RIGIDITY METHOD FOR AUTOMORPHIC FORMS OVER FUNCTION FIELDS

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Contents

These lectures will give an introduction to the rigidity method for constructing automorphic forms for semisimple groups over function fields. The rigidity method leads to explicit constructions of local systems that are Langlands parameters of rigid automorphic forms. The examples of local systems produced in this way have applications to algebraic geometry and number theory.

We will emphasize on the "engineering" aspect of the theory, leaving out most of the proofs: we will give principles on how to design rigid automorphic data and methods for computing the resulting local systems. For more details and proofs, we refer to [36].

1. Automorphic forms over function fields

In this section we recall some basic concepts for automorphic forms over a function field.

1.1. The setting.

1.1.1. *Function field.* Let k be a finite field. Let X be a projective, smooth and geometrically connected curve X over k. Let $F = k(X)$ be the field of rational functions on X.

Let $|X|$ be the set of closed points of X (places of F). For each $x \in |X|$, let F_x denote the completion of F at the place x. The valuation ring and residue field of F_x are denoted by $\mathcal{O}_x$ and k_x. The maximal ideal of $\mathcal{O}_x$ is denoted by $\mathfrak{m}_x$. The ring of adèles is the restricted product

$$\mathbb{A}_F := \prod_{x\in|X|}{}' F_x$$

where for almost all x, the x-component of an element $a \in \mathbb{A}_F$ lies in $\mathcal{O}_x$. There is a natural topology on $\mathbb{A}_F$ making it a locally compact topological ring.

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1.1.2. *Groups.* For simplicity of the presentation, we restrict ourselves to the following situation:

G is a connected split reductive group over k;

In addition, when talking about automorphic forms, G is assumed to be semisimple.

We will fix a split maximal torus T and a Borel subgroup $B \supset T$. They give a based root system

$$(\mathbb{X}^*(T), \Phi(G,T), \Delta_B, \mathbb{X}_*(T), \Phi^\vee(G,T), \Delta_B^\vee)$$

where Δ_B are the simple roots and $\Delta_B^\vee$ the simple coroots. It makes sense to talk about $G(R)$ whenever R is a k-algebra. For example, $G(F)$, $G(\mathcal{O}_x)$ and $G(F_x)$, etc.

The Langlands dual group $\widehat{G}$ to G is a connected reductive group over $\overline{\mathbb{Q}}_\ell$, equipped with a maximal torus $\widehat{T}$ and a Borel subgroup $\widehat{B} \supset \widehat{T}$, such that the corresponding based root system

$$(\mathbb{X}^*(\widehat{T}), \Phi(\widehat{G}, \widehat{T}), \Delta_{\widehat{B}}, \mathbb{X}_*(\widehat{T}), \Phi^\vee(\widehat{G}, \widehat{T}), \Delta_{\widehat{B}}^\vee)$$

is identified with the based root system $(\mathbb{X}_*(T), \Phi^\vee(G,T), \Delta_B^\vee, \mathbb{X}^*(T), \Phi(G,T), \Delta_B)$ obtained from that of G by switching roots and coroots.

1.1.3. *Adelic and level groups.* The group $G(\mathbb{A}_F)$ of $\mathbb{A}_F$-points of G can also be expressed as the restricted product

$$G(\mathbb{A}_F) = {\prod_{x \in |X|}}' G(F_x)$$

where most components lie in $G(\mathcal{O}_x)$. This is a locally compact topological group. The diagonally embedded $G(F)$ inside $G(\mathbb{A}_F)$ is a discrete subgroup. The quotient $G(F)\backslash G(\mathbb{A}_F)$ is a locally compact space.

Let $K_x \subset G(F_x)$ be a compact open subgroup, one for each $x \in |X|$, such that for almost all x, $K_x = G(\mathcal{O}_x)$. Let $K = \prod_x K_x$ be the compact open subgroup of $G(\mathbb{A}_F)$. The double coset space $G(F)\backslash G(\mathbb{A}_F)/K$ has discrete topology.

1.1.4. *Automorphic forms.* Let $\mathcal{A}_K = C(G(F)\backslash G(\mathbb{A}_F)/K, \overline{\mathbb{Q}}_\ell)$ be the vector space of $\overline{\mathbb{Q}}_\ell$-valued functions on $G(\mathbb{A}_F)$ that are left invariant under $G(F)$ and right invariant under K. Let $\mathcal{A}_{K,c} = C_c(G(F)\backslash G(\mathbb{A}_F)/K, \overline{\mathbb{Q}}_\ell) \subset \mathcal{A}_K$ be the subspace that are supported on finitely many double cosets.

Elements in $\mathcal{A}_{K,c}$ will be our working definition of (compactly supported) *automorphic forms for $G(\mathbb{A}_F)$ with level K*. For more precise definition, we refer to [3, Definition 5.8].

1.1.5. *Action of the Hecke algebra.* Let $\mathcal{H}_K$ be the vector space of $\overline{\mathbb{Q}}_\ell$-valued, compactly supported functions on $G(\mathbb{A}_F)$ that are left and right invariant under K. Similarly one can define the local Hecke algebra $\mathcal{H}_{K_x}$. The vector space $\mathcal{H}_K$ (resp. $\mathcal{H}_{K_x}$) has an algebra structure under convolution, such that the characteristic function of $\mathbf{1}_K$ (resp. $\mathbf{1}_{K_x}$) is the unit element. Then $\mathcal{H}_K$ is the restricted tensor product of the local Hecke algebras $\mathcal{H}_{K_x}$ with respect to their algebra units. The algebra $\mathcal{H}_K$ acts on $\mathcal{A}_K$ and $\mathcal{A}_{K,c}$ by the rule:

$$(f \star h)(x) = \sum_{g \in G(\mathbb{A}_F)/K} f(xgK)h(Kg^{-1}K), \quad \forall f \in \mathcal{A}_K, h \in \mathcal{H}_K.$$

1.1.6. **Exercise.** Give a formula for the algebra structure of $\mathcal{H}_K$.

1.1.7. *Cusp forms.* As our working definition, a *cusp form* for $G(\mathbb{A}_F)$ with level K is an element $f \in \mathcal{A}_{K,c}$ that generates a finite-dimensional $\mathcal{H}_K$-submodule. The official definition of a cusp form uses the vanishing of constant terms and looks quite different. It is shown in [24,] that these two notions are equivalent.

A cusp form $f \in \mathcal{A}_{K,c}$ is called a *Hecke eigenform* if it is an eigenfunction for $\mathcal{H}_{K_x}$, for almost all $x \in |X|$.

1.2. **Weil's interpretation.** We allow G to be reductive in this subsection. Let $K^\natural = \prod_x G(\mathcal{O}_x)$ in the following discussion.

1.2.1. *Case of* GL_n. When $G = \mathrm{GL}_n$, Weil has given a geometric interpretation of the double coset $G(F)\backslash G(\mathbb{A}_F)/K^\natural$ in terms of vector bundles of rank n over X. Let $\mathrm{Vec}_n(X)$ be the groupoid of rank n vector bundles over X. Let us give the map $e : G(\mathbb{A}_F) \to \mathrm{Vec}_n(X)$. First, for any finite subset $S \subset |X|$, we shall define a map $e_S : \prod_{x\in S} G(F_x) \to \mathrm{Vec}_n(X)$ as follows. Let $j : X - S \hookrightarrow X$ be the open inclusion. Let $(g_x)_{x\in S} \in \prod_{x\in S} G(F_x)$ and let $\Lambda_x \subset F_x^{\oplus n}$ be the $\mathcal{O}_x$-submodule $g_x\mathcal{O}_x^{\oplus n}$. Define $e_S((g_x)_{x\in S})$ to be the quasi-coherent subsheaf $\mathcal{V} \subset j_*\mathcal{O}_X^{\oplus n}$ such that, for any affine open $U \subset X$,

$$\Gamma(U, \mathcal{V}) = \Gamma(U - S, \mathcal{O}_X^{\oplus n}) \cap (\prod_{x\in|U|\cap S} \Lambda_x).$$

Here the intersection is taken inside $\prod_{x\in|U|\cap S} F_x^{\oplus n}$, and $\Gamma(U - S, \mathcal{O}_X^{\oplus n})$ is diagonally embedded into it by taking completion at each $x \in |U| \cap S$.

1.2.2. **Exercise.** (1) Show that $\mathcal{V}$ constructed above is a vector bundle of rank n.
(2) Suppose $S \subset S'$, $g_x \in G(F_x)$ for each $x \in S$, and $g_x \in G(\mathcal{O}_x)$ for each $x \in S' - S$. Let $g_S = (g_x)_{x\in S}$ and $g_{S'} = (g_x)_{x\in S'}$. Then there is a canonical isomorphism $e_S(g_S) = e_{S'}(g_{S'})$. Using this to show the e_S for various S give a well-defined map $e : G(\mathbb{A}_F) \to \mathrm{Vec}_n(X)$.
(3) Show that e is left invariant under $G(F)$ and right invariant under $K^\natural$.
(4) Show that e is an equivalence of groupoids.

1.2.3. *General* G. For general G, there is a similar interpretation of $G(F)\backslash G(\mathbb{A}_F)/K^\natural$ in terms of G-torsors over X. Recall that a (right) G-torsor over X is a scheme $Y \to X$ together with a fiberwise action of G that looks like $G \times X$ (with G acting on itself by right translation) étale locally over X. An isomorphism between G-torsors Y and Y' is a G-equivariant isomorphism $Y \cong Y'$ over X.

1.2.4. **Exercise.** When $G = \mathrm{GL}_n$, show that there is an equivalence of categories between G-torsors over X and vector bundles of rank n on X. You will need descent theory to show that étale local triviality is the same as Zariski local triviality for sheaves of $\mathcal{O}_X$-modules.

Similarly, $\mathrm{Bun}_{\mathrm{SL}_n}$ classifies pairs $(\mathcal{V}, \iota)$ where $\mathcal{V}$ is a vector bundle of rank n over X and $\iota : \wedge^n\mathcal{V} \xrightarrow{\sim} \mathcal{O}_X$ is a trivialization of the determinant of $\mathcal{V}$.

1.2.5. **Exercise.** Show that PGL_n-torsors over X are the same as rank n vector bundles over X modulo tensoring with line bundles. For this, you will need the fact that the Brauer group of X is trivial.

1.2.6. **Exercise.** For $G = \mathrm{Sp}_{2n}$ and SO_n, give an interpretation of G-torsors using vector bundles with bilinear forms.

Let $\mathrm{Bun}_G(k)$ be the groupoid of G-torsors over X: this is a category whose objects are G-torsors over X and morphisms are isomorphisms between G-torsors. The groupoid $\mathrm{Bun}_G(k)$ is in fact the groupoid of k-points of an algebraic stack Bun_G. For a k-algebra R the groupoid $\mathrm{Bun}_G(R)$ is the groupoid of G-torsors over $X \otimes_k R$.

Weil observed that there is a natural equivalence of groupoids

$$e : G(F)\backslash G(\mathbb{A}_F)/K^\natural \xrightarrow{\sim} \mathrm{Bun}_G(k). \tag{1.1}$$

So this is not just a bijection of sets, but for any double coset $[g] = G(F)gK^\natural$, the automorphism group of $e([g])$ (as a G-torsor) is isomorphic to $G(F) \cap gK^\natural g^{-1}$.

The construction of e is similar to the case of $G = \mathrm{GL}_n$, using modification of the trivial G-torsor at finitely many points S given by $(g_x)_{x\in S} \in \prod_{x\in S} G(F_x)$.

1.2.7. *Birkhoff decomposition.* We consider the case $X = \mathbb{P}^1$. Grothendieck proves that every vector bundle on $\mathbb{P}^1$ is isomorphic to a direct sum of line bundles $\oplus_i \mathcal{O}(n_i)$, and the multiset $\{n_i\}$ is well-defined. This implies that the underlying set of $\mathrm{Bun}_{\mathrm{GL}_n}(k)$ is in bijection with the $\mathbb{Z}^n/S_n$.

More generally, there is a canonical bijection of sets

$$|\mathrm{Bun}_G(k)| \cong |\mathbb{X}_*(T)/W|.$$

By the bijection (1.1), we see that $|G(F)\backslash G(\mathbb{A}_F)/K^\natural|$ is in bijection with $\mathbb{X}_*(T)/W$. We can construct the bijection as follows.

1.2.8. **Exercise.** Let t be an affine coordinate of $\mathbb{A}^1 \subset \mathbb{P}^1$.

(1) Let $G = T$ and $\lambda \in \mathbb{X}_*(T)$. Viewing λ as a homomorphism $\lambda : \mathbb{G}_m \to T$, and $t \in F_0$ (local field at $0 \in |\mathbb{P}^1|$), let $t^\lambda := \lambda(t) \in T(F_0)$. Now view t^λ as an element in $T(\mathbb{A}_F)$ that is t^λ at the place 0 and 1 elsewhere. The assignment $\lambda \to t^\lambda$ defines a homomorphism $\mathbb{X}_*(T) \to T(\mathbb{A}_F)$. Show that it induces a bijection $\mathbb{X}_*(T) \xrightarrow{\sim} |T(F)\backslash T(\mathbb{A}_F)/K_T^\natural|$.

(2) For general G, the above construction gives a map $\mathbb{X}_*(T) \to |T(F)\backslash T(\mathbb{A}_F)/K_T^\natural| \to |G(F)\backslash G(\mathbb{A}_F)/K^\natural|$. Show that this map is W-invariant, and it induces a bijection between the set of W-orbits on $\mathbb{X}_*(T)$ and the underlying set of $G(F)\backslash G(\mathbb{A}_F)/K^\natural$.

1.2.9. *Interpretation of Hecke operators as modifications.* By Weil's interpretation, $\mathcal{A}_{K^\natural,c}$ can be identified with the space of $\overline{\mathbb{Q}}_\ell$-valued points on the set of isomorphisms classes of G-torsors over X, i.e.,

$$\mathcal{A}_{K^\natural} = C(\mathrm{Bun}_G(k), \overline{\mathbb{Q}}_\ell).$$

Fix $x \in |X|$ and let $K_x = G(\mathcal{O}_x)$. The algebra $\mathcal{H}_{K_x}$ is called the *spherical Hecke algebra* for the group $G(F_x)$. It has a $\overline{\mathbb{Q}}_\ell$-basis consisting of characteristic functions $\mathbf{1}_{K_x g K_x}$ of double cosets. We know that $\mathcal{H}_{K_x}$ acts on $\mathcal{A}_{K^\natural}$. We give an interpretation of the action of $\mathbf{1}_{K_x g K_x}$ on $\mathcal{A}_{K^\natural} = C(\mathrm{Bun}_G(k), \overline{\mathbb{Q}}_\ell)$ in terms of G-torsors.

