi_\tau}$, never mind the fact that $\mathrm{PGL}_2\times\mathrm{SO}_5$ is not a dual pair in a symplectic group.

We need an intermediate step: the Shimura correspondence, or rather its automorphic description by Waldspurger [31, 33]. Using the theta correspondence for $\mathrm{Mp}_2\times\mathrm{SO}_3$, Waldspurger was able to provide a classification of the constituents of $\mathcal{A}(\mathrm{Mp}_2)$ in the style of Arthur's conjecture.

More precisely, τ gives rise to a packet of cuspidal representations on Mp_2, whose structure is exactly the same as that of the Saito-Kurokawa packets. Namely, for each place v, one has a local packet of irreducible representations of $\mathrm{Mp}_2(k_v)$:

$$\tilde{A}_{\tau_v}=\begin{cases}\{\sigma_{\tau_v}^+,\sigma_{\tau_v}^-\}, \text{ if } \tau_v \text{ is discrete series;}\\ \{\sigma_{\tau_v}^+\}, \text{ if } \tau_v \text{ is not discrete series.}\end{cases}$$

We call these the Waldspurger packets. One can form the global packet as a restricted tensor product of the local ones, and one gets a submodule

$$\tilde{\mathcal{A}}_\tau=\bigoplus_{\pi\in\tilde{A}_\tau:\epsilon_\sigma=\epsilon(1/2,\tau)}\sigma\subset\mathcal{A}_{cusp}(\mathrm{Mp}_2).$$

.

Observe the formal similarity between the structure of the Waldspurger packets and the Saito-Kurokawa ones. Given this, and taking note that one has a dual pair

$$\mathrm{Mp}_2\times\mathrm{SO}_5$$

(which is the next step of the SO_{2n+1} Rallis tower), it is then not surprising that the local Saito-Kurokawa packets can be constructed as local theta lifts of the local Waldspurger packet: one sets

$$\pi_{\tau_v}^{\epsilon_v} = \theta_{\psi_v}(\sigma_{\tau_v}^{\epsilon_v}).$$

These local theta lifts are nonzero because we are in the stable range. Likewise, the Saito-Kurokawa submodule $\mathcal{A}_{\psi_\tau}$ can be constructed as the global theta lift of the submodule $\tilde{\mathcal{A}}_\tau \subset \mathcal{A}_{cusp}(\mathrm{Mp}_2)$. This was first studied by Piatetski-Shapiro [28], but see [9] for a more refined discussion.

4.11. U_3**: Howe-PS example.** Now we carry out the same analysis for a family of nontempered A-parameters of $G = \mathrm{U}_3$ (relative to E/k) which will explain the Howe-PS example we discussed.

The Langlands dual group of $G = \mathrm{U}_3$ is $\mathrm{GL}_3(\mathbb{C})$, but we need to work with the L-group ${}^LG = \mathrm{GL}_3(\mathbb{C}) \rtimes \mathrm{Gal}(E/k)$. The A-parameters of G are then

$$\psi : L_k \times \mathrm{SL}_2(\mathbb{C}) \longrightarrow {}^LG.$$

Thankfully, by [11], the equivalence class of ψ is determined by the equivalence class of its restriction to L_E, so we can simply consider

$$\psi : L_E \longrightarrow G^\vee = \ GL_3(\mathbb{C}).$$

In other words, an A-parameter of $G = \mathrm{U}_3$ is simply a 3-dimensional semisimple representation of L_E. But this representations needs to satisfy an extra condition: it should be conjugate orthogonal. In addition, for it to be elliptic, ψ should be multiplicity-free.

Clearly, one has a subgroup

$$\mathrm{GL}_1(\mathbb{C}) \times \mathrm{GL}_2(\mathbb{C}) \subset \mathrm{GL}_3(\mathbb{C}).$$

We are going to build a discrete A-parameter

$$\psi : L_E \times \mathrm{SL}_2(\mathbb{C}) \longrightarrow GL_1(\mathbb{C}) \times \mathrm{GL}_2(\mathbb{C}) \subset \mathrm{GL}_3(\mathbb{C}),$$

so that $\psi(\mathrm{SL}_2(\mathbb{C})) = \mathrm{SL}_2(\mathbb{C}) \subset \mathrm{GL}_2(\mathbb{C})$. As a 3-dimensional representation, ψ takes the form

$$\psi = \mu \oplus \chi \otimes S_2$$

where S_2 denotes the 2-dimensional irreducible representation of $SL_2(\mathbb{C})$. The conjugate-orthogonal condition amounts to requiring that

- μ is a conjugate-orthogonal 1-dimensional character of L_E, which by our (basic hypothesis) corresponds to an automorphic character of $\mathbb{A}_E^\times$ trivial on $\mathbb{A}_k^\times$;
- χ is a conjugate -symplectic 1-dimensional character if L_E, which corresponds by our (basic hypothesis) to an automorphic character of $\mathbb{A}_E^\times$ whose restriction to $\mathbb{A}_k^\times$ is $\omega_{E/k}$.