Take the example $G = \mathrm{GL}_n$ and

$$g_{i,x} = \mathrm{diag}(\underbrace{t_x, \cdots, t_x}_{i}, \underbrace{1, \cdots, 1}_{n-i})$$

(t_x is a uniformizer at x). For two vector bundles $\mathcal{E}$ and $\mathcal{E}'$ of rank n on X, we use the notation $\mathcal{E} \xrightarrow[x]{i} \mathcal{E}'$ to mean that $\mathcal{E} \subset \mathcal{E}' \subset \mathcal{E}(x)$ and $\dim_{k_x}(\mathcal{E}'/\mathcal{E}) = i$. Then for any $f \in \mathcal{A}_{K^\natural}$, viewed as a function on $\mathrm{Vec}_n(X) \cong \mathrm{Bun}_{\mathrm{GL}_n}(k)$, we have

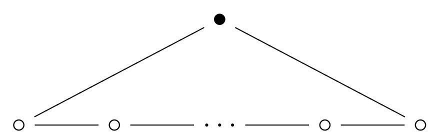
$$(f \star \mathbf{1}_{K_x g_{i,x} K_x})(\mathcal{E}) = \sum_{\mathcal{E} \xrightarrow[x]{i} \mathcal{E}'} f(\mathcal{E}').$$

1.3. **Level structures.**

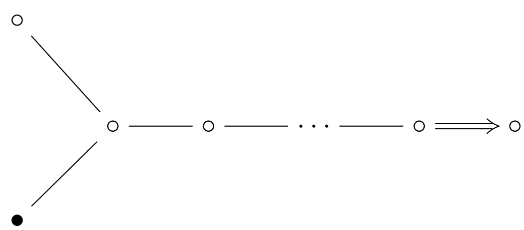
1.3.1. *Parahoric subgroups.* Let $x \in |X|$. Let $I \subset G(\mathcal{O}_x)$ be the preimage of a Borel subgroup $B(k_x) \subset G(k_x)$ under the reduction map $G(\mathcal{O}_x) \to G(k_x)$. This is an example of an Iwahori subgroup of $G(F_x)$. General Iwahori subgroups are conjugates of I in $G(F_x)$. Parahoric subgroups of $G(F_x)$ always contain an Iwahori subgroup with finite index. A precise definition of parahoric subgroups involves a fair amount of Bruhat-Tits theory [5]. The conjugacy classes of parahoric subgroups under $G^{\mathrm{sc}}(F_x)$ (G^{sc} is the simply-connected cover of G) are in bijection with proper subsets of the vertices of the extended Dynkin diagram $\widetilde{\mathrm{Dyn}}(G)$ of G. Empty subset corresponds to Iwahori subgroups. The set of vertices of the finite Dynkin diagram corresponds to conjugates of $G(\mathcal{O}_x)$.

Extended Dynkin diagram of G is obtained from the Dynkin diagram of G by adding one vertex corresponding to the affine simple root α_0, which we mark as a black dot:

$\widetilde{\mathrm{Dyn}}(A_n)$

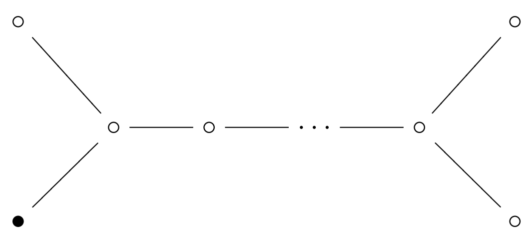


$\widetilde{\mathrm{Dyn}}(B_n)$

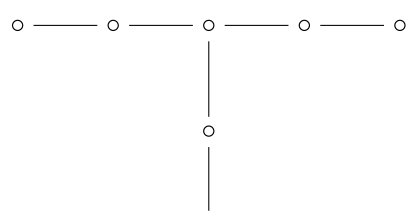


$\widetilde{\mathrm{Dyn}}(C_n)$

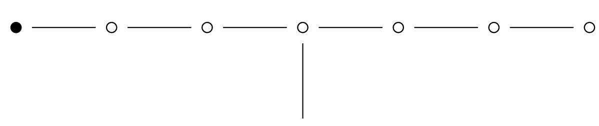
$\widetilde{\mathrm{Dyn}}(D_n)$



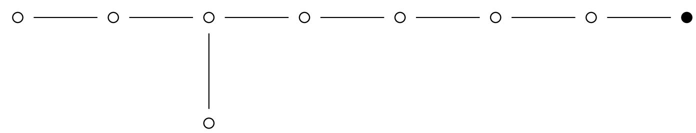
$\widetilde{\mathrm{Dyn}}(E_6)$



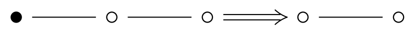
$\widetilde{\mathrm{Dyn}}(E_7)$



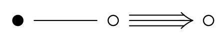
$\widetilde{\mathrm{Dyn}}(E_8)$



$\widetilde{\mathrm{Dyn}}(F_4)$



$\widetilde{\mathrm{Dyn}}(G_2)$



1.3.2. **Example.** For $G = \mathrm{SL}_n$, we can understand parahoric subgroups using chains of lattices. A lattice in F_x^n is an $\mathcal{O}_x$-submodule of rank n. The relative dimension of a lattice Λ is defined to be

$$\dim[\Lambda : \mathcal{O}_x^n] = \ell_x(\Lambda/\Lambda \cap \mathcal{O}_x^n) - \ell_x(\mathcal{O}_x^n/\Lambda \cap \mathcal{O}_x^n)$$

where $\ell_x(-)$ is the length of a $\mathcal{O}_x$-module (since we are in the function field case, $\ell_x(-) = \dim_{k_x}(-)$). A *periodic lattice chain* is a chain of lattices $\Lambda_\bullet = \{\Lambda_i\}_{i\in I}$ in F_x^n indexed by a non-empty subset of $I \subset \mathbb{Z}$ such that

- If $i < j$ then $\Lambda_i \subset \Lambda_j$;
- The relative dimension of Λ_i is i.
- I is stable under translation by $n\mathbb{Z}$, and $\Lambda_{i-n} = \mathfrak{m}_x\Lambda_i$.

To a periodic lattice chain $\Lambda_\bullet$, its simultaneous stabilizer

$$P_{\Lambda_\bullet} = \{g \in \mathrm{SL}_n(F_x) | g\Lambda_i = \Lambda_i, \forall i \in I\}$$

is a parahoric subgroup of $\mathrm{SL}_n(F_x)$. Conversely, every parahoric subgroup of $\mathrm{SL}_n(F_x)$ arises this way for a unique periodic lattice chain $\Lambda_\bullet$.

A periodic lattice chain is *complete* if $I = \mathbb{Z}$. Iwahori subgroups of $\mathrm{SL}_n(F_x)$ are precisely the stabilizers of complete periodic lattice chains.

1.3.3. **Exercise.** Assume $\mathrm{char}(k) \neq 2$. Use lattice chains to describe parahoric subgroups of $G(F_x)$ when $G = \mathrm{Sp}(V)$ or $\mathrm{SO}(V)$, where V is a vector space over F_x equipped with a symplectic form or symmetric bilinear form $\langle \cdot, \cdot \rangle$. For a lattice $\Lambda \subset V$, denote $\Lambda^\vee = \{v \in V | \langle v, \Lambda\rangle \subset \mathcal{O}_x\}$, another lattice in V.

(1) Let $G = \mathrm{Sp}(V)$. Let $\Lambda \subset V$ be a lattice such that $\mathfrak{m}_x\Lambda^\vee = \Lambda$. Then its stabilizer P_Λ is a parahoric subgroup of $G(F_x)$ whose conjugacy class corresponds to the complement of the right end of $\widetilde{\mathrm{Dyn}}(G)$. Show that $\Lambda^\vee/\Lambda$ as a k_x-vector space carries a canonical symmetric bilinear form, and give a canonical surjection $P_\Lambda \to \mathrm{SO}(\Lambda^\vee/\Lambda)$.

(2) The same construction "works" for $\mathrm{SO}(V)$ when $\dim V = 2n$. But which subdiagram of $\widetilde{\mathrm{Dyn}}(G)$ does P_Λ correspond?

(3) Let $G = \mathrm{SO}(V)$ and $\dim V = 2n + 1$. Let $\Lambda \subset V$ be a lattice such that $\mathfrak{m}_x\Lambda^\vee \subset \Lambda$ with quotient of length 1. Then its stabilizer P_Λ is a parahoric subgroup corresponding to the complement of the right end of $\widetilde{\mathrm{Dyn}}(G)$. Show that $\Lambda^\vee/\Lambda$ as a k_x-vector space carries a canonical symmetric bilinear form, and give a canonical surjection $P_\Lambda \to \mathrm{SO}(\Lambda^\vee/\Lambda)$.

1.3.4. *Loop groups.* Later we shall also need to view $G(F_x)$ and $G(\mathcal{O}_x)$ as infinite-dimensional groups over k. More precisely, for a k_x-algebra R we let $L_xG(R) = G(R\widehat{\otimes}_{k_x}F_x)$ and let $L_x^+G(R) = G(R\widehat{\otimes}_{k_x}\mathcal{O}_x)$. Here $R\widehat{\otimes}_{k_x}F_x$ or $R\widehat{\otimes}_{k_x}\mathcal{O}_x$ means the completion of the tensor product with respect to the $\mathfrak{m}_x$-adic topology on F_x or $\mathcal{O}_x$. If we choose a uniformizer t_x of F_x, then $L_xG(R) = G(R(\!(t_x)\!))$ and $L_x^+G = G([\![t_x]\!])$. The functor L_xG (resp. L_x^+G) is representable by a group indscheme (resp. group scheme of infinite type) over k_x. There is a reduction map $L_x^+G \to G\otimes_k k_x$.

More generally, any parahoric subgroup of $G(F_x)$ can be equipped with the structure of a pro-algebraic group (an inverse limit of algebraic groups). Bruhat and Tits [5] constructed certain smooth models $\mathcal{P}$ of G over $\mathcal{O}_x$ called Bruhat-Tits group schemes. The parahoric subgroups are exactly of the form $\mathcal{P}(\mathcal{O}_x)$ for Bruhat-Tits group schemes $\mathcal{P}$. To such $\mathcal{P}$ we can attach a pro-algebraic subgroup $\mathbf{P} \subset L_x G$ whose R-points is $\mathcal{P}(R\widehat{\otimes}_{k_x}\mathcal{O}_x)$. Then k_x-points $\mathbf{P}(k_x)$ of $\mathbf{P}$ is a parahoric subgroup of $G(F_x)$.

Viewed as a pro-algebraic group, the parahoric subgroup $\mathbf{P} \subset L_x G$ has a maximal reductive quotient $L_{\mathbf{P}}$ which is a connected reductive group over k_x. We denote $\mathbf{P}^+ = \ker(\mathbf{P} \to L_{\mathbf{P}})$ its pro-unipotent radical.

If the conjugacy class of $\mathbf{P}$ corresponds to the subset J of the vertices the extended Dynkin diagram $\widetilde{\mathrm{Dyn}}(G)$ of G, then the Dynkin diagram of $L_{\mathbf{P}}$ is given by the subdiagram $\widetilde{\mathrm{Dyn}}(G)$ spanned by J.

For example, the preimage $\mathbf{I}_x$ of $B \otimes_k k_x$ under the reduction map $L_x^+ G \to G \otimes_k k_x$ is called the *standard Iwahori subgroup* of $L_x G$ (with respect to B), and any parahoric subgroup of $L_x G$ contains a conjugate of $\mathbf{I}_x$.

1.3.5. Bun_G *with level structures.* One can generalize Bun_G to G-torsors with level structures. Fix a finite set $S \subset |X|$, and for each $x \in S$ let $\mathbf{K}_x \subset L_x G$ be a proalgebraic subgroup commensurable with $L_x^+ G$. Then we may talk about G-torsors over X with $\mathbf{K}_S$-level structures: these are G-torsors $\mathcal{E}$ over X together with trivializations $\iota_x : \mathcal{E}|_{\mathrm{Spec}\mathcal{O}_x} \cong G|_{\mathrm{Spec}\mathcal{O}_x}$ (for each $x \in S$) up to left multiplication by $\mathbf{K}_x$ (via the intuitive action if $\mathbf{K}_x \subset L_x^+ G$, but it requires some thought to define the action in general). We shall denote the corresponding moduli stack by $\mathrm{Bun}_G(\mathbf{K}_S)$.

Let $K_x = \mathbf{K}_x(k_x) \subset G(F_x)$ be the corresponding compact open subgroup. Then the isomorphism (1.1) generalizes to an equivalence of groupoids

$$G(F)\backslash G(\mathbb{A}_F)/(\prod_{x \notin S} G(\mathcal{O}_x) \times \prod_{x \in S} K_x) \xrightarrow{\sim} \mathrm{Bun}_G(\mathbf{K}_S)(k). \tag{1.2}$$

1.3.6. **Exercise.** Let $G = \mathrm{GL}_n$ and let $\mathbf{K}_x$ be an Iwahori subgroup of $L_x G$ for each $x \in S$. Give an interpretation of $\mathrm{Bun}_G(\mathbf{K}_S)(k)$ in terms of vector bundles on X with extra structure along S.

2. Rigid automorphic data

We will define the notion of an automorphic datum and its geometric version. We will upgrade the notion of automorphic functions to automorphic sheaves, and define what it means for such a geometric automorphic datum to be rigid.

2.1. Automorphic data.

Let $Z \subset G$ be the center of G. This is a finite group scheme over k.

2.1.1. **Definition.** Let $S \subset |X|$ be finite. An *automorphic datum* for G unramified outside S is a collection $\{(K_x, \chi_x)\}_{x \in S}$ where

- K_x is a compact open subgroup of $G(F_x)$;
- $\chi_x : K_x \to \overline{\mathbb{Q}}_\ell^\times$ is a finite order character.

We denote the automorphic datum by (K_S, χ_S).

2.1.2. **Definition.** Let $\omega : Z(F)\backslash Z(\mathbb{A}_F) \to \overline{\mathbb{Q}}_\ell^\times$ be a central character. and (K_S, χ_S) be an automorphic datum. We say ω is *compatible with* (K_S, χ_S) if $\omega|_{Z(F_x)} = 1$ for $x \notin S$ and $\omega|_{Z(F_x)\cap K_x} = \chi_x|_{Z(F_x)\cap K_x}$ for $x \in S$.

2.1.3. **Definition.** Let (K_S, χ_S) be an automorphic datum. A $\overline{\mathbb{Q}}_\ell$-valued function f on $G(F)\backslash G(\mathbb{A}_F)$ is called a (K_S, χ_S)-*typical automorphic form* if

(1) For every $x \in S$ and $h_x \in K_x$, $f(gh_x) = \chi_x(h_x)f(g)$ for all $g \in G(F)\backslash G(\mathbb{A}_F)$.
(2) For every $x \notin S$, f is right $G(\mathcal{O}_x)$-invariant.

Clearly, if f is a nonzero (K_S, χ_S)-typical automorphic form that is eigen under the center $Z(\mathbb{A}_F)$ with central character ω, then ω has to be compatible with (K_S, χ_S).

Let $\mathcal{A}(K_S, \chi_S)$ be the $\overline{\mathbb{Q}}_\ell$-vector space of (K_S, χ_S)-typical automorphic forms. Similarly we have the space $\mathcal{A}_c(K_S, \chi_S)$ of compactly supported (K_S, χ_S)-typical automorphic forms.

Let $K_x^+ = \ker(\chi_x)$ for $x \in S$. If we define

$$K^+ = \prod_{x\in S} K_x^+ \times \prod_{x\notin S} G(\mathcal{O}_x),$$

then we have $\mathcal{A}(K_S, \chi_S) \subset \mathcal{A}_{K^+}$ and $\mathcal{A}_c(K_S, \chi_S) \subset \mathcal{A}_{K^+,c}$.