Thus, such a ψ is specified by the pair (χ, μ) satisfying the above properties.

4.11.1. **Component groups and A-packets.** The global component groups of ψ is

$$S_\psi = \mu_2$$

and the local component groups are:

$$S_{\psi_v} = \begin{cases} \mu_2, & \text{if } v \text{ is inert in } E; \\ 1, & \text{if } v \text{ splits in } E. \end{cases}$$

So the local A-packets have the form

$$A_{\psi_v} = \begin{cases} \{\pi_v^+, \pi_v^-\}, \text{ if } v \text{ is inert in } E; \\ \{\pi_v^+\}, \text{ if } v \text{ splits in } E. \end{cases}$$

Moreover, for almost all inert places v, the representation π_v^+ is the unramified representation contained in the principal series representation

$$\mathrm{Ind}_B^G \chi | - |^{1/2} \otimes \tilde{\mu},$$

where $\tilde{\mu}$ is μ regarded as a character of E^1, via the standard isomorphism $E_v^\times / k_v^\times \cong E_v^1$. Observe that the global A-packet A_ψ has infinitely many elements in this case.

4.11.2. **Multiplicity formula.** To work out the multiplicity formula, we need to work out the quadratic character ϵ_ψ. A short and instructive computation shows that ϵ_ψ is the trivial character of $S_\psi = \mu_2$ if and only if

$$\epsilon_E(1/2, \chi\mu^{-1}) = 1.$$

So Arthur's conjecture predicts that

$$\mathcal{A}_{\psi_{\chi,\mu}} \cong \bigoplus_{\pi \in A_\psi : \epsilon(\pi) = \epsilon_E(1/2, \chi\mu^{-1})} \pi.$$

4.11.3. **Construction via theta lifts.** In view of Adam's conjecture above, we should expect that the theta correspondence for $\mathrm{U}_1 \times \mathrm{U}_3$ that we discussed in Lectures 2 and 3 can be used to construct the local A-packets and the submodule $\mathcal{A}_{\psi_{\chi,\mu}}$.

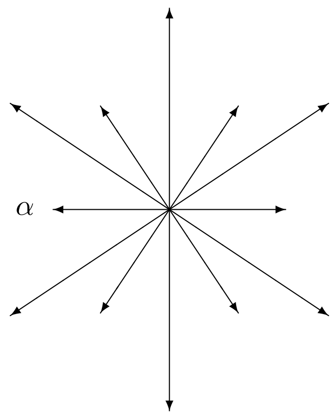
Let us fix our skew-Hermitian space W over k and consider an inert place v of k. Recall that there are two rank 1 Hermitian spaces V_v^+ and V_v^- over k_v for such an inert place v. Roughly speaking, the local A-packet $A_{\psi_{\chi_v, \mu_v}}$ should be constructed as the local theta lift of a particular character of $\mathrm{U}(V_v^+)$ and $\mathrm{U}(V_v^-)$, under the two theta correspondences for $\mathrm{U}(V_v^+) \times \mathrm{U}(W)$ and $\mathrm{U}(V_v^-) \times \mathrm{U}(W)$. To make this precise, we need to answer a few questions:

- Which pair of splitting characters (χ_V, χ_W) should we use for the theta correspondence?
- Having fixed (χ_V, χ_W), which character of $\mathrm{U}(V_v^+) = \mathrm{U}(V_v^-)$ should we start with?
- How should we label the two representations in the local A-packet, i.e. which of these two theta lifts is π_v^+, so as to achieve the predicted multiplicity formula?

Based on our understanding of the theta correspondence from Lectures 2 and 3, can you answer these questions? It will be a good exercise to reflect on them and I will leave these for discussion in the problem session.

4.12. **Example of G_2.** We conclude this section by describing 2 families of A-parameters of the split exceptional group G_2.

4.12.1. **Some structural facts.** The Langlands dual group of G is $G_2(\mathbb{C})$. We list a couple of relevant facts about the structure of $G_2(\mathbb{C})$, referring to its root system here for justification:



- The root system of G_2 contains a mutually orthogonal pair of long and short roots, giving rise to a commuting pair of SL_2's (as in the case of $\mathrm{Sp}_4(\mathbb{C})$)
$$(SL_{2,l} \times SL_{2,s})/\mu_2^{\Delta} \subset G_2.$$
The difference is that in the Sp_4 case, the two roots involved have the same length (they are both long), whereas here they are of different length. Hence, these two SL_2's are not conjugate to each other. Further, the centralizer of one of these SL_2's is the other SL_2.
- The 6 long roots of the G_2 root system gives an A_2 root system, reflecting the fact that $G_2(\mathbb{C})$ contains a subgroup $\mathrm{SL}_3(\mathbb{C})$. Observe that $\mathrm{SL}_{2,l}(\mathbb{C}) \times_{\mu_2} T \subset \mathrm{SL}_3(\mathbb{C})$ (where T is the diagonal torus of $\mathrm{SL}_{2,s}$) but $\mathrm{SL}_{2,s}$ is not contained in SL_3 (even after conjugation). Moreover, the normalizer of $\mathrm{SL}_3(\mathbb{C})$ in $G_2(\mathbb{C})$ contains $\mathrm{SL}_3(\mathbb{C})$ with index 2. Indeed, an element in the non-identity component is given by the longest Weyl group element of G_2. This is also the element $(w, w) \in \mathrm{SL}_{2,l} \times_{\mu_2} \mathrm{SL}_{2,s}$, where w is the standard Weyl element in SL_2. The conjugation action of this element on $\mathrm{SL}_3(\mathbb{C})$ is an outer automorphism. Hence one has containment
$$\mathrm{SL}_{2,l} \times_{\mu_2} N_{\mathrm{SL}_2}(T) \subset N_{G_2}(\mathrm{SL}_3) = \mathrm{SL}_3 \rtimes \mathbb{Z}/2\mathbb{Z} \subset G_2(\mathbb{C}).$$
- The smallest faithful (irreducible) algebraic representation of G_2 is 7-dimensional; in fact one has
$$G_2 \hookrightarrow \mathrm{SO}_7.$$
The weights of this 7-dimensional representation are the short roots and the zero vector.

  When restricted to the subgroup SL_3, this 7-dimensional representations decomposes as :
$$(std_3) \oplus 1 \oplus (std_3)^{\vee}$$
where (std_3) is a 3-dimensional irreducible representation of SL_3 and $(std_3)^{\vee}$ is its dual.

When restricted to the subgroup $\mathrm{SL}_{2,l} \times_{\mu_2} \mathrm{SL}_{2,s}$, it decomposes as:

$$(std_2) \otimes (std_2) \oplus 1 \otimes \mathrm{Sym}^2(std_2)$$

where (std_2) denotes the 2-dimensional representation of SL_2.

- consider the adjoint action of G_2 on its Lie algebra $\mathfrak{g}_2$. When restricted to the subgroup SL_3,
$$\mathfrak{g}_2 = \mathfrak{sl}_3 \oplus (std)_3 \oplus (std_3)^\vee.$$
When restricted to the subgroup $\mathrm{SL}_{2,l} \times_{\mu_2} \mathrm{SL}_{2,s}$, one has
$$\mathfrak{g}_2 = \mathfrak{sl}_{2,l} \oplus \mathfrak{sl}_{2,s} \oplus (std_2) \otimes \mathrm{Sym}^3(std_2).$$

You can find out more about the structure of G_2 (for example, its maximal parabolic subgroups) from Aaron Pollack's lecture notes.

4.12.2. **Some A-parameters.** Now suppose that τ is a cuspidal representation of PGL_2, which by our (basic hypothesis) corresponds to an irreducible representation

$$\rho_\tau : L_F \longrightarrow SL_2(\mathbb{C}).$$

Using ρ_τ, we can build 2 different nontempered A-parameters of G_2, depending on whether $\mathrm{SL}_2(\mathbb{C})$ is mapped to $\mathrm{SL}_{2,l}$ or $\mathrm{SL}_{2,s}$:

$$\psi_{\tau,s} : L_k \times \mathrm{SL}_2(\mathbb{C}) \longrightarrow \mathrm{SL}_{2,l} \times \mathrm{SL}_{2,s} \subset G_2(\mathbb{C})$$

or

$$\psi_{\tau,l} : L_k \times \mathrm{SL}_2(\mathbb{C}) \longrightarrow \mathrm{SL}_{2,s} \times \mathrm{SL}_{2,l} \subset G_2(\mathbb{C}).$$

We call $\psi_{\tau,s}$ the short root A-parameter and $\psi_{\tau,l}$ the long root A-parameter associated to τ.

4.12.3. **Short root A-parameters.** Now let's work out the consequences of Arthur's conjecture for the short root A-parameter. We have:

$$S_{\psi_{\tau,s}} \cong \mu_2$$

and for a place v of k,

$$S_{\psi_{\tau,s,v}} = \begin{cases} \mu_2, \text{ if } \tau_v \text{ is discrete series (i.e. } \rho_{\tau,v} \text{ is irreducible);} \\ 1 \text{ if } \tau_v \text{ is not discrete series (i.e. } \rho_{\tau,v} \text{ is reducible).} \end{cases}$$

4.12.4. **Local short root A-packets.** Now Arthur's conjecture predicts that for each place v, the local A-packets $A_{\tau,s,v}$ has the form:

$$A_{\tau,s,v} = \begin{cases} \{\pi^+_{\tau_v}\}, & \text{if } \tau_{\mathrm{v}} \text{ is not discrete series} \\ \{\pi^+_{\tau_v}, \pi^-_{\tau_v}\}, & \text{if } \tau_{\mathrm{v}} \text{ is discrete series.} \end{cases}$$

Here, $\pi^+_{\tau_v}$ is indexed by the trivial character of $S_{\tau,v}$. Moreover, we know what π^+_v has to be for almost all v. Indeed, for almost all v where τ_v is unramified, π^+_v is the unramified representation with Satake parameter $s_{\psi_{\tau,v}}$, and this representation is the unramified constituent of

$$I_P(\tau_v, 1/2) = \mathrm{Ind}_P^{G_2} \tau \cdot |\det|^{1/2}$$

where P is the Heisenberg parabolic subgroup of G_2 with Levi factor GL_2.