2.1.4. **Example.** Let $X = \mathbb{P}^1$, $G = \mathrm{SL}_2$ and $S = \{0, 1, \infty\}$. For $x \in S$, we let $K_x = I_x$ be the standard Iwahori subgroup of $G(F_x)$, i.e.,

$$I_x = \left\{ \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in G(\mathcal{O}_x) \middle| c \in \mathfrak{m}_x \right\}.$$

For each $x \in S$, we choose a character $\chi_x : k^\times \to \overline{\mathbb{Q}}_\ell^\times$ and view it as a character of I_x by sending the above matrix to $\chi_x(\overline{a})$, where $\overline{a} \in k^\times$ is the image of $a \in \mathcal{O}_x^\times$.

A central character compatible with (K_S, χ_S) exists if and only if

$$\prod_{x\in S} \chi_x(-1) = 1, \tag{2.1}$$

in which case it is unique: take $\omega_x = \chi_x|_{Z(F_x)}$ for $x \in S$ and trivial otherwise.

2.1.5. **Example** (Kloosterman automorphic datum)**.** In [18], an automorphic datum for $X = \mathbb{P}^1$ is introduced for any split reductive group and some quasi-split ones, which provided the first examples of rigid automorphic data. Below let $S = \{0, \infty\}$ and assume G to be semisimple and split.

At 0 we take K_0 to be an Iwahori subgroup with quotient torus identified with T. Let $\chi_0 : T(k) \to \overline{\mathbb{Q}}_\ell^\times$ be a character, and view it as a character of K_0 via $K_0 \to T(k)$.

At ∞, let $\tau = t^{-1}$, a uniformizer of $F_\infty \cong k(\!(\tau)\!)$. Let $I_\infty \subset G(\mathcal{O}_\infty)$ be the Iwahori subgroup that is the preimage of $B(k)$ under the reduction map $G(\mathcal{O}_\infty) \to G(k)$. Again we can have a natural map $I_\infty \to B(k) \to T(k)$; let I_∞^+ be its kernel. We take $K_\infty = I_\infty^+$.

Let $\Delta_B = \{\alpha_1, \cdots, \alpha_r\}$ be simple roots of G with respect to B and T. Let $U_{\alpha_i} \subset G$ be the root subgroups, each isomorphic to $\mathbb{G}_a$. Let $-\theta \in \Phi(G, T)$ be the lowest root, and $U_{-\theta}$ be the root subgroup. Let U_{α_0} be the additive group whose k-points are $U_{-\theta}(\tau F_\infty)/U_{-\theta}(\tau F_\infty)$. There is a canonical surjective map

$$\mathrm{pr} : K_\infty = I_\infty^+ \twoheadrightarrow \prod_{i=0}^{r} U_{\alpha_i}(k). \tag{2.2}$$

We denote the kernel of this map by I_∞^{++}.

To define χ_∞, pick an isomorphism $U_{\alpha_i} \cong \mathbb{G}_a$, and take the sum of these maps to give a homomorphism

$$\varphi : \prod_{i=0}^{r} U_{\alpha_i}(k) \to \mathbb{G}_a(k) = k.$$

Finally, composing with a nontrivial additive character ψ of k to get a character

$$\chi_\infty : K_\infty \xrightarrow{\mathrm{pr}} \prod_{i=0}^{r} U_{\alpha_i}(k) \xrightarrow{\varphi} k \xrightarrow{\psi} \overline{\mathbb{Q}}_\ell^\times.$$

For example, when $G = \mathrm{SL}_2$, we have

$$I_\infty^+ = \left\{ \begin{pmatrix} a & b \\ c & d \end{pmatrix} \middle| a, d \in 1 + \tau\mathcal{O}_\infty, b \in \mathcal{O}_\infty, c \in \tau\mathcal{O}_\infty \right\}$$

and, after identifying U_{α_0} and U_{α_1} with $\mathbb{G}_a$ in an obvious way, the map pr maps the above matrix to $(b \mod \tau, c/\tau \mod \tau) \in k^2$. The automorphic datum (K_S, χ_S) is called the *Kloosterman automorphic datum*.

There is a unique central character ω compatible with (K_S, χ_S): $\omega_0 = \chi_0|_{Z(k)}$ and $\omega_\infty = \chi_0^{-1}|_{Z(k)}$ (we identify $Z(F_0)$ and $Z(F_\infty)$ with $Z(k)$).

The datum (K_∞, χ_∞) picks up those representations of $G(F_\infty)$ that contain nonzero eigenvectors under I_∞^+ on which I_∞^+ acts through the character $\psi \circ \phi$. These representations were first discovered by Gross and Reeder [16, §9.3], and are called *simple supercuspidal representations*. When G is simply-connected, any such representation is given by compact induction $\mathrm{ind}_{Z(k)I_\infty^+}^{G(F_\infty)}(\omega_\infty \boxtimes \psi \circ \phi)$ for some character $\omega_\infty : Z(k) \to \overline{\mathbb{Q}}_\ell^\times$.

2.2. **A naive definition of rigidity.** We would like to call an automorphic datum (K_S, χ_S) *rigid* if $\dim \mathcal{A}_c(K_S, \chi_S) = 1$, i.e., compactly supported (K_S, χ_S)-typical automorphic forms are unique up to a scalar. For example, the automorphic datum in Example 2.1.4 turns out to be rigid in this sense.

However, there are several issues with this definition:

- When G is not simply-connected, Bun_G has several connected components, and it is more natural to require (K_S, χ_S)-typical automorphic forms to be unique up to scalar on each connected component. With this modification, the Kloosterman automorphic data in Example 2.1.5 are rigid.
- It is more natural to fix a central character.

But these are only technical issues. The more serious issue is the following:

- The space $\mathcal{A}_c(K_S, \chi_S)$ may be small simply because the field k is small. It would be more natural to ask that the dimension of $\mathcal{A}_c(K_S, \chi_S)$ be "independent of k".

What does it mean to vary k while keeping the automorphic datum (K_S, χ_S)? To make senses of it, we need to reformulate automorphic datum in geometric terms.

2.3. **Sheaf-to-function correspondence.**

2.3.1. *The dictionary.* Let X be a scheme of finite type over a finite field k and let $\mathcal{F}$ be a constructible complex of $\overline{\mathbb{Q}}_\ell$-sheaves for the étale topology of X. For each closed point $x \in X$, the geometric Frobenius element Frob_x at x acts on the geometric stalk $\mathcal{F}_{\overline{x}}$, which is a complex of $\overline{\mathbb{Q}}_\ell$-vector spaces. We consider the function

$$\begin{array}{rcl} f_{\mathcal{F},k} : X(k) & \to & \overline{\mathbb{Q}}_\ell \\ x & \mapsto & \displaystyle\sum_{i \in \mathbb{Z}} (-1)^i \mathrm{Tr}(\mathrm{Frob}_x, \mathrm{H}^i \mathcal{F}_{\overline{x}}) \end{array}$$

Similarly we can define a function $f_{\mathcal{F},k'} : X(k') \to \overline{\mathbb{Q}}_\ell$ for any finite extension k' of k. The correspondence

$$\mathcal{F} \mapsto \{f_{\mathcal{F},k'}\}_{k'/k}$$

is called the *sheaf-to-function* correspondence. This construction enjoys various functorial properties. For a morphism $\phi : X \to Y$ over k, the derived push forward $f_!$ transforms into integration of functions along the fibers (this is a consequence of the Lefschetz trace formula for the Frobenius endomorphism); the derived pullback f^* transforms into pullback of functions. It also transforms tensor product of sheaves into pointwise multiplication of functions.

2.3.2. **Definition.** Let L be a connected algebraic group over k with the multiplication map $m : L \times L \to L$ and the identity point $e : \mathrm{Spec}(k) \to L$. A *rank one character sheaf* $\mathcal{K}$ on L is a local system of rank one on L equipped with two isomorphisms

$$\mu : m^*\mathcal{K} \xrightarrow{\sim} \mathcal{K} \boxtimes \mathcal{K},$$
$$u : \overline{\mathbb{Q}}_\ell \xrightarrow{\sim} e^*\mathcal{K}.$$

These isomorphisms should be compatible in the sense that

$$\mu|_{L\times\{e\}} = \mathrm{id}_{\mathcal{K}} \otimes u : \mathcal{K} = \overline{\mathbb{Q}}_\ell \otimes_{\overline{\mathbb{Q}}_\ell} \mathcal{K} \xrightarrow{\sim} e^*\mathcal{K} \otimes \mathcal{K},$$
$$\mu|_{\{e\}\times L} = u \otimes \mathrm{id}_{\mathcal{K}} : \mathcal{K} = \mathcal{K} \otimes_{\overline{\mathbb{Q}}_\ell} \overline{\mathbb{Q}}_\ell \xrightarrow{\sim} \mathcal{K} \otimes e^*\mathcal{K}.$$

Since L is connected, the isomorphism μ in Definition 2.3.2 automatically satisfies the usual cocycle relation on L^3. A local system $\mathcal{K}$ of rank one on L being a character sheaf is a property rather than extra structure on $\mathcal{K}$. Let $\mathrm{CS}_1(L)$ be the category (groupoid) of rank one character sheaves $(\mathcal{K}, \mu, u)$ on L, then it carries a symmetric monoidal structure given by the tensor product of character sheaves with the unit object given by the constant sheaf. The groupoid $\mathrm{CS}_1(L)$ has trivial automorphisms, therefore it is equivalence to its underlying set. Tensor product equips $\mathrm{CS}_1(L)$ with the structure of an abelian group.

If $f : L \to L'$ is a homomorphism between connected algebraic groups, then pullback along f induces a group homomorphism $f^* : \mathrm{CS}_1(L') \to \mathrm{CS}_1(L)$.

One can define the group $\mathrm{CS}_1(L)$ of rank one character sheaves for L a connected proalgebraic group. In fact, write L as the inverse limit of finite-dimensional quotients L_i, and define $\mathrm{CS}_1(L)$ as the direct limit of $\mathrm{CS}_1(L_i)$.

We can similarly define the notion of rank one character sheaves over $\overline{k}$ and form the group $\mathrm{CS}_1(L/\overline{k})$. The base change map $\mathrm{CS}_1(L) \to \mathrm{CS}_1(L/\overline{k})$ is injective, and the image consists of $\mathrm{Gal}(\overline{k}/k)$-invariants.

For each $\mathcal{K} \in \mathrm{CS}_1(L)$, the sheaf-to-function correspondence gives a function $f_{\mathcal{K}} : L(k) \to \overline{\mathbb{Q}}_\ell^\times$ which is in fact a group homomorphism because of the isomorphism μ. This way we obtain a homomorphism

$$f_L : \mathrm{CS}_1(L) \to \mathrm{Hom}(L(k), \overline{\mathbb{Q}}_\ell^\times). \tag{2.3}$$

One can show that f_L is always injective. The following result gives descriptions of $\mathrm{CS}_1(L)$ in various special cases.

2.3.3. **Theorem** ([36, Appendix A])**.** (1) *Let L be a connected commutative algebraic group over k. Then f_L is an isomorphism of abelian groups*

$$f_L : \mathrm{CS}_1(L) \xrightarrow{\sim} \mathrm{Hom}(L(k), \overline{\mathbb{Q}}_\ell^\times).$$

(2) *Let L be a connected reductive group over k and $L^{\mathrm{sc}} \to L$ be the simply-connected cover of its derived group. Then f_L induces an isomorphism of abelian groups*

$$\mathrm{CS}_1(L) \xrightarrow{\sim} \mathrm{Hom}(L(k)/L^{\mathrm{sc}}(k), \overline{\mathbb{Q}}_\ell^\times).$$

Let T be a maximal torus in L and $T^{\mathrm{sc}} \subset L^{\mathrm{sc}}$ be its preimage in L^{sc}. Then we also have

$$\mathrm{CS}_1(L) \xrightarrow{\sim} \mathrm{Hom}(T(k)/T^{\mathrm{sc}}(k), \overline{\mathbb{Q}}_\ell^\times).$$

The construction of an inverse to f_L uses the Lang map $L \to L$ sending $g \mapsto F_L(g)g^{-1}$, where F_L is the Frobenius endomorphism of L (relative to k).

2.3.4. **Example.** When $L = \mathbb{G}_m$ is the multiplicative group, local systems in $\mathrm{CS}_1(\mathbb{G}_m)$ are called *Kummer sheaves*. They are in bijection with characters $\chi : k^\times \to \overline{\mathbb{Q}}_\ell^\times$. Let $[q-1] : \mathbb{G}_m \to \mathbb{G}_m$ be the $(q-1)^{\mathrm{th}}$ power map, which is the Lang map for $\mathbb{G}_m$. Then the Kummer sheaf associated with the character χ is $\mathcal{K}_\chi := ([q-1]_!\overline{\mathbb{Q}}_\ell)_\chi$, the direct summand of $[q-1]_!\overline{\mathbb{Q}}_\ell$ on which $\mathbb{G}_m(k) = k^\times$ acts through χ.

More generally, when L is a torus, any object in $\mathrm{CS}_1(L)$ is obtained as a direct summand of $[n]_!\overline{\mathbb{Q}}_\ell$ where $[n] : T \to T$ is the n-th power morphism and n is prime to $\mathrm{char}(k)$. If L is a split torus, it suffices to take $n = q-1$.

2.3.5. **Example.** When $L = \mathbb{G}_a$ is the additive group, local systems in $\mathrm{CS}_1(\mathbb{G}_a)$ are called *Artin-Schreier sheaves*. They are in bijection with characters $\psi : k \to \overline{\mathbb{Q}}_\ell^\times$. Let $\lambda_{\mathbb{G}_a} : \mathbb{G}_a \to \mathbb{G}_a$ be the $\mathbb{G}_a(k)$-torsor given by $a \mapsto a^q - a$. Then the Artin-Schreier sheaves associated with ψ is $\mathrm{AS}_\psi := (\lambda_{\mathbb{G}_a,!}\overline{\mathbb{Q}}_\ell)_\psi$, the direct summand of $\lambda_{\mathbb{G}_a,!}\overline{\mathbb{Q}}_\ell$ on which $\mathbb{G}_a(k) = k$ acts through ψ.

More generally, let $L = V$ be a vector space over k viewed as a vector group. Fix a nontrivial characters $\psi : k \to \overline{\mathbb{Q}}_\ell^\times$. Then objects in $\mathrm{CS}_1(V)$ are of the form $\mathrm{AS}_\phi := \phi^*\mathrm{AS}_\psi$ for a unique covector $\phi \in V^*$, viewed as a homomorphism $\phi : V \to \mathbb{G}_a$.

2.3.6. **Exercise.** Let L be a reductive group over k such as GL_n, PGL_n or SO_n. Describe elements in $\mathrm{CS}_1(L)$ using coverings of L.

2.4. **Geometric automorphic data.** We resume with the setting in §1.1. Also, for notational simplicity, in the rest of the notes, we assume

$$S \subset |X| \text{ is a finite set consisting of } k\text{-rational points.} \tag{2.4}$$

2.4.1. **Definition.** A pair $(\mathbf{K}_S, \mathcal{K}_S)$ is a *geometric automorphic datum* with respect to S if

(1) $\mathbf{K}_S$ is a collection $\{\mathbf{K}_x\}_{x\in S}$, where $\mathbf{K}_x \subset L_xG$ is a connected proalgebraic subgroup contained in some parahoric subgroup.
(2) $\mathcal{K}_S$ is a collection $\{\mathcal{K}_x\}_{x\in S}$ where $\mathcal{K}_x \in \mathrm{CS}_1(\mathbf{K}_x)$. (i.e., $\mathcal{K}_x$ is the pullback of a rank one character sheaf from a finite-dimensional quotient of $\mathbf{K}_x$.)

2.4.2. **Remark.** (1) It is possible to include a geometric central character into the definition of a geometric automorphic datum, but we omit it here. For example, in case $\mathbf{K}_x$ contains Z for all $s \in S$, one datum to add is an trivialization of the character sheaf

$$\bigotimes_{x\in S} \mathcal{K}_x|_Z \in \mathrm{CS}_1(Z).$$

This is a geometric analogue of a compatibility condition in Definition 2.1.2, but it is extra datum rather than a condition.

(2) A geometric automorphic datum $(\mathbf{K}_S, \mathcal{K}_S)$ gives rise to a (partial) automorphic datum (K_S, χ_S) where $K_x = \mathbf{K}_x(k)$, and χ_x is the character of K_x determined by $\mathcal{K}_x$ under the sheaf-to-function correspondence (see (2.3)).

Since k is a finite field, the character sheaf $\mathcal{K}_x$ is uniquely determined by its associated character $\chi_x : K_x = \mathbf{K}_x(k) \to \overline{\mathbb{Q}}_\ell^\times$. Therefore we also call the pair $(\mathbf{K}_S, \chi_S)$ a geometric automorphic datum, provided χ_x does arise from an object in $\mathrm{CS}_1(\mathbf{K}_x)$ for all $x \in S$.

Typical examples of $\mathbf{K}_x$ are the *Moy-Prasad groups*.

2.4.3. *Base change of geometric automorphic data.* Let k'/k be a finite extension. Let S' be the preimage of S in $X \otimes_k k'$ (by (2.4), S' also consists of k'-rational points). Given a geometric automorphic datum $(\mathbf{K}_S, \mathcal{K}_S)$, we may define a corresponding geometric automorphic datum $(\mathbf{K}_{S'}, \chi_{S'})$ for G and the function field $F' = F \otimes_k k'$. For each $y \in S'$ with image $x \in S$, let $\mathbf{K}_y = \mathbf{K}_x \otimes_k k'$, and let $\mathcal{K}_y \in \mathrm{CS}_1(\mathbf{K}_y)$ be the pullback of $\mathcal{K}_x$ along the projection $\mathbf{K}_y \to \mathbf{K}_x$. We have the space $\mathcal{A}_c(k'; K_{S'}, \chi_{S'})$ of compactly supported $(K_{S'}, \chi_{S'})$-typical automorphic forms defined for the situation $X', G \otimes_k k', K_{S'}$ and $\chi_{S'}$ (the last two come from $\mathbf{K}_{S'}$ and $\mathcal{K}_{S'}$).

2.4.4. **Definition.** A geometric automorphic datum $(\mathbf{K}_S, \mathcal{K}_S)$ is called *weakly rigid*, if there is a constant N such that for *every* finite extension k'/k,

$$\dim_{\overline{\mathbb{Q}}_\ell} \mathcal{A}_c(k'; K_{S'}, \chi_{S'}) \leq N,$$

and, for some finite extension k'/k, $\mathcal{A}_c(k'; K_{S'}, \chi_{S'}) \neq 0$.

2.5. **Automorphic sheaves.** Suggested by Weil's interpretation in §1.2 and the sheaf-to-function correspondence §2.3, we shall seek to upgrade the function space $\mathcal{A}_c(K_S, \chi_S)$ into a category of sheaves on the moduli stack of G-torsors over X with level structures. This idea was due to Drinfeld who worked out the case $G = \mathrm{GL}_2$, and for general G it was formulated as the geometric Langlands correspondence by Laumon [25] and Beilinson–Drinfeld [2].

2.5.1. *The category of automorphic sheaves.* Consider the situation in §2.4. Let $\mathbf{K}_x^+ \subset \mathbf{K}_x$ be a connected normal subgroup of finite codimension such that the rank one character sheaf $\mathcal{K}_x$ on $\mathbf{K}_x$ is pulled back from the quotient $L_x = \mathbf{K}_x/\mathbf{K}_x^+$. Let $\mathrm{Bun}_G(\mathbf{K}_S)$ and $\mathrm{Bun}_G(\mathbf{K}_S^+)$ be the moduli stack of G-torsors over X with the respective level structures, as defined in §1.3.5. The morphism $\mathrm{Bun}_G(\mathbf{K}_S^+) \to \mathrm{Bun}_G(\mathbf{K}_S)$ is an $L_S := \prod_{x\in S} L_x$-torsor. The tensor product $\mathcal{K}_S := \boxtimes_{x\in S}\mathcal{K}_x$ is an object in $\mathrm{CS}_1(L_S)$. It makes sense to talk about $\overline{\mathbb{Q}}_\ell$-complexes of sheaves on $\mathrm{Bun}_G(\mathbf{K}_S^+)$ which are $(L_S, \mathcal{K}_S)$-equivariant.

Without giving the detailed definition, we have the full subcategory

$$D_c(\mathbf{K}_S, \mathcal{K}_S) \subset D^b_{(L_S, \mathcal{K}_S)}(\mathrm{Bun}_G(\mathbf{K}_S^+), \overline{\mathbb{Q}}_\ell)$$

consisting of bounded constructible $\overline{\mathbb{Q}}_\ell$-complexes on $\mathrm{Bun}_G(\mathbf{K}_S^+)$ equipped with $(L_S, \mathcal{K}_S)$-equivariant structures that are supported on finite-type substacks of $\mathrm{Bun}_G(\mathbf{K}_S^+)$. Objects in the category $D_c(\mathbf{K}_S, \mathcal{K}_S)$ are called $(\mathbf{K}_S, \mathcal{K}_S)$-*typical automorphic sheaves* with respect to the geometric automorphic datum $(\mathbf{K}_S, \mathcal{K}_S)$.

Let

$$K := \prod_{x\notin S} G(\mathcal{O}_x) \times \prod_{x\in S} K_x;$$
$$K^+ := \prod_{x\notin S} G(\mathcal{O}_x) \times \prod_{x\in S} K_x^+$$

According to (1.2), we have an $L_S(k)$-equivariant equivalence of groupoids

$$\mathrm{Bun}_G(\mathbf{K}_S^+)(k) \cong G(F)\backslash G(\mathbb{A}_F)/K^+,$$

that allows us to identify $(L_S(k), \chi_S)$-eigenfunctions on both sides. Therefore the sheaf-to-function correspondence gives a map

$$\mathrm{Ob}\ D_c(\mathbf{K}_S, \mathcal{K}_S) \to \mathcal{A}_c(K_S, \chi_S).$$

Similarly, for the base-changed situation to k'/k including $k' = \overline{k}$, we get the category $D_c(k'; \mathbf{K}_{S'}, \chi_{S'})$ for the base changed situation and a map

$$\mathrm{Ob}\ D_c(k'; \mathbf{K}_{S'}, \mathcal{K}_{S'}) \to \mathcal{A}_c(k'; K_{S'}, \chi_{S'}).$$

2.5.2. *Relevant points.* Consider a point $\mathcal{E} \in \mathrm{Bun}_G(\mathbf{K}_S)(\overline{k})$, which represents a G-torsor over $X_{\overline{k}}$ with $\mathbf{K}_x$-level structures at $x \in S$. The automorphism group $\mathrm{Aut}(\mathcal{E})$ of the point $\mathcal{E}$ is an affine algebraic group over $\overline{k}$. For each $x \in S$, restricting an automorphism of $\mathcal{E}$ gives an element in $\mathbf{K}_x \otimes_k \overline{k}$ (this depends on the choice a trivialization of $\mathcal{E}$ around x), and thus a homomorphism

$$\mathrm{ev}_{S,\mathcal{E}} : \mathrm{Aut}(\mathcal{E}) \to \prod_{x\in S} \mathbf{K}_x \otimes \overline{k} \to \prod_{x \in S} L_x \otimes \overline{k} =: L_S \otimes \overline{k} \tag{2.5}$$

which is well-defined up to conjugacy if one changes the trivialization of $\mathcal{E}$ around x.

2.5.3. **Definition.** Let $(\mathbf{K}_S, \mathcal{K}_S)$ be a geometric automorphic datum. A point $\mathcal{E} \in \mathrm{Bun}_G(\mathbf{K}_S)(\overline{k})$ is called *relevant to* $(\mathbf{K}_S, \mathcal{K}_S)$ if the restriction of $\mathrm{ev}^*_{S,\mathcal{E}}\mathcal{K}_S$ to the neutral component $\mathrm{Aut}(\mathcal{E})^\circ$ of $\mathrm{Aut}(\mathcal{E})$ is trivial as an element in $\mathrm{CS}_1(\mathrm{Aut}(\mathcal{E})^\circ)$. Otherwise the point $\mathcal{E}$ is called *irrelevant*.

We have a similar characterization of relevant points in terms of automorphic sheaves.

2.5.4. **Lemma.** *Let $\mathcal{E} \in \mathrm{Bun}_G(\mathbf{K}_S)(\overline{k})$. Then $\mathcal{E}$ is relevant for the geometric automorphic datum $(\mathbf{K}_S, \mathcal{K}_S)$ if and only if there exists $\mathcal{F} \in D_c(\mathbf{K}_S, \mathcal{K}_S)$ such that the geometric stalk of $\mathcal{F}$ at $\mathcal{E}$ is nonzero. The same is true when "stalk" is replaced by "costalk".*

Proof sketch. Let $A = \mathrm{Aut}(\mathcal{E})$ and $i_{\mathcal{E}} : \mathrm{pt}/A \hookrightarrow \mathrm{Bun}_G(\mathbf{K}_S)$ be the embedding. Let $\mathcal{L} := \mathrm{ev}^*_{S,\mathcal{E}}\mathcal{K}_S \in \mathrm{CS}_1(A)$. Then $i^*_{\mathcal{E}}\mathcal{F}$ is an $(A, \mathcal{L})$-equivariant sheaf on a point. If $\mathcal{L}|_{A^\circ}$ is nontrivial, such a sheaf must be zero. If $\mathcal{L}|_{A^\circ}$ is trivial, such a sheaf can be understood using twisted representations of $\pi_0(A)$, see §4.5.1. □

2.5.5. **Definition.** A geometric automorphic datum $(\mathbf{K}_S, \mathcal{K}_S)$ is called *geometrically rigid*, if for every connected component of $\mathrm{Bun}_G(\mathbf{K}^+_S)$, there is exactly one relevant $\overline{k}$-point.

Now we explain the relation between geometric rigidity and weakly rigidity.

The following lemma gives another characterization of relevant points in terms of automorphic functions.

2.5.6. **Lemma.** *Let $[g] \in G(F)\backslash G(\mathbb{A}_F)/K = \mathrm{Bun}_G(\mathbf{K}_S)(k)$, viewed as a $\overline{k}$-point of $\mathrm{Bun}_G(\mathbf{K}_S)$.*

1. *If $[g]$ is irrelevant, then any $f \in \mathcal{A}_c(K_S, \chi_S)$ must vanish on the double coset $G(F)gK$. Similar statement holds when k is replaced by a finite extension k'.*
2. *If $[g]$ is relevant, then for some finite extension k'/k, there exists a nonzero $f \in \mathcal{A}_c(k'; K_{S'}, \chi_{S'})$ supported on the double coset preimages of $G(F')gK'$ (K' is defined as K for the base changed situation).*

From the lemma we get:

2.5.7. **Theorem.** *Let $(\mathbf{K}_S, \mathcal{K}_S)$ be a geometric automorphic datum.*

1. *The geometric automorphic datum $(\mathbf{K}_S, \mathcal{K}_S)$ is weakly rigid if and only if there is a finite nonzero number of relevant $\overline{k}$-point on $\mathrm{Bun}_G(\mathbf{K}_S)$.*
2. *In particular, if $(\mathbf{K}_S, \mathcal{K}_S)$ is geomertically rigid, then it is weakly rigid.*
3. *If $(\mathbf{K}_S, \chi_S)$ is weakly rigid, then for any finite extension k'/k, any $(K'_S, \chi_{S'})$-typical automorphic form f is cuspidal.*

2.5.8. **Example.** Consider the geometric automorphic datum arising from Example 2.1.4. The moduli stack $\mathrm{Bun}_G(\mathbf{I}_S)$ in this case classifies $(\mathcal{V}, \iota, \{\ell_x\}_{x\in S})$ where $\mathcal{V}$ is a rank two vector bundle over X, $\iota : \wedge^2(\mathcal{V}) \cong \mathcal{O}_X$ and ℓ_x is a line in the fiber $\mathcal{V}_x$.

We claim that when χ_S satisfies the following genericity condition:

For any map $\epsilon : S \to \{\pm 1\}$, $\prod_{x\in S} \chi_x^{\epsilon(x)}$ is a nontrivial character on $k^\times$

then $(\mathbf{I}_S, \mathcal{K}_S)$ is weakly rigid. Indeed, the only relevant point in $\mathrm{Bun}_G(\mathbf{I}_S)$, which corresponds to the trivial bundle $\mathcal{O}_X^2$ with three lines $\{\ell_x\}_{x\in S}$ in generic position (i.e., three distinct lines in k^2).

2.5.9. **Example.** Consider the geometric automorphic datum arising from Example 2.1.5. Let's assume G is simply-connected so that $\mathrm{Bun}_G(\mathbf{K}_S)$ is connected.

Let B and B^{op} be opposite Borel subgroups of G containing T. Let $I_0 \subset G(F_0)$ be the Iwahori that is the preimage of $B^{op}(k) \subset G(k)$; let $I_\infty^+ \subset G(F_\infty)$ be the Iwahori that is the preimage of $B(k) \subset G(k)$.

Let $\Gamma_0 = I_0 \cap G(\overline{k}[\tau, \tau^{-1}])$ ($\tau = 0$ corresponds to $t = \infty$), $\Gamma_\infty^+ = I_\infty^+ \cap \Gamma_\infty^+$. Then $\Gamma_0 \cap \Gamma_\infty^+ = \{1\}$. The $\overline{k}$-points of $\mathrm{Bun}_G(\mathbf{K}_S)$ are in bijection with double cosets

$$\Gamma_\infty^+ \backslash G(\overline{k}[\tau, \tau^{-1}])/\Gamma_0. \tag{2.6}$$

There is a bijection between the affine Weyl group $W_{\mathrm{aff}} = \mathbb{X}_*(T) \rtimes W$ of G and the double cosets (2.6). To each $(\lambda, w) \in W_{\mathrm{aff}}$, the corresponding coset representative is $\tau^\lambda \dot{w}$ for any lifting $\dot{w}$ to $N_G(T)$.