4.12.5. **Global short root A-packets.** Let S_τ be the set of places v where τ_v is discrete series, so that the global A-packet has $2^{\# S_\tau}$ elements. To describe the multiplicity of $\pi_\eta \in A_{\tau,s}$ in $L^2_{\psi_\tau}$, we need to know the quadratic character $\epsilon_{\psi_{\tau,s}}$ of $S_{\psi_{\tau,s}}$. It turns out that $\epsilon_{\psi_{\tau,s}}$ is the non-trivial character of $S_{\psi_\tau} \cong \mu_2$ if and only if $\epsilon(1/2,\tau) = -1$.

Now if $\pi = \otimes_v \pi_v^{\epsilon_v} \in A_{\tau,s}$, then the multiplicity associated to π by Arthur's conjecture is:

$$m(\pi) = \begin{cases} 1, & \text{if } \epsilon_\pi := \prod_v \epsilon_v = \epsilon(1/2,\tau); \\ 0, & \text{if } \epsilon_\pi = -\epsilon(1/2,\tau). \end{cases}$$

Thus, Arthur's conjecture predicts that:

$$\mathcal{A}_{\psi_{\tau,s}} \cong \bigoplus_{\pi \in A_\tau : \epsilon_\pi = \epsilon(1/2,\tau)} \pi.$$

4.12.6. **Construction of short root A-packets.** Observe that the structure of these A-packets is thus entirely the same as that of the Saito-Kurakawa packets for SO_5. Since the Saito-Kurokawa packets were constructed as theta liftings of Waldspurger's packets on Mp_2, one might guess that one can construct the short root A-packets of G_2 by lifting from the corresponding packets on Mp_2. But is $\mathrm{Mp}_2 \times G_2$ a reductive dual pair?

Well, it turns out that one may consider the dual pair

$$\mathrm{Mp}_2 \times \mathrm{SO}_7.$$

Recalling that $G_2 \hookrightarrow \mathrm{SO}_7$, we may consider theta lifts from Mp_2 to SO_7, followed by restriction of representations from SO_7 to G_2. Somewhat amazingly, this restriction does not lose much information. In other words, one may consider the commuting pair

$$\mathrm{Mp}_2 \times G_2$$

and restrict the Weil representation of $\mathrm{Mp}_2 \times \mathrm{SO}_7$ to it. Such a construction was first conceived by Rallis and Schiffman, but the full analysis was completed in [10]. In this way, it was shown in [10] that one may construct the A-packets and the corresponding submodule in $\mathcal{A}_2(G_2)$.

4.12.7. **Long root A-parameters.** The main project for this course is the analysis and construction of the long root A-packets of G_2. In particular, the first task of the project is to work out the prediction of Arthur's conjecture for the long root A-parameter $\psi_{\tau,l}$, and then specialize to the case when τ is dihedral.

We list the expected answers here, leaving it as a series of guided exercises:

- the global and local component groups are the same as for the short root A-parameters; so the local A-packets have 2 or 1 elements depending on whether τ_v is discrete series or not.
- the quadratic character $\epsilon_{\psi_{\tau,l}}$ is trivial if and only if

$$\epsilon(1/2,\tau,\mathrm{Sym}^3) = \epsilon(1/2,\mathrm{Sym}^3\rho_\tau) = 1.$$

  So we see that the Sym^3-epsilon factor appears in the Arthur multiplicity formula.

4.13. **Dihedral long root A-parameters.** We now suppose that τ is a dihedral cuspidal representation relative to a quadratic field extension E/k. This can be interpreted in one of the following equivalent ways:

- $\rho_\tau \cong \mathrm{Ind}_{W_E}^{W_k}\chi$ for some 1-dimensional character χ of the global Weil group W_E (which is supposedly a quotient of L_E).
- $\tau \otimes \omega_{E/k} \cong \tau$.

The fact that $\det \rho_\tau = 1$ implies that when regarded as an automorphic character of $\mathbb{A}_E^\times$, $\chi|_{\mathbb{A}_k^\times} = \omega_{E/k}$, i.e. χ is a conjugate-symplectic automorphic character. The image of ρ_τ is contained in the normalizer $N_{\mathrm{SL}_2}(T)$, where T is a maximal torus of SL_2. Now we observe:

- When τ is dihedral as above, the long root A-parameter $\psi_{\tau,l}$ factors as:
$$\psi_{\tau,l} : L_k \twoheadrightarrow W_k \longrightarrow N_{\mathrm{SL}_{2,s}}(T) \times_{\mu_2} \mathrm{SL}_{2,l}(\mathbb{C}) \subset \mathrm{SL}_3(\mathbb{C}) \rtimes \mathbb{Z}/2\mathbb{Z} \subset G_2(\mathbb{C}).$$
This follows from one of the structural facts we recall about $G_2(\mathbb{C})$.
- $\mathrm{SL}_3(\mathbb{C}) \rtimes \mathbb{Z}/2\mathbb{Z} \subset \mathrm{GL}_3(\mathbb{C}) \rtimes \mathbb{Z}/2\mathbb{Z} = {}^L\mathrm{U}_3$.
- Hence the long root A-parameter $\psi_{\tau,l}$ of G_2 factors through the L-group of U_3, thus giving rise to an A-parameter for U_3. Moreover, when restricted to W_E, one obtains a 3-dimensional representation of $W_E \times \mathrm{SL}_2(\mathbb{C})$ of the form
$$\psi_{\tau,l}|_{W_E} = \chi^{-2} \oplus \chi \otimes S_2.$$
In other words, one obtains a Howe-PS A-parameter for U_3.

Said in another way, one could start with a Howe-PS A-parameter ψ for U_3 with $\psi(L_E) \subset \mathrm{SL}_3(\mathbb{C})$ (or equivalently, giving rise to representations of PU_3). The composition of ψ with the natural inclusion
$$\mathrm{SL}_3(\mathbb{C}) \rtimes \mathbb{Z}/2\mathbb{Z} = N_{G_2}(\mathrm{SL}_3) \hookrightarrow G_2(\mathbb{C})$$
then gives a long root A-parameter whose associated τ is dihedral with respect to E/k.

This suggests the following question:

Question: Is it possible to construct the local and global long root A-packets of G_2 by lifting from the Howe-PS packets for U_3, and then verify the Arthur multiplicity formula?

Addressing this question will be the main project for this course.

5. **Bonus Lecture: Arthur's Conjecture for Classical Groups via Theta Lifts**

As we mentioned in Lecture 4, Arthur's conjecture has been established for quasi-split classical groups by Arthur [3] and Mok [25], using the stable (twisted) trace formula. By the same technique, one expects the conjecture for non-quasi-split groups can be similarly obtained, though not without some hard work. For non-quasi-split unitary groups, this has been carried out in a manuscript of Kaletha-Minguez-Shin-White [20], at least for the part of $\mathcal{A}_2$ corresponding to tempered A-parameters.

On the other hand, in this series of lectures, we have seen that the theta correspondence is a useful tool for constructing certain nontempered A-packets of a classical group, as theta lifts of tempered cusp forms on smaller groups. This is a natural procedure, in which one assumes inductively that one already understands the cusp forms on the smaller group, and then tries to propagate this understanding to larger groups. These constructions are also quite efficient and often carries with it other useful information of the constructed automorphic forms, such as their Fourier coefficients. However, in view of Adam's conjecture, one would not expect such construction to yield the whole of the nontempered discrete spectrum of classical groups. Indeed, Adams conjecture predicts that one would only obtain A-parameters on the bigger group of the following form:

$$(\text{A-parameters of smaller group}) \oplus \chi \otimes S_r$$

for some 1-dimensional character χ. Such A-parameters are of course far from being the most general. This is again not surprising, for there is no reason to expect that the automorphic discrete spectrum of a group can be totally understood in terms of the spectra of smaller groups. Indeed, one does not expect the study of the tempered discrete spectrum to be reducible to those of smaller groups: it should contain genuinely new spectral information.

In this bonus section, we will describe a series of recent work by Ichino and myself, as well as by two of my students, Rui Chen and Jialiang Zou, which (somewhat surprisingly) establishes Arthur's conjecture (at least the tempered part) for non-quasi-split classical groups, using the theta correspondence to transport one's knowledge from the quasi-split case established by Arthur and Mok. To be more precise, the approach was first conceived in [12] to show (partially) the Arthur conjecture for metaplectic groups $\mathrm{Mp}(W)$, before it was adapted to the case of non-quasi-split classical groups in the thesis work of Chen and Zou.

5.1. **The idea.**

We shall illustrate with the case of unitary groups. The main idea is simple: instead of lifting from smaller groups to a fixed (non-quasi-split) unitary group $\mathrm{U}(V)$, why don't we construct $\mathcal{A}_2(\mathrm{U}(V))$ by theta lifting from a larger unitary group $\mathrm{U}(W)$ which is quasi-split? More precisely, if you would like to construct and understand $\mathcal{A}_\psi(\mathrm{U}(V))$ for a particular A-parameter ψ, you can do so as follows:

- take a much larger unitary group $\mathrm{U}(W)$, so that $\mathrm{U}(V) \times \mathrm{U}(W)$ is in the stable range, and consider the A-parameter
$$\psi' = \chi_V \cdot (\chi_W^{-1}\psi \oplus S_r)$$
of $\mathrm{U}(W)$ (as suggested by Adam's conjecture). By the work of Arthur and Mok, one already "knows" the submodule $\mathcal{A}_{\psi'}(\mathrm{U}(W))$ in terms of local and global A-packets with the multiplicity formula.