We show that the unit coset is the only relevant point for the Kloosterman automorphic datum. Let $\widetilde{w} = (\lambda, w) \in W_{\mathrm{aff}}$ and let $\mathcal{E}_{\widetilde{w}} \in \mathrm{Bun}_G(\mathbf{K}_S)(\overline{k})$ be corresponding point. Then $\mathrm{Aut}(\mathcal{E}_w) = \Gamma_\infty^+ \cap \mathrm{Ad}(\widetilde{w})\Gamma_0$ is a finite-dimensional unipotent group.

To describe $\mathrm{Aut}(\mathcal{E}_w)$ we need some terminology from affine Lie algebras. The Lie algebra $\mathfrak{g} \otimes \overline{k}[\tau, \tau^{-1}]$ of $G(\overline{k}[\tau, \tau^{-1}])$ decomposes as a direct sum of affine root spaces under the action of T and the scaling on τ. For $\alpha \in \Phi(G, T)$, we say that $\mathfrak{g}_\alpha \tau^n$ is the affine root space of the affine root $\alpha + n$. We say $\alpha + n > 0$ if and only if $n > 0$ or $(n = 0) \wedge (\alpha > 0)$. There is an action of W_{aff} on affine roots: (λ, w) sends $\alpha + n$ to $w\alpha + n + \langle \lambda, w\alpha \rangle$. The extra affine simple root $\alpha_0 = -\theta + 1$.

Then $\mathrm{Aut}(\mathcal{E}_w)$ is the unipotent group whose Lie algebra is spanned by the affine root spaces $\mathfrak{g}_\alpha \tau^n$ such that

$$\alpha + n > 0, \widetilde{w}^{-1}(\alpha + n) < 0.$$

When $\widetilde{w} \neq 1$, we show $\mathcal{E}_{\widetilde{w}}$ is irrelevant. one of the affine simple roots α_i will be sent to a negative affine root under $\widetilde{w}^{-1}$. Therefore $\mathrm{LieAut}(\mathcal{E}_{\widetilde{w}})$ contains α_i, and the root subgroup $U_{\alpha_i} \subset \mathrm{Aut}(\mathcal{E}_{\widetilde{w}})$. The evaluation map ev_S only maps nontrivially to K_∞. Composing with φ to $\prod_j U_{\alpha_j}$, we see that

$$\mathrm{Aut}(\mathcal{E}_{\widetilde{w}}) \xrightarrow{\mathrm{ev}_\infty} K_\infty = I_\infty^+ \xrightarrow{\varphi} \prod_j U_{\alpha_j}$$

maps U_{α_i} identically to the U_{α_i}-factor in the product. Therefore the Artin-Schreier sheaf AS_ψ pulls back nontrivially to $\mathrm{Aut}(\mathcal{E}_{\widetilde{w}})$ (which is connected). This shows $\mathcal{E}_{\widetilde{w}}$ is irrelevant.

2.5.10. **Exercise.** For the Kloosterman automorphic datum and $G = \mathrm{SL}_2$, give an interpretation of $\mathrm{Bun}_G(\mathbf{K}_S)$ in terms of bundles with extra data. Classify points in this moduli stack, and show "by hand" that only the open point is relevant.

3. Constructing rigid automorphic datum

We will give several principles on how to construct automorphic data that have a good chance of being weakly rigid. We also give a survey of most of the examples of rigid automorphic data in the literature.

3.1. **Numerical condition for rigidity.** All known examples of weakly rigid geometric automorphic datum $(\mathcal{K}_S, \mathcal{K}_S)$ satisfy

$$\dim \mathrm{Bun}_G(\mathbf{K}_S) = 0.$$

When $(\mathcal{K}_S, \mathcal{K}_S)$ is geometrically rigid and its relevant points have finite stabilizer, the above equality automatically holds.

Suppose $X = \mathbb{P}^1$. We give a formula for $\dim \mathrm{Bun}_G(\mathbf{K}_S)$. Let $\mathfrak{k}_x \subset \mathfrak{g}(F_x)$ be the Lie algebra of $\mathbf{K}_x$. We have the relative dimension $[\mathfrak{g}(\mathcal{O}_x) : \mathfrak{k}_x]$ over k_x (see Example 1.3.2).

3.1.1. **Lemma.** *Let g_X be the genus of X. We have*

$$\dim \mathrm{Bun}_G(\mathbf{K}_S) = \sum_{x\in S}[\mathfrak{g}(\mathcal{O}_x) : \mathfrak{k}_x] + (g_X - 1)\dim G$$

Therefore when $X = \mathbb{P}^1$, $\dim \mathrm{Bun}_G(\mathbf{K}_S) = 0$ is equivalent to

$$\sum_{x\in S}[\mathfrak{g}(\mathcal{O}_x) : \mathfrak{k}_x] = \dim G. \tag{3.1}$$

This is a numerical condition that the automorphic datum wants to satisfy.

3.1.2. **Example.** Suppose $X = \mathbb{P}^1$ and all $\mathbf{K}_S$ are parahoric subgroups with Levi quotient L_x. Assume $\mathbf{K}_x$ contains the standard Iwahori $\mathbf{I}_x$. Then

$$[\mathfrak{g}(\mathcal{O}_x) : \mathfrak{k}_x] = \dim L_x^+G/\mathbf{I}_x - \dim \mathbf{K}_x/\mathbf{I}_x = \frac{\dim G - \dim L_x}{2} = \frac{|\Phi(G,T)| - |\Phi(L_x,T)|}{2}.$$

One can read of the root system of L_x from the extended Dynkin diagram $\widetilde{\mathrm{Dyn}}(G)$, so the condition (3.1) is easy to verify in this case.

Here are some interesting examples in exceptional groups when $S = \{0, 1, \infty\}$. In each case, $\mathbf{K}_0, \mathbf{K}_1$ and $\mathbf{K}_\infty$ are the parahorics that correspond to the *complement* of the vertices v_0, v_1 and v_∞ respectively. We color these vertices in the extended Dynkin diagram by black.

(1) Tetrahedron Example. Let $G = G_2$:

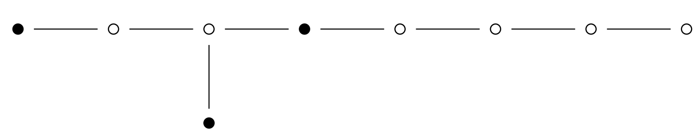
$$\circ \;\text{---}\; \bullet \Rrightarrow \blacksquare$$

Here the vertex labelled $\blacksquare$ appears twice in $\{v_0, v_1, v_\infty\}$.

(2) Octahedron Example. Let $G = F_4$:

$$\circ \;\text{---}\; \bullet \;\text{---}\; \bullet \Rightarrow \bullet \;\text{---}\; \circ$$

(3) Icosahedron Example. Let $G = E_8$:



3.1.3. **Exercise.** Construct more examples with all $\mathbf{K}_S$ are parahoric subgroups satisfying (3.1).

3.2. **Designing rigid automorphic data.** The choice of local data $(\mathbf{K}_x, \mathcal{K}_x)$ is guided under the local Langlands correspondence by the structure theory of p-adic groups on the one hand, and local Galois representations on the other hand.

3.2.1. *Moy Prasad filtration.* For a parahoric subgroup $\mathbf{P}$ in the loop group $LG = G((t))$, Moy and Prasad defined a natural filtration on it indexed by $\frac{1}{m}\mathbb{Z}_{\geq 0}$:

$$\mathbf{P} = \mathbf{P}_0 \supset \mathbf{P}_{\geq 1/m} \supset \mathbf{P}_{\geq 2/m} \supset \cdots .$$

Here the number m can be calculated as follows. Let $\{n_i\}_{0\leq i\leq r}$ be the usual labelling on the extended Dynkin diagram such that

$$n_0 = 1, \quad \theta = \sum_{i=1}^{r} n_i\alpha_i.$$

Let $J \subset \{0, 1, \cdots, r\}$ corresponds to the subset of vertices in $\widetilde{\mathrm{Dyn}}(G)$ corresponding to $\mathbf{P}$. Then

$$m = \sum_{i \in J} n_i.$$

Let us assume $\mathbf{P}$ contains the standard Iwahori subgroup $\mathbf{I}$. Let $x_{\mathbf{P}} \in \mathbb{X}_*(T)_{\mathbb{Q}}$ be determined by the condition

$$\langle \alpha_i, x \rangle = \begin{cases} 0, & i \in J \\ 1/m, & i \notin J \end{cases} \quad 1 \le i \le r.$$

This is the barycenter of the facet in the building of LG corresponding to $\mathbf{P}$. Then the Moy-Prasad filtration on $\mathbf{P}$ can be described as follows:

(1) $\mathbf{P}_{\ge 1/m} = \mathbf{P}^+$ is the pro-unipotent radical of $\mathbf{P}$.
(2) The quotient $V_{i/m} = \mathbf{P}_{\ge i/m}/\mathbf{P}_{\ge (i+1)/m}$ is a vector group over k_x. The affine roots that appear in $V_{i/m}$ are those $\alpha + n$ (where $\alpha \in \Phi(G, T) \cup \{0\}$, $n \in \mathbb{Z}$) such that

$$\langle \alpha, x_{\mathbf{P}} \rangle + n = i/m.$$

Here we also allow α to be zero, in which case the affine root space for n is $t^n \mathrm{Lie}(T)$.

3.2.2. **Exercise.** Describe the Moy-Prasad filtration for the standard Iwahori subgroup.

3.2.3. *Local monodromy.* Let $x \in S$. Under the local Langlands correspondence, an irreducible representation π_x of $G(F_x)$ gives rise to a homomorphism $\rho_x : \mathrm{Gal}(F_x^{\mathrm{sep}}/F_x) \to \widehat{G}(\overline{\mathbb{Q}}_\ell)$. We will discuss the global Langlands correspondence in §4.1. If an automorphic representation π of $G(\mathbb{A}_F)$ gives rise to a homomorphism $\rho : \mathrm{Gal}(F^{\mathrm{sep}}/F) \to \widehat{G}(\overline{\mathbb{Q}}_\ell)$, then ρ_x will should be the restriction of ρ to the decomposition group $\mathrm{Gal}(F_x^{\mathrm{sep}}/F_x)$ at x. If ρ comes from a $\widehat{G}$-local system E on $U = X - S$, then ρ_x records the $\widehat{G}$-local system on $\mathrm{Spec} F_x$ obtained by restricting E to the formal punctured disk at x.

The local Galois group $\mathrm{Gal}(F_x^{\mathrm{sep}}/F_x)$ is an extension of the form

$$1 \to \mathcal{I}_x \to \mathrm{Gal}(F_x^{\mathrm{sep}}/F_x) \to \mathrm{Gal}(\overline{k}/k_x) \to 1.$$

The normal subgroup $\mathcal{I}_x$ of $\mathrm{Gal}(F_x^{\mathrm{sep}}/F_x)$ is the *inertia group* at x. We have a normal subgroup $\mathcal{I}_x^w \lhd \mathcal{I}_x$ called the *wild inertia group* such that the quotient $\mathcal{I}_x^t := \mathcal{I}_x/\mathcal{I}_x^w$ is the maximal prime-to-p quotient of $\mathcal{I}_x$, called the *tame inertia group*. We have a canonical isomorphism of $\mathrm{Gal}(\overline{k}/k_x)$-modules

$$\mathcal{I}_x^t \stackrel{\sim}{\to} \varprojlim_{(n,p)=1} \mu_n(\overline{k}).$$

The $\widehat{G}$-local system ρ is said to be *tame at* $x \in S$ if $\rho_x|_{\mathcal{I}_x}$ factors through the tame inertia group $\mathcal{I}_x^t$.

For representation σ of $\mathrm{Gal}(F_x^{\mathrm{sep}}/F_x)$ on an n-dimensional $\overline{\mathbb{Q}}_\ell$-vector space V, there is a multiset of n non-negative rational numbers called the *slopes* of σ. The slopes measure how wildly ramified ρ_x is. Tame representations have all slopes equal to 0. The sum of all slopes (with multiplicities) is a non-negative integer called the Swan conductor $\mathrm{Sw}(\sigma)$. The Artin conductor of σ is defined as

$$a(\sigma) = \dim V/V^{\sigma(\mathcal{I}_x)} + \mathrm{Sw}(\sigma).$$

Of particular importance to us is when $\sigma = \mathrm{Ad}(\rho_x)$, which is the composition $\rho_x : \mathrm{Gal}(F_x^{\mathrm{sep}}/F_x) \to \widehat{G}(\overline{\mathbb{Q}}_\ell) \to \mathrm{GL}(\widehat{\mathfrak{g}})$.

3.2.4. *Matching automorphic data with local monodromy.* The local Langlands correspondence (which is still a conjecture beyond GL_n) allows one to predict the inertial part of the local Galois representation from the datum $(\mathbf{K}_x, \mathcal{K}_x)$; conversely, if we want to design an automorphic datum to match a given representation $\rho_x|_{\mathcal{I}_x} : \mathcal{I}_x \to \widehat{G}(\overline{\mathbb{Q}}_\ell)$, the local Langlands correspondence tells us which $(\mathbf{K}_x, \mathcal{K}_x)$ to choose.

We give some principles of this sort.

(1) If $(\mathbf{K}_S, \mathcal{K}_S)$ is weakly rigid, and it corresponds to a local $\widehat{G}$-Galois representation ρ_x at $x \in S$, then we expect the matching of local numerical invariants:
$$[\mathfrak{g}(\mathcal{O}_x) : \mathfrak{k}_x] \stackrel{?}{=} a(\mathrm{Ad}(\rho_x)).$$
For intuition behind this expectation, see §5.1.3.

(2) If $\mathbf{K}_x$ is a parahoric subgroup and $\mathcal{K}_x$ is pulled back from the Levi quotient L_x, then ρ_x should be tame. Moreover, one can determine the semisimple part of $\rho_x|\mathcal{I}_x$ as follows. We may assume $T \subset L_x$. Restrict $\mathcal{K}_x \in \mathrm{CS}_1(L_x)$ to T we get a Kummer local system which is in bijection with characters $\chi : T(k_x) \to \overline{\mathbb{Q}}_\ell^\times$. Equivalently, χ can be viewed as a homomorphism $k_x^\times \otimes_{\mathbb{Z}} \mathbb{X}_*(T) \to \overline{\mathbb{Q}}_\ell^\times$, hence giving $\chi' : k_x^\times \to \mathrm{Hom}(\mathbb{X}^*(T), \overline{\mathbb{Q}}_\ell^\times) = \widehat{T}(\overline{\mathbb{Q}}_\ell)$. By local class field theory we have $\mathcal{I}_x^{ab} \cong \mathcal{O}_x^\times$ hence a canonical tame quotient $\mathcal{I}_x \twoheadrightarrow k_x^\times$. Then χ' can be viewed as a homomorphism
$$\widetilde{\chi}' : \mathcal{I}_x \twoheadrightarrow k_x^\times \xrightarrow{\chi'} \widehat{T}(\overline{\mathbb{Q}}_\ell).$$
Then $\widetilde{\chi}'$ is conjugate to the semisimple part of $\rho_x|_{\mathcal{I}_x}$.