- use global theta lifting to transport $\mathcal{A}_{\psi'}(\mathrm{U}(W))$ back to $\mathrm{U}(V)$; one should get the submodule $\mathcal{A}_{\psi}$.

5.2. **Low rank representations.** From a purely representation theoretic point of view, this idea is not so bad, because as we have noted during this course, theta correspondence is easier in the stable range. Indeed, there is no obstruction to the nonvanishing of local or global theta lifts from the smaller $\mathrm{U}(V)$ to the larger $\mathrm{U}(W)$. Moreover, in his thesis and subsequent work [22, 23], Jianshu Li has studied these theta lifts in stable range, showing that the lifting preserves unitarity of representations and also gives an exhaustive construction of the so-called low rank representations of $\mathrm{U}(W)$. In other words, the representations with A-parameter ψ' are low rank and have very rigid properties, as far as their "Fourier coefficients" go. Hence they are amenable to other ways of studying them, in addition to the viewpoint of A-packets, as we shall see later.

In fact, at the local level, Chen and Zou [5, 6] exploited this idea to establish the local Langlands correspondence for non-quasi-split unitary and even orthogonal groups by transporting the LLC for quasi-split groups.

5.3. **Some Issues.** There will however be some potential issues that one can readily point out:

- Since the A-parameter ψ' of $\mathrm{U}(W)$ is nontempered, the understanding of the associated A-packet coming from Arthur's work is far from complete. For example, Arthur does not know from his approach how many representations are in the local A-packets (remember that the representations π_η can be reducible or 0). If one hopes to construct $\mathcal{A}_\psi$ from $\mathcal{A}_{\psi'}$, this is an undesirable situation, since one cannot then hope to have a good understanding of $\mathcal{A}_\psi$ without first understanding the input $\mathcal{A}_{\psi'}$.

  Thankfully, this issue is somewhat alleviated by the independent work of Moeglin (as clarified by Bin Xu and Hiraku Atobe) who gave an independent and explicit construction local A-packets using Jacquet module techniques. In particular, through her work, one has a rather good understanding of $\mathcal{A}_{\psi'}$.

- By the tower property of theta correspondence, one does not expect many constituents of $\mathcal{A}_{\psi'}$ to be cuspidal; indeed, they should be noncuspidal when $\mathrm{U}(W)$ is sufficiently large. Given this, how can one perform the global theta lifting on them? Recall that we need the integral defining the gobal theta lifting to converge, and there may be no reason for this convergence when the input is noncuspidal.

  In the next subsection, we shall explain the novel idea that allows one to bypass this difficulty.

5.4. **Jianshu Li's Inequalities.** In the stable range, it turns out that one can study the global theta lifting from the L^2-point of view, exploiting the fact that the representations on the larger group has low rank. This innovative approach was carried out by Jianshu Li in his 1998 paper [23]. Let us recall his results here.

Theorem 5.1. *Assume that* $\mathrm{U}(V)\times\mathrm{U}(W)$ *is in the stable range with* $\mathrm{U}(V)$ *the smaller group. Suppose that one has a direct integral decomposition*

$$L^2([\mathrm{U}(V)]) = \int_{\widehat{\mathrm{U}(V)(\mathbb{A})}} m_\pi \cdot \pi \, d\mu(\pi)$$

for some measure $d\mu$ *on the unitary dual* $\widehat{\mathrm{U}(V)(\mathbb{A})}$ *of* $\mathrm{U}(V)(\mathbb{A})$*, and some multiplicity function* $m(-)$*. Then* $L^2([\mathrm{U}(W)])$ *contains as a submodule the direct integral*

$$\int_{\widehat{\mathrm{U}(V)(\mathbb{A})}} m_\pi \cdot \theta^{abs}(\pi) \, d\mu(\pi)$$

where $\theta^{abs}(\pi) = \otimes_v \theta(\pi_v)$ *is the (abstract) theta lift of* π *(which is nonzero since we are in the stable range).*

It is worth noting that the proof of this theorem does not involve the usual integral defining the global theta lifting, but involves harmonic analysis of the part of the L^2 spectrum of $\mathrm{U}(W)$ involving low rank representations. The properties of low rank representations thus play an indispensable role here.

As a corollary of this theorem and other results, we have the following inequalities:

Corollary 5.1. *For* $\pi \in \mathrm{Irr}(\mathrm{U}(V)(\mathbb{A}))$*, let*

$$m(\pi) = \dim \mathrm{Hom}_{\mathrm{U}(V)}(\pi, \mathcal{A}(\mathrm{U}(V))) \quad \textit{and} \quad m_2(\pi) = \dim \mathrm{Hom}_{\mathrm{U}(V)}(\pi, \mathcal{A}_2(\mathrm{U}(V))).$$

Then one has

$$m_2(\pi) \le m_2(\theta^{abs}(\pi)) \le m(\theta^{abs}(\pi)) \le m(\pi).$$

Here, the first inequality is a consequence of the theorem, the second inequality is obvious whereas the third follows by other considerations (of Fourier coefficients).