How about the unipotent part of $\rho_x|_{\mathcal{I}_x}$? When $(\mathbf{K}_S, \mathcal{K}_S)$ is weakly rigid, the unipotent part of $\rho_x|_{\mathcal{I}_x}$ tends to be as nontrivial as possible, given that it commutes with its semisimple part which is determined by

(3) If all slopes of ρ_x are $\le \lambda$ for some $\lambda \in \mathbb{Q}$, then there should exist a parahoric subgroup $\mathbf{P} \subset L_x G$ such that $\mathbf{K}_x$ contains $\mathbf{P}_{\ge\lambda}$. Moreover, one can take $\mathbf{K}_x^+$ to contain $\mathbf{P}_{>\lambda}$.

For example, when $G = \mathrm{PGL}_n$, we will see in §4.6 that the Galois representation arising from the Kloosterman automorphic datum is the classical Kloosterman sheaf of Deligne. Its slopes at ∞ are all equal to $1/n$. On the other hand, $\mathbf{K}_\infty = \mathbf{I}_\infty^+$ is the $1/n$ step in the Moy-Prasad filtration of $\mathbf{I}_\infty$. The more epipelagic examples 3.3.3 also follow this pattern.

(4) More generally, we expect $\mathbf{K}_x$ to be a *Yu group*, following a construction of J-K. Yu [33]. Roughly, if $\lambda_0 < \lambda_1 < \lambda_2 < \cdots$ are the different slopes that appear in thee local adjoint Galois representation $\mathrm{Ad}(\rho_x)$, one should try to find a sequence of twisted Levi subgroups
$$G^0 \subset G^1 \subset G^2 \subset \cdots \subset G$$
and Moy-Prasad groups $\mathbf{P}^i_{\ge\lambda_i/2} \subset L_x G^i$ (again this is oversimplified!), and take $\mathbf{K}_x$ to be the subgroup generated by the $\mathbf{P}^i_{\ge\lambda_i/2}$. There is also a recipe for matching the wild part of $\mathcal{K}_x$ (which corresponds to characters of $\mathbf{P}^i_{\ge\lambda_i}/\mathbf{P}^i_{>\lambda_i}$) with the restriction of ρ_x to the wild inertia.

3.3. More examples.

3.3.1. **Example** (Hypergeometric automorphic data)**.** In [23], Kamgarpour and Yi constructed automorphic datum realizing Katz's hypergeometric local systems. Let $G = \mathrm{PGL}_n$, $X = \mathbb{P}^1$. There are two kinds of such data corresponding to tame and wild hypergeometric sheaves.

The tame case generalizes Example 2.1.4. In this case, $S = \{0, 1, \infty\}$. Both $\mathbf{K}_0$ and $\mathbf{K}_\infty$ are taken to be an Iwahori subgroup. Let $\mathcal{K}_0$ and $\mathcal{K}_\infty$ be arbitrary Kummer sheaves on the quotient tori of $\mathbf{K}_0$ and $\mathbf{K}_\infty$. Take $\mathbf{K}_1 \subset L_1^+ G$ to be parahoric subgroup that is the preimage of a parabolic subgroup with block sizes $(n-1,1)$ under the evaluation map $L_1^+G \to G$ (there are two conjugacy classes of such parabolic subgroups, but their preimages in L_1^+G are conjugate under $\mathrm{PGL}_n(F_1)$). Let $\mathcal{K}_1$ be any Kummer sheaf pulled back from the abelianization map $\mathbf{K}_1 \to \mathbb{G}_m$.

In the wild case, take $S = \{0, \infty\}$. Let $\mathbf{K}_0$ be an Iwahori subgroup and $\mathcal{K}_0$ be a Kummer sheaf pulled back from its quotient torus. To define $(\mathbf{K}_\infty, \mathcal{K}_\infty)$, we recall that the hypergeometric sheaf of rank n we are trying to get under the Langlands correspondence will have d slopes equal to $1/d$ (for some $1 \le d \le n$), and $n-d$ slopes equal to 0. Therefore it is natural to ask $\mathbf{K}_\infty$ to contain $\mathbf{P}^+ = \mathbf{P}_{1/d}$ for some parahoric subgroup with the first nontrivial filtration step $1/d$, together with a bit more to accommodate a Kummer sheaf on $\mathbb{G}_m^{n-d}$ for the tame part. Let $V = k^n$ and think of PGL_n as $\mathrm{PGL}(V)$. Choose a partial flag

$$V_\bullet: \quad 0 = V_0 \subset V_{\le 1} \subset V_{\le 2} \subset \cdots V_{\le d} = V$$

with $V_i = V_{\le i}/V_{\le i-1}$ nonzero. Fix a decomposition $V_i = \ell_i \oplus V_i'$ $(1 \le i \le d)$. Let $\mathbf{P} \subset L_\infty G$ be the preimage of the parabolic $P_{V_\bullet} \subset \mathrm{PGL}(V)$ stabilizing the above partial flag. Then its Levi quotient $L_{\mathbf{P}} \cong (\prod_{i=1}^d \mathrm{GL}(V_i))/\Delta\mathbb{G}_m$. Let $\mathbf{P}^{++} = \mathbf{P}_{2/d}$. Then the next subquotient in the Moy-Prasad filtration has the form

$$\mathbf{P}^+/\mathbf{P}^{++} \cong \bigoplus_{i=1}^d \mathrm{Hom}(V_i, V_{i-1})$$

as an $L_{\mathbf{P}}$-module. Here V_0 is understood to be V_d.

Let $\varphi : \mathbf{P}^+/\mathbf{P}^{++} \to \mathbb{G}_a$ to be a linear function whose component $\varphi_i : V_i \to V_{i-1}$ sends the line ℓ_i isomorphically to ℓ_{i-1}, and is zero on V_i'. Let L_φ be the stabilizer of φ under $L_{\mathbf{P}}$; then $L_\varphi \cong \prod_i \mathrm{GL}(V_i')$. Let $B_\varphi \subset L_\varphi$ be a Borel subgroup. Then its quotient torus T_φ has dimension $n-d$. Finally, let $\mathbf{K}_\infty$ be the preimage of B_φ under the quotient $\mathbf{P} \to L_{\mathbf{P}}$.

To define $\mathcal{K}_\infty$, we first extend $\varphi^*\mathrm{AS}_\psi \in \mathrm{CS}_1(\mathbf{P}^+)$ to a rank one character sheaf $\mathcal{K}_\varphi$ on $\mathbf{K}_\infty$ (which is possible because B_φ fixes φ). Let $\mathcal{K}_\rho$ be the pullback of any Kummer sheaf on T_{n-d} via $\mathbf{K}_\infty \to B_\varphi \to T_\varphi$. Finally take $\mathcal{K}_\infty = \mathcal{K}_\varphi \otimes \mathcal{K}_\rho \in \mathrm{CS}_1(\mathbf{K}_\infty)$.

The construction of $(\mathbf{K}_\infty, \mathcal{K}_\infty)$ in the wild case can be generalized to other types of groups, see Example 3.3.4.

For a specific choice of the partial flag $V_\bullet$ where the dimensions of V_i are distributed as evenly as possible, it was proved in [23] that $(\mathbf{K}_S, \mathcal{K}_S)$ is geometrically rigid.

3.3.2. **Example** (Airy automorphic datum)**.** In the work of Kamgarpour, Jakob and Yi [19], a series of geometrically rigid examples with $X = \mathbb{P}^1$ and $S = \{\infty\}$ are introduced. When $G = \mathrm{SL}_2$ the corresponding local system, or rather its de Rham version, is the D-module given by the Airy equation.

We explain the construction for $G = \mathrm{SL}_2$. Let K_∞ be the subgroup of I_∞ consisting of the following matrices (where $a_i, b_i, c_i \in k$)

$$\begin{pmatrix} 1 + a_1\tau + \cdot & b_0 + b_1\tau + \cdot \\ b_0\tau + c_2\tau^2 + \cdots & 1 - a_1\tau + \cdots \end{pmatrix}$$

There is a homomorphism $\varphi : K_\infty \to k$ sending the above matrix to $b_1 + c_2$ (check this is a group homomorphism!). Let χ_∞ be the pullback of a nontrivial additive character via φ. It is clear how to turn (K_∞, χ_∞) into a geometric automorphic datum $(\mathbf{K}_\infty, \mathcal{K}_\infty)$.

A more conceptual way of defining K_∞ is the following. First take the Moy-Prasad filtration $I_{3/2}$ of I_∞ and consider the homomorphism $\varphi_{3/2} : I_{3/2} \to I_{3/2}/I_2 \cong k^2 \xrightarrow{sum} k$. We may ask if $\varphi_{3/2}$ can be extended to a larger subgroup of $I^+ = I_{1/2}$. First $\varphi_{3/2}$ can be extended to $\varphi_1 : I_1 \to k$ by letting it to be trivial on the diagonal matrices. Next, we may view φ_1 as an element (not uniquely determined) in the dual of the Lie algebra $\mathfrak{g}((\tau))$, which can be identified with $\mathfrak{g}((\tau))\frac{d\tau}{\tau}$ using the Killing form and the residue pairing. We choose $\widetilde{\varphi}_1 = \begin{pmatrix} 0 & \tau^{-2} \\ \tau^{-1} & 0 \end{pmatrix} \frac{d\tau}{\tau}$. Let $H(F_\infty)$ be the centralizer of $\widetilde{\varphi}_1$ under the adjoint action of $G(F_\infty)$. We see that

$$H(F_\infty) = \left\{ \begin{pmatrix} a & b \\ b\tau & a \end{pmatrix} \middle| a^2 - b^2\tau = 1, a, b \in F_\infty \right\}.$$

In fact $b \in \mathcal{O}_\infty$ and $a \in 1 + \tau\mathcal{O}_\infty$. Finally $K_\infty = H(F_\infty)I_1$.

3.3.3. **Example** (Epipelagic automorphic data)**.** In [35] we give generalizations of Kloosterman sheaves. Let $X = \mathbb{P}^1$ and $S = \{0, 1\}$. Let $\mathbf{P}_\infty \subset L_\infty G$ be a parahoric subgroup of specific types. These parahoric subgroups are singled out by Reeder and Yu [29] to define their *epipelagic representations*. Their conjugacy classes are in bijection with regular elliptic conjugacy classes in the Weyl group W. Consider the next step $\mathbf{P}_\infty^{++} = \mathbf{P}_{\infty,2/m}$ in the Moy-Prasad filtration of $\mathbf{P}_\infty$, and $V_{\mathbf{P}} := \mathbf{P}_\infty^{+}/\mathbf{P}_\infty^{++}$ is a finite-dimensional representation of the Levi $L_{\mathbf{P}}$.

We take $\mathbf{K}_\infty = \mathbf{P}_\infty^{+}$. To define $\mathcal{K}_\infty$, we pick a linear function $\varphi : V_{\mathbf{P}} \to k$ satisfying the stability condition: it has closed orbit and finite stabilizer under $L_{\mathbf{P}_\infty}$. Define $\mathcal{K}_\infty$ to be the pullback of AS_ψ under the composition

$$\mathbf{K}_\infty = \mathbf{P}_\infty^{+} \twoheadrightarrow V_{\mathbf{P}} \xrightarrow{\varphi} \mathbb{G}_a.$$

For $\mathbf{K}_0$, we take it to be the parahoric subgroup $\mathbf{P}_0 \subset L_0G$ opposite to $\mathbf{P}_\infty$: the intersection $\mathbf{P}_0 \cap \mathbf{P}_\infty$ is a common Levi subgroup $L_{\mathbf{P}}$ of both. We can take any $\mathcal{K}_0 \in \mathrm{CS}_1(L_{\mathbf{P}})$. The geometric automorphic datum $(\mathbf{K}_S, \mathcal{K}_S)$ is geometrically rigid. To check this, one has to use the stability property of φ.

One can let ϕ varies in an open subset of the dual space of $V_{\mathbf{P}}$ and get a family version of geometric automorphic datum. The resulting $\widehat{G}$-local systems "glue" together to give a $\widehat{G}$-local system over an open subset of $V_{\mathbf{P}}^* \times (\mathbb{P}^1 - \{0, \infty\})$.

3.3.4. **Example** (Euphotic automorphic data, [20])**.** This is a further generalization of Epipelagic automorphic data that weakens the assumption that $\varphi \in V_{\mathbf{P}}^*$ is stable. Let $X = \mathbb{P}^1$ and $S = \{0, \infty\}$. Assume G is simply-connected.

We start with any parahoric $\mathbf{P}_\infty$ and any linear function φ on $V_{\mathbf{P}} = \mathbf{P}_\infty^{+}/\mathbf{P}_\infty^{++}$ with a closed orbit under $L_{\mathbf{P}}$. Let L_φ be the stabilizer of φ in $L_{\mathbf{P}}$, and let $B_\varphi \subset L_\varphi$ be a Borel subgroup. Then $\mathbf{K}_\infty = \mathbf{P}_\infty^{+}B_\varphi$. We take $\mathcal{K}_\infty$ to be the tensor product of $\varphi^*\mathrm{AS}_\psi$ (extended to a character sheaf on $\mathbf{K}_\infty$ by letting it be trivial on B_φ) and a Kummer sheaf $\mathcal{K}_\chi$ from the quotient torus of B_φ.

At 0 we take $\mathbf{K}_0$ to be a parahoric subgroup contained in the parahoric $\mathbf{P}_0$ opposite to $\mathbf{P}_\infty$. It corresponds to the choice of a parabolic subgroup $Q \subset L_{\mathbf{P}}$. We require that

(3.2) The group B_φ acts on $L_{\mathbf{P}}/Q$ with an open orbit with finite stabilizer.

This is to match the numerical condition $\dim \mathrm{Bun}_G(\mathbf{K}_S) = 0$. Let $\mathcal{K}_0$ be trivial.

Such automorphic datum $(\mathbf{K}_S, \mathcal{K}_S)$ is not guaranteed to be weakly rigid. In [20], for $P_\infty = G(\mathcal{O}_\infty)$, we give a complete list of pairs (φ, Q) (where $\varphi \in \mathfrak{g}^*$ is semisimple, and $Q \subset G$ is a parabolic subgroup) satisfying the condition (3.2). We prove that in these cases, for a generic choice of the Kummer sheaf $\mathcal{K}_\chi$ on B_φ, $(\mathbf{K}_S, \mathcal{K}_S)$ is weakly rigid, but not always geometrically rigid.

Here is an example that is weakly rigid but not geometrically rigid.

**** to be added ****

3.3.5. **Example.** In [34], we constructed a geometrically rigidity automorphic datum for $X = \mathbb{P}^1_k$ and $S = \{0, 1, \infty\}$ for groups G of type $A_1, D_{2n}, E_7, E_8, G_2$ that only involve mutiplicative characters. One interesting observations is that the construction makes sense over any field k with $\mathrm{char}(k) \neq 2$. In particular, in [34] it was applied to $k = \mathbb{Q}$ to obtain the first examples of motives with E_7 and E_8 as motivic Galois groups, and to solve the inverse Galois problem for $E_8(\mathbb{F}_\ell)$ for sufficiently large prime ℓ.

Assume G is simply-connected, and the longest element w_0 in the Weyl group W of G acts by inversion on T.

Up to conjugacy, there is a unique parahoric subgroup $\mathbf{P} \subset L_0G$ such that its reductive quotient $L_{\mathbf{P}}$ is isomorphic to the fixed point subgroup G^τ of a Cartan involution corresponding to the split real form of G. For example, we can take $\mathbf{P}$ to be the parahoric subgroup corresponding to the facet containing the element $\rho^\vee/2$ in the T-apartment of the building of L_0G ($\rho^\vee$ is half the sum of positive coroots of G).