5.5. **Assigning A-parameters.** Let us illustrate how the corollary above is useful in attaching A-parameters to near equivalence classes. Suppsoe that $\mathcal{C} \subset \mathcal{A}_2(\mathrm{U}(V))$ is a nonzero near equivalence class, say

$$\mathcal{C} \cong \bigoplus_{i \in I} m(\pi_i)\pi_i.$$

By the corollary, we see that $\mathcal{A}_2(\mathrm{U}(W))$ contains as a submodule

$$\theta^{abs}(\mathcal{C}) := \bigoplus_{i \in I} m(\pi_i) \cdot \theta^{abs}(\pi_i).$$

Now all the summands in $\theta^{abs}(\mathcal{C})$ are nonzero and nearly equivalent with each other. Since $\mathrm{U}(W)$ is quasi-split, one knows by Arthur-Mok that $\theta^{abs}(\mathcal{C})$ is associated to an elliptic A-parameter ψ'. By using our knoweldge of the unramified theta correspondence and poles of standard L-functions (coming from he doubling zeta integrals), one can show that ψ' has the desired form

$$\psi' = \chi_V \cdot (\chi_W^{-1}\psi \oplus S_r)$$

for some elliptic A-parameter ψ for $\mathrm{U}(V)$. One then attaches ψ to $\mathcal{C}$.

In this way, Chen and Zou [7] showed:

Theorem 5.2. *For any (non-quasi-split) unitary group* $\mathrm{U}(V)$*, one has a decomposition*

$$\mathcal{A}_2(\mathrm{U}(V)) = \bigoplus_{\psi} \mathcal{A}_\psi$$

where ψ *runs over the elliptic A-parameters of* $\mathrm{U}(V)$ *and* $\mathcal{A}_\psi$ *is the associated near equivalence class.*

Another way to formulate this theorem is that it gives the weak Langlands functorial lifting from $\mathrm{U}(V)$ to $\mathrm{GL}(V)$, with image given by the expected description.

5.6. **Does equality hold?** It remains then to understand each near equivalence $\mathcal{A}_\psi$, and in particular to describe it in the language of local A-packets and the multiplicity formula. In some sense, one would like to transport the structure of $\mathcal{A}_{\psi'}$ back to $\mathcal{A}_\psi$. However, since we only have equalities in the above corollary, it means that one has

$$\theta^{abs}(\mathcal{A}_\psi) \subset \mathcal{A}_{\psi'}.$$

In other words, in transferring from $\mathrm{U}(W)$ back to $\mathrm{U}(V)$, we might lose some information. For example, it is possible that $\mathcal{A}_\psi = 0$ but $\mathcal{A}_{\psi'} \neq 0$.

Thus, we see that it is important to know when the equality $m_2(\pi) = m_2(\theta^{abs}(\pi))$ holds. For this, one has:

Proposition 5.2. *Equality holds in Jianshu Li's inequalities in the following cases:*

- ψ *is a tempered A-parameter;*
- *The Witt index of* V *is* 0 *or* 1*.*

In these cases, what was shown is that

$$m_2(\pi) = m(\pi),$$

so that equalities hold througout Jianshu Li's inequalities. So for these cases, it is reasonable to expect that the structure of $\mathcal{A}_\psi$ can be faithfully inherited from that of $\mathcal{A}_{\psi'}$.

5.7. **Results of Chen-Zou.** In their paper [7], Chen and Zou showed the following theorem:

Theorem 5.3. *(i) The submodule* $\mathcal{A}_\psi$ *can be described as in Arthur's conjecture for any tempered* ψ*.*

(ii) Arthur's conjecture holds for $\mathrm{U}(V)$ *if* $\mathrm{U}(V)$ *has k-rank* ≤ 1*.*

To go beyond this, it seems one needs to answer the following question:

Question: Is it the case that one always has:

$$m_2(\pi) = m_2(\theta^{abs}(\pi)) = m(\theta^{abs}(\pi)) = m(\pi)?$$

This is undoubtedly an interesting question to ponder.

6. **The Project**

We consider a long root A-parameter $\psi_{\tau,l}$ of G_2 from the end of Lecture 4, with $\tau = \tau(\chi)$ a cuspidal representation of PGL_2 which is dihedral with respect to E/k and associated with a conjugate-symplectic automorphic character χ of $E^\times$. We have observed that $\psi_{\tau,l}$ factors through the L-group of PU_3, giving an A-parameter of PU_3 of Howe-PS type. More precisely, if we consider a Howe-PS A-parameter

$$\psi_\chi : W_k \times \mathrm{SL}_2(\mathbb{C}) \longrightarrow {}^L PU_3 = \mathrm{SL}_3(\mathbb{C}) \rtimes \mathbb{Z}/2\mathbb{Z}$$

with

$$\psi_\chi|_{W_E \times \mathrm{SL}_2(\mathbb{C})} = \chi^{-2} \oplus \chi \cdot S_2,$$

then we obtain a long root A-parameter of G_2 by composition:

$$\psi_{\tau(\chi),l} : W_k \times \mathrm{SL}_2(\mathbb{C}) \longrightarrow \mathrm{SL}_3(\mathbb{C}) \rtimes \mathbb{Z}/2\mathbb{Z} \hookrightarrow G_2(\mathbb{C}).$$

This suggests the following question:

Question: Is it possible to construct the associated long root A-packet of G_2 by lifting from the corresponding Howe-PS packets for PU_3, and then verify the Arthur multiplicity formula?