The Dynkin diagram of the reductive quotient $L_{\mathbf{P}} \cong G^\tau$ of $\mathbf{P}$ is obtained by removing one or two nodes from the extended Dynkin diagram of G. We tabulate the type of $L_{\mathbf{P}}$ and the nodes to be removed in each case.

G	$L_{\mathbf{P}}$	nodes to be removed
B_{2n}	$B_n \times D_n$	the $(n+1)$-th counting from the short node
B_{2n+1}	$B_n \times D_{n+1}$	the $(n+1)$-th counting from the short node
C_n	$A_{n-1} \times \mathbb{G}_m$	the two ends
D_{2n}	$D_n \times D_n$	the middle node
E_7	A_7	the end of the leg of length 1
E_8	D_8	the end of the leg of length 2
F_4	$A_1 \times C_3$	second from the long node end
G_2	$A_1 \times A_1$	middle node

Fact: If G is not of type C, then $L_{\mathbf{P}}^{\mathrm{sc}} \to L_{\mathbf{P}}$ is a double cover. Even if G is of type C_n, $L_{\mathbf{P}} \cong \mathrm{GL}_n$ still admits a unique nontrivial double cover.

Therefore, in all cases, there is a canonical nontrivial double cover $v : \widetilde{L}_{\mathbf{P}} \to L_{\mathbf{P}}$. In particular,

$$(v_!\overline{\mathbb{Q}}_\ell)_{\mathrm{sgn}} \in \mathrm{CS}_1(L_{\mathbf{P}}) \tag{3.3}$$

(here sgn denotes the nontrivial character of $\ker(v)(k) = \{\pm 1\}$).

Let $\mathbf{K}_0 = \mathbf{P} \subset L_0G$ be a parahoric subgroup of the type defined above. Let $\mathcal{K}_0$ be the pullback of the local system (3.3). Let $\mathbf{K}_\infty = \mathbf{P}_\infty \subset L_\infty G$ be the parahoric subgroup of the same type as $\mathbf{P}$. Let $\mathbf{K}_1 = \mathbf{I}_1 \subset L_1G$ be an Iwahori subgroup. Let $\mathcal{K}_\infty$ and $\mathcal{K}_1$ be trivial.

The central character compatible with (K_S, χ_S) above is unique, and it exists if and only if $\chi|_{Z(k)} = 1$, which can always be achieved by passing to a quadratic extension k'/k.

In [34], it is shown that if G is either simply-laced or of type G_2 (again assume $w_0 = -1$, so G is of type A_1, D_{2n}, E_7, E_8 and G_2), then the geometric automorphic datum $(\mathcal{K}_S, \mathcal{K}_S)$ defined above is weakly rigid. Indeed, there is a unique relevant $\overline{k}$-point in $\mathrm{Bun}_G(\mathbf{K}_S)$ which is the unique open point.

The moduli stack $\mathrm{Bun}_G(\mathbf{P}_0, \mathbf{P}_\infty)$ contains an open substack isomorphic to $\mathrm{pt}/L_{\mathbf{P}}$. The preimage of $\mathrm{pt}/L_{\mathbf{P}}$ in $\mathrm{Bun}_G(\mathbf{K}_S)$ is isomorphic to $L_{\mathbf{P}}\backslash G/B$. Since $L_{\mathbf{P}}$ is a symmetric subgroup of G, it acts on the flag variety of G with an open orbit $O \subset G/B$. We thus get an open point

$$j : L_{\mathbf{P}}\backslash O \hookrightarrow \mathrm{Bun}_G(\mathbf{K}_S).$$

When G is either simply-laced or of type G_2, this turns out to be the unique relevant point.

The stabilizer A_u of $L_{\mathbf{P}}$ on any point $u \in O$ is canonically isomorphic to $T[2]$. The category $D_c(\mathbf{K}_S, \mathcal{K}_S)$ is more interesting. Taking the preimage of A_u in the double cover $\widetilde{L}_{\mathbf{P}}$ we get a central extension

$$1 \to \mu_2 \to \widetilde{A}_u \to T[2] \to 1.$$

For simplicity let's assume $G = E_8$. Then $\widetilde{A}_u$ is a finite Heisenberg 2-group with μ_2 as the center. It has a unique irreducible $\overline{\mathbb{Q}}_\ell$-representation V with nontrivial central character. This representation gives an irreducible local system on the preimage of the open point $L_{\mathbf{P}}\backslash O$ in $\mathrm{Bun}_G(\mathbf{K}_S^+)$, whose extension to $\mathrm{Bun}_G(\mathbf{K}_S^+)$ by zero gives an object $\mathcal{F} \in D_c(\overline{k}; \mathbf{K}_S, \mathcal{K}_S)$. It turns out $\mathcal{F}$ is the unique simple perverse sheaf in $D_c(\overline{k}; \mathbf{K}_S, \mathcal{K}_S)$.

When G is not of type E_8, the center of $\widetilde{A}_u$ is larger. One can construct a simple perverse sheaf for each central character of $\widetilde{A}_u$ whose restriction to μ_2 is nontrivial.

4. Computing eigen local systems

In this section we explain how to get $\widehat{G}$-local systems out of rigid automorphic data.

4.1. **From eigenforms to local systems.**

We give a very brief review of how Langlands correspondence works over function fields in the automorphic to Galois direction.

4.1.1. *Eigenvalues.* Given a Hecke eigenform f, then for $x \in |X| - S$ (where S is a finite set of places), we have a character $\sigma_x : \mathcal{H}_{G(\mathcal{O}_x)} \to \overline{\mathbb{Q}}_\ell$ of the spherical Hecke algebra $\mathcal{H}_{G(\mathcal{O}_x)}$ such that

$$f \star h_x = \sigma_x(h_x) f, \quad \forall h_x \in \mathcal{H}_{G(\mathcal{O}_x)}.$$

We recall the Satake isomorphism:

$$\mathcal{H}_{G(\mathcal{O}_x)} \cong \overline{\mathbb{Q}}_\ell[\mathbb{X}_*(T)]^W.$$

Each homomorphism $\sigma_x : \mathcal{H}_{G(\mathcal{O}_x)} \to \overline{\mathbb{Q}}_\ell$ gives rise to a W-orbit on $\widehat{T}$, which is the same datum as a semisimple conjugacy class $[\phi_x] \in \widehat{G}$. What property does the collection $\{\phi_x\}_{x \in |X|-S}$ satisfy?

In the simplest case $G = \mathrm{PGL}_2$ and $\widehat{G} = \mathrm{SL}_2$, σ_x is determined by its value at the characteristic function of $G(\mathcal{O}_x)\begin{pmatrix} t_x & 0 \\ 0 & 1 \end{pmatrix}G(\mathcal{O}_x)$; this value is equal to the number $a_x = \mathrm{Tr}(\phi_x)$. The $\{a_x\}$ is the function field analogues of Fourier coefficients a_p for a Hecke eigen modular form for PGL_2 over $\mathbb{Q}$. Can we get arbitrary collection of traces $\{a_x\}_{x \in |X|-S}$ from eigenforms?

The Langlands correspondence predicts that the collection $\{\phi_x\}$ of conjugacy classes in $\widehat{G}$ must come from a single global object, namely a $\widehat{G}$-local system on $X - S$.

4.1.2. *Local systems.* A rank n $\overline{\mathbb{Q}}_\ell$-local system on U for the étale topology is slightly technical to define: it arises from an inverse system of locally constant $\mathbb{Z}/\ell^m\mathbb{Z}$ sheaves on U free of rank n. Let $\mathrm{Loc}(U, \overline{\mathbb{Q}}_\ell)$ be the tensor category of $\overline{\mathbb{Q}}_\ell$-local systems on U of varying rank. This is a tensor category linear over $\overline{\mathbb{Q}}_\ell$.

Let $u \in U$ be a geometric point $u \in U$. We have the étale fundamental group $\pi_1(U, u)$, which is a profinite group. A rank n local system E on U with $\overline{\mathbb{Q}}_\ell$-coefficients gives rise to a continuous representation of $\pi_1(U, u)$ on a n-dimensional $\overline{\mathbb{Q}}_\ell$-vector space (with ℓ-adic topology), called the monodromy representation of E. This representation is constructed by looking at the action of $\pi_1(U, u)$ on the stalk E_u.

The above discussion gives an equivalence of tensor categories

$$\omega_u : \mathrm{Loc}(U, \overline{\mathbb{Q}}_\ell) \xrightarrow{\sim} \mathrm{Rep}_{\mathrm{cont}}(\pi_1(U, u), \overline{\mathbb{Q}}_\ell). \tag{4.1}$$

Note that $\pi_1(U, u)$ is a quotient of the absolute Galois group $\mathrm{Gal}(F^{\mathrm{sep}}/F)$ of the function field $F = k(X)$. Therefore, a rank n local system on U gives rise to an n-dimensional continuous representation of $\mathrm{Gal}(F^{\mathrm{sep}}/F)$.

4.1.3. *$\widehat{G}$-local systems.* With the choice of a geometric point $u \in U$, a $\widehat{G}$-local system on U is a continuous homomorphism

$$\rho : \pi_1(U, u) \to \widehat{G}(\overline{\mathbb{Q}}_\ell). \tag{4.2}$$

Therefore, a GL_n-local system is the same datum as a rank n local system on U.

A more canonical definition can be given using the Tannakian formalism. Let $\mathrm{Rep}(\widehat{G}, \overline{\mathbb{Q}}_\ell)$ be the tensor category of algebraic representations of $\widehat{G}$ on finite-dimensional $\overline{\mathbb{Q}}_\ell$-vector spaces. Then a $\widehat{G}$-local system on U is the same datum as a tensor functor

$$E : \mathrm{Rep}(\widehat{G}, \overline{\mathbb{Q}}_\ell) \to \mathrm{Loc}(U, \overline{\mathbb{Q}}_\ell). \tag{4.3}$$

The two notions of H-local systems are equivalent. Given a representation ρ as in (4.2), we define a tensor functor E by assigning to $V \in \mathrm{Rep}(H, \overline{\mathbb{Q}}_\ell)$ the local system with monodromy representation

$$\rho_V : \pi_1(U, u) \xrightarrow{\rho} \widehat{G}(\overline{\mathbb{Q}}_\ell) \to \mathrm{GL}(V).$$

Conversely, given a tensor functor E as in (4.3), using the equivalence (4.1), E can be viewed as a tensor functor $\mathrm{Rep}(\widehat{G}, \overline{\mathbb{Q}}_\ell) \to \mathrm{Rep}_{\mathrm{cont}}(\pi_1(U, u), \overline{\mathbb{Q}}_\ell)$. Tannakian formalism [7] then implies that such a tensor functor comes from a group homomorphism ρ as in (4.2), well-defined up to conjugacy.

4.1.4. **Example.** For $\widehat{G} = \mathrm{SO}_n$, a $\widehat{G}$-local system on U is the same thing as a rank n local system E on U together with a self-adjoint isomorphism $b : E \xrightarrow{\sim} E^\vee$.

Below we will get more technical because we will be using sheaf theory on algebraic stacks such as Bun_G and its Hecke correspondences to compute the $\widehat{G}$-local systems attached to an eigenform. The ideas come from the geometric Langlands program originated from the works of Drinfeld, Deligne, Laumon, etc. The main result is Theorem 4.3.3, which roughly says that for rigid automorphic datum, one can construct the corresponding local system explicitly.

4.2. **The Satake category.** The Satake category is an upgraded version of the spherical Hecke algebra $\mathcal{H}_{G(\mathcal{O}_x)}$ under the sheaf-to-function correspondence. In this subsection we G is split reductive.

Let LG and L^+G be the group objects over k whose R-points are $G(R(\!(t)\!))$ and $G(R[\![t]\!])$ respectively. The quotient $\mathrm{Gr} = LG/L^+G$ is called the *affine Grassmannian* of G, and it is an infinite union of projective schemes. Then L^+G acts on Gr via left translation. The L^+G-orbits on Gr are indexed by dominant coweights $\lambda \in \mathbb{X}_*(T)^+$. The orbit containing the element $t^\lambda \in T(k(\!(t)\!))$ is denoted by Gr_λ and its closure is denoted by $\mathrm{Gr}_{\leq\lambda}$. We have $\dim \mathrm{Gr}_\lambda = \langle 2\rho, \lambda\rangle$, where 2ρ is the sum of positive roots in G. Each $\mathrm{Gr}_{\leq\lambda}$ is a projective but usually singular variety over k. We denote the intersection complex of $\mathrm{Gr}_{\leq\lambda}$ by IC_λ.

Let $\mathrm{Sat}_{\overline{k}} = \mathrm{Perv}_{(L^+G)_{\overline{k}}}(\mathrm{Gr}_{\overline{k}})$ be the category of $(L^+G)_{\overline{k}}$-equivariant perverse sheaves on $\mathrm{Gr}_{\overline{k}} = \mathrm{Gr} \otimes_k \overline{k}$ (with $\overline{\mathbb{Q}}_\ell$-coefficients) supported on finitely many $(L^+G)_{\overline{k}}$ orbits. In [26], [12] and [28], it was shown that when k is algebraically closed, $\mathrm{Sat}_{\overline{k}}$ carries a natural tensor structure, such that the global cohomology functor $h = H^*(\mathrm{Gr}_{\overline{k}}, -) : \mathrm{Sat}_{\overline{k}} \to \mathrm{Vec}(\overline{\mathbb{Q}}_\ell)$ is a fiber functor. It is also shown that the Tannakian group of the tensor category $\mathrm{Sat}_{\overline{k}}$ is the Langlands dual group $\widehat{G}$. The Tannakian formalism gives the *geometric Satake equivalence* of tensor categories

$$\mathrm{Sat}_{\overline{k}} \cong \mathrm{Rep}(\widehat{G}, \overline{\mathbb{Q}}_\ell).$$

Similarly we define $\mathrm{Sat}_k = \mathrm{Perv}_{L^+G}(\mathrm{Gr})$; this is also a tensor category with fiber functor $H^*(\mathrm{Gr}_{\overline{k}}, -)$, but its Tannakian group is larger than $\widehat{G}$ (one can show its Tannakian group is the algebraic envelope of $\widehat{G} \times \mathrm{Gal}(\overline{k}/k)$). Define $\mathrm{Sat}^0_k \subset \mathrm{Sat}_k$ to be the full subcategory consisting of direct sums of the IC_λ's (for various $\lambda \in \mathbb{X}_*(T)^+$). It turns out that Sat^0_k is closed under the tensor structure, and we have an equivalence of tensor categories

$$\mathrm{Sat}^0_k \cong \mathrm{Rep}(\widehat{G}, \overline{\mathbb{Q}}_\ell).$$

For $V \in \mathrm{Rep}(\widehat{G}, \overline{\mathbb{Q}}_\ell)$, we denote the corresponding object in Sat^0_k by IC_V.