It turns out that we have a dual reductive pair

$$\mathrm{PU}_3 \times G_2 \subset E_6^{E/k}$$

where $E_6^{E/k}$ is a quasi-split exceptional group of type E_6. One may consider the minimal representation of $E_6^{E/k}$ and restrict it to this dual pair, obtaining a local theta correspondence. In fact, one should consider the disconnected dual pair

$$(\mathrm{PU}_3 \rtimes \mathbb{Z}/2\mathbb{Z}) \times G_2 \subset E_6^{E/k} \rtimes \mathbb{Z}/2\mathbb{Z}$$

to obtain a nice local theta correspondence (in the classical case, the analog is that one should consider $\mathrm{O}(V) \times \mathrm{Sp}(W)$ instead of $\mathrm{SO}(V) \times \mathrm{Sp}(W)$).

The study of this local theta correspondence (for the disconnected dual pair) was recently carried out by Gan-Savin [14] at split p-adic places and by Bakić-Savin [4] at the inert p-adic places. In particular, they proved the Howe duality theorem (on the finite length of $\Theta(\pi)$ and the irreducibility of $\theta(\pi)$). Using their results, one can make a definition of the local A-packets as local theta lifts of the corresponding Howe-PS packet (extended to $\mathrm{PU}_3 \rtimes \mathbb{Z}/2\mathbb{Z}$), and understand quite precisely the representations obtained, at least in the p-adic case.

What about the archimedean case?

Mission: Your mission, should you choose to accept it, is to study the global theta correspondence (addressing nonvanishing and cuspidality) and establish the desired Arthur multiplicity formula.

No worries: this is not exactly Mission Impossible (as you can see, this document did not self-destruct after 5 seconds).

There are a few preliminary groundwork that one can do to familiarize oneself with this project:

- understand some structure theory of G_2: parabolic subgroups and their unipotent radical and their internal modules, some natural subgroups like SL_3 and SO_4. One can learn a lot about G_2 by studying Aaron Pollack's notes for this AWS.
- work out in detail the consequences of Arthur's conjecture for long and short root A-parameters of G_2, for example the computation of the quadratic character ϵ_ψ. While the answers have been given in Lecture 4, the details were left as exercises there.
- work out and understand in detail the construction of Howe-PS A-packets vis theta lifts from U_1. This was also left as a challenging exercise in Lecture 4. Which members of the Howe-PS packets are cuspidal?
- classify the cuspidal representations of $\mathrm{PU}_3 \rtimes \mathbb{Z}/2\mathbb{Z}$ in terms of those of PU_3. This is necessary because we will be working with the disconnected dual pair. In the local setting, the paper of Bakić-Savin [4] addresses the same question on the level of abstract irreducible representations.
- understand the local results of Bakić-Savin [4]; the proofs should largely not be needed.
- figure out what is known for the theta correspondence of $\mathrm{PU}_3 \times G_2$ in the archimedean case. The paper of Huang-Pandzic-Savin [19] resolves this in the case when PU_3 is compact (so not quasi-split), obtaining quaternionic discrete series representations of G_2 as theta lifts. This is another point of contact with Aaron Pollack's lectures. If one is willing to work with the Howe-PS A-packets on anisotropic U_3, then one can use this exceptional theta lifting to construct quaternionic modular forms.
- understand the following phenomenon: the local Howe-PS A-packets have two elements for half the places of k (namely the inert ones), but the corresponding dihedral long root A-packets has two elements only for finitely many v's. How does one reconcile this difference, especially in view of our hope to construct the latter form the former?

For the project itself, the following are some problems to resolve:

(i) Show the cuspidality (or not) of global theta lifts.

(ii) Show the nonvanishing of global theta lifts by computing Fourier coefficients. These Fourier coefficients are the same ones which appear in Pollack's lectures. This computation should reduce to the torus period on PU_3,

(iii) The result from (ii) leads one to consider questions like:

- how to classify maximal tori in U_3?
- which torus periods are supported by the Howe-PS representations?

This is a mini-project which could stand on its own, and has nothing to do with G_2. It involves understanding the Howe-PS A-packets and their construction as theta lifts from U_1 really well. To address the torus period problem involves using an argument with see-saw dual pairs.

(iv) Verify the AMF: how does the Sym^3 epsilon factor show up?

(v) Bonus: can one construct quaternionic modular forms (in the sense of Pollack's lectures) by this exceptional theta lifting, and compute the Fourier coefficients of the theta lifts?

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