4.2.1. **Example.** For $G = \mathrm{GL}_n$, $\mathrm{Gr}(k)$ parametrizes $\mathcal{O} = k[\![t]\!]$-lattices in $k(\!(t)\!)^n$. For $\lambda = (1, \cdots, 1, 0, \cdots, 0)$ with i 1's and $n-i$ 0's, $\mathrm{Gr}_\lambda = \mathrm{Gr}_{\leq\lambda} \cong \mathrm{Gr}(n, i)$. In fact, $\mathrm{Gr}_\lambda(k)$ consists of lattices Λ such that $t^{-1}\mathcal{O}^n \supset \Lambda \supset \mathcal{O}^n$ and $\dim_k(\Lambda/\mathcal{O}^n) = i$.

4.2.2. **Example.** For $G = \mathrm{Sp}(V)$, V a symplectic space of dimension $2n$, $\mathrm{Gr}(k)$ parametrizes $\mathcal{O} = k[\![t]\!]$-lattices Λ in $V \otimes k(\!(t)\!)$ such that the symplectic form restricts to a perfect pairing on Λ (these are called self-dual lattices). Let $\Lambda_0 = V \otimes \mathcal{O}$ be the standard self-dual lattice. For $\lambda = (1, 0, \cdots, 0) \in \mathbb{Z}^n$, $\mathrm{Gr}_\lambda(k)$ consists of self-dual lattices Λ such that $\dim_k(\Lambda/\Lambda \cap \Lambda_0) = 1$. The orbit closure $\mathrm{Gr}_{\leq\lambda} = \mathrm{Gr}_\lambda \cup \{\Lambda_0\}$. There is a map $\mathrm{Gr}_\lambda \to \mathbb{P}(V)$ sending Λ to the line in V that is the image of $\Lambda \mod t^{-1} \subset t^{-1}V$. The fiber over a point $\ell \in \mathbb{P}(V)$ is the line $\mathrm{Hom}(\ell, V/\ell^\perp) \cong \ell^{\otimes -2}$ (where $\ell^\perp$ is defined using the symplectic form on V). Therefore Gr_λ is isomorphic to the total space of $\mathcal{O}(2)$ over $\mathbb{P}^{2n-1}$.

4.2.3. **Exercise.** For $G = \mathrm{SO}(V)$ with $\dim V = 2n$ or $2n+1$, describe Gr_λ for $\lambda = (1, 0, \cdots, 0) \in \mathbb{Z}^n$.

4.3. **Geometric Hecke operators.** We consider the situation of §2.5.1. In particular, we have a geometric automorphic datum $(\mathbf{K}_S, \mathcal{K}_S)$, and moduli stacks $\mathrm{Bun}_G(\mathbf{K}_S)$ and $\mathrm{Bun}_G(\mathbf{K}^+_S)$.

4.3.1. *Hecke correspondence.* Consider the following diagram

$$\begin{tikzcd} & \mathrm{Hk}_G(\mathbf{K}^+_S) \arrow[dl, "\overleftarrow{h}"'] \arrow[dr, "\overrightarrow{h}"] \arrow[rr, "\pi"] & & U := X - S \\ \mathrm{Bun}_G(\mathbf{K}^+_S) & & \mathrm{Bun}_G(\mathbf{K}^+_S) & \end{tikzcd} \tag{4.4}$$

Here, the stack $\mathrm{Hk}_G(\mathbf{K}^+_S)$ classifies the data $(x, \mathcal{E}, \mathcal{E}', \tau)$ where $x \in U := X - S$, $\mathcal{E}, \mathcal{E}' \in \mathrm{Bun}_G(\mathbf{K}^+_S)$ and $\tau : \mathcal{E}|_{X-\{x\}} \stackrel{\sim}{\to} \mathcal{E}'|_{X-\{x\}}$ is an isomorphism of G-torsors over $X - \{x\}$ preserving the $\mathbf{K}^+_x$-level structures at each $x \in S$. The morphisms $\overleftarrow{h}$, $\overrightarrow{h}$ and π send $(x, \mathcal{E}, \mathcal{E}', \tau)$ to $\mathcal{E}, \mathcal{E}'$ and x respectively.

For $x \in U$, we denote its preimage under π by $\mathrm{Hk}_{G,x}(\mathbf{K}^+_S)$. We have an evaluation morphism

$$\mathrm{ev}_x : \mathrm{Hk}_{G,x}(\mathbf{K}^+_S) \to L^+_x G \backslash L_x G / L^+_x G. \tag{4.5}$$

In fact, for a point $(x, \mathcal{E}, \mathcal{E}', \tau) \in \mathrm{Hk}_{G,x}(\mathbf{K}^+_S)$, if we fix trivializations of $\mathcal{E}$ and $\mathcal{E}'$ over $\mathrm{Spec}\mathcal{O}_x$, the isomorphism τ restricted to $\mathrm{Spec}F_x$ is an isomorphism between the trivial G-torsors over $\mathrm{Spec}F_x$, hence given by a point $g_\tau \in L_x G$. Changing the trivializations of $\mathcal{E}|_{\mathrm{Spec}\mathcal{O}_x}$ and $\mathcal{E}'|_{\mathrm{Spec}\mathcal{O}_x}$ will result in left and right multiplication of g_τ by elements in $L^+_x G$. Therefore we have a well-defined morphism ev_x as above between stacks.

As x moves along U, we may identify the target of (4.5) as $L^+G\backslash LG/L^+G$ by choosing a local coordinate t at x. Modulo the ambiguity caused by the choice of the local coordinates, we obtain a well defined morphism

$$\mathrm{ev} : \mathrm{Hk}_G(\mathbf{K}^+_S) \to \left[\frac{L^+G\backslash LG/L^+G}{\mathrm{Aut}^+}\right], \tag{4.6}$$

where Aut^+ is the group scheme over k of continuous ring automorphisms of $k[\![t]\!]$, and it acts on LG and L^+G via its action on $k[\![t]\!]$.

4.3.2. *Geometric Hecke operators.* For each object $V \in \mathrm{Rep}(\widehat{G}, \overline{\mathbb{Q}}_\ell)$, the corresponding object $\mathrm{IC}_V \in \mathrm{Sat}$ under the geometric Satake equivalence defines a complex on the quotient stack $\left[\frac{L^+G\backslash LG/L^+G}{\mathrm{Aut}^+}\right]$. The geometric Hecke operator associated with V is the functor

$$\begin{aligned} \mathbb{T}_V : D_{(L_S,\mathcal{K}_S)}(\mathrm{Bun}_G(\mathbf{K}_S^+) \times U) &\to D_{(L_S,\mathcal{K}_S)}(\mathrm{Bun}_G(\mathbf{K}_S^+) \times U) \\ \mathcal{F} &\mapsto (\overrightarrow{h} \times \pi)_! \left((\overleftarrow{h} \times \pi)^* \mathcal{F} \otimes \mathrm{ev}^* \mathrm{IC}_V \right). \end{aligned}$$

The composition of these functors are compatible with the tensor structure of Sat: there is a natural isomorphism of functors

$$\mathbb{T}_V \circ \mathbb{T}_W \cong \mathbb{T}_{V\otimes W}, \quad \forall V, W \in \mathrm{Rep}(\widehat{G}, \overline{\mathbb{Q}}_\ell)$$

which is compatible with the associativity constraint of the tensor product in $\mathrm{Rep}(\widehat{G}, \overline{\mathbb{Q}}_\ell)$ and the associativity of composition of functors $\mathbb{T}_{V_1} \circ \mathbb{T}_{V_2} \circ \mathbb{T}_{V_3}$ in the obvious sense.

4.3.3. **Theorem** ([20, Corollary A.4.2], imprecise statement)**.** *Let $(\mathbf{K}_S, \mathcal{K}_S)$ be a geometrically rigid geometric automorphic datum. Then one can decompose $\mathrm{Perv}_c(\overline{k}; \mathbf{K}_S, \mathcal{K}_S)$ (perverse sheaves in $D_c(\overline{k}; \mathbf{K}_S, \mathcal{K}_S)$) into a finite direct sum of subcategories stable under the geometric Hecke operators, such that each direct summand gives rise to a $\widehat{G}$-local system on U.*

The proof uses the structure of an E_2-action of $\mathrm{Rep}(\widehat{G})$ on $\mathrm{Perv}_c(\overline{k}; \mathbf{K}_S, \mathcal{K}_S)$ that "spreads over" the curve U. Such a structure comes from versions of geometric Hecke operators modifying at two or more points on U.

A special case is when $\mathrm{Perv}_c(\overline{k}; \mathbf{K}_S, \mathcal{K}_S)$ has a single simple object $\mathcal{F}$ (which we may assume is defined over k). In this case, Theorem 4.3.3 says the following: $\mathcal{F}$ is a *Hecke eigensheaf* in the following sense. For every $V \in \mathrm{Rep}(\widehat{G}, \overline{\mathbb{Q}}_\ell)$, there is an isomorphism

$$\varphi_V : \mathbb{T}_V(\mathcal{F} \boxtimes \overline{\mathbb{Q}}_\ell) \cong \mathcal{F} \boxtimes E_V$$

for some local system $E_V \in \mathrm{Loc}(U, \overline{\mathbb{Q}}_\ell)$. Moreover the assignment $V \mapsto E_V$ gives a tensor functor $E : \mathrm{Rep}(\widehat{G}, \overline{\mathbb{Q}}_\ell) \to \mathrm{Loc}(U)$, hence a $\widehat{G}$-local system E.

4.4. **Computing local systems – a simple case.** Fix a geometric automorphic datum $(\mathbf{K}_S, \mathcal{K}_S)$.

4.4.1. *Assumptions.* We assume G is simply-connected and $(\mathbf{K}_S, \mathcal{K}_S)$ is geometrically rigid. Since $\mathrm{Bun}_G(\mathbf{K}_S)$ is connected, it has a unique relevant point $\mathcal{E}$. We further assume that $\mathrm{Aut}(\mathcal{E})$ is trivial. This implies that $\mathcal{E}$ is an open point in $\mathrm{Bun}_G(\mathbf{K}_S)$.

4.4.2. Let $\mathfrak{G}$ be the group ind-scheme over U whose fiber over $x \in U$ is the automorphism group of $\mathcal{E}|_{X-\{x\}}$ as a G-torsor with $\mathbf{K}_S$-level structures. The evaluation map along S (well-defined up to conjugacy) can be extended to $\mathfrak{G}$:

$$\mathrm{ev}_{S,\mathcal{E}} : \mathfrak{G} \to \prod_{x\in S} \mathbf{K}_x \otimes \overline{k} \to L_S \otimes \overline{k}.$$

For any dominant coweight $\lambda \in \mathbb{X}_*(T)^+$, let $\mathfrak{G}^{\leq\lambda} \subset \mathfrak{G}$ be the closed subscheme whose fiber over $x \in U$ are those automorphisms of $\mathcal{E}|_{X-\{x\}}$ with modification type $\leq \lambda$ at x. Thus $\mathfrak{G}_x^{\leq\lambda}$ can be identified with an open subscheme of $\mathrm{Gr}_{\leq\lambda}$.

4.4.3. **Proposition.** *Under the assumption that G is simply-connected and* $\mathrm{Aut}(\mathcal{E})$ *is trivial,* $\mathrm{Perv}_c(\overline{k}; \mathbf{K}_S, \mathcal{K}_S)$ *contains a unique simple object $\mathcal{F}$, and it is a Hecke eigensheaf. The corresponding $\widehat{G}$-local system E is described as follows.*

For a dominant coweight $\lambda \in \mathbb{X}_*(T)^+$ and the corresponding irreducible representation V_λ of $\widehat{G}$, we have

$$E_{V_\lambda} \cong \pi^\lambda_!(\mathrm{ev}^*_{S,\mathcal{E}}\mathcal{K}_S \otimes \mathrm{IC}_{\mathfrak{G}^{\leq\lambda}}[-1])$$

where $\pi^\lambda : \mathfrak{G}^{\leq\lambda} \to U$ is the projection (it contains the implicit statement that the right side is concentrated in degree zero).

In particular, if λ is minuscule so that $\mathfrak{G}^{\leq\lambda} = \mathfrak{G}^\lambda$ is smooth of relative dimension $\langle\lambda, 2\rho\rangle$ over U, then

$$E_{V_\lambda} \cong \mathbf{R}^{\langle\lambda,2\rho\rangle}\pi^\lambda_!\mathrm{ev}^*_{S,\mathcal{E}}\mathcal{K}_S.$$

4.5. **Computing local systems – general case.** Now we drop the condition that G be simply-connected (still assumed to be split semisimple) and the automorphism groups of the relevant points be trivial. The discussion in the general case is a bit technical.

4.5.1. *Twisted representations.* Let Γ be a group and $\xi \in Z^2(\Gamma, \overline{\mathbb{Q}}_\ell^\times)$ a cocycle such that $\xi_{1,\gamma} = \xi_{\gamma,1} = 1$ for all $\gamma \in \Gamma$. A ξ-twisted representation of Γ is a finite-dimensional $\overline{\mathbb{Q}}_\ell$-vector space V with automorphisms $T_\gamma : V \to V$, one for each $\gamma \in \Gamma$, such that $T_1 = \mathrm{id}_V$ and

$$T_{\gamma\delta} = \xi_{\gamma,\delta}T_\gamma T_\delta, \forall\gamma, \delta \in \Gamma.$$

Let $\mathrm{Rep}_\xi(\Gamma, \overline{\mathbb{Q}}_\ell)$ be the category of ξ-twisted representations of Γ. This is a $\overline{\mathbb{Q}}_\ell$-linear abelian category which, up to equivalence, only depends on the cohomology class $[\xi] \in \mathrm{H}^2(\Gamma, \overline{\mathbb{Q}}_\ell^\times)$.

A natural source of 2-cocycles on Γ come from rank one character sheaves. We did not define rank one character sheaves for disconnected groups but the same definition works, except now they may have nontrivial automorphisms. Isomorphism classes in $\mathrm{CS}_1(\Gamma)$ are in bijection with $\mathrm{H}^2(\Gamma, \overline{\mathbb{Q}}_\ell^\times)$.

Suppose $\mathcal{E} \in \mathrm{Bun}_G(\mathbf{K}_S)(\overline{k})$ be a relevant point for $(\mathbf{K}_S, \mathcal{K}_S)$. Let $A_\mathcal{E} = \mathrm{Aut}(\mathcal{E})$ and $\Gamma_\mathcal{E} = \pi_0(\mathrm{Aut}(\mathcal{E}))$. Since $\mathrm{ev}^*_{S,\ \mathcal{E}}\mathcal{K}_S$ is trivial on $\mathrm{Aut}^\circ(\mathcal{E})$, it descends to a rank one character sheaf on $\Gamma_\mathcal{E}$ and gives a cocycle $\xi \in Z^2(\Gamma_\mathcal{E}, \overline{\mathbb{Q}}_\ell^\times)$ satisfying $\xi_{1,\gamma} = \xi_{\gamma,1} = 1$ for all $\gamma \in \Gamma_\mathcal{E}$, whose cohomology class is well-defined.

4.5.2. **Lemma.** *Let* $\mathrm{Perv}_c(\overline{k}; \mathbf{K}_S, \mathcal{K}_S)_\mathcal{E}$ *be the category of perverse sheaves in* $\mathrm{Perv}_c(\overline{k}; \mathbf{K}_S, \mathcal{K}_S)$ *that have vanishing stalks outside the preimage of the relevant point $\mathcal{E}$. Then* $\mathrm{Perv}_c(\overline{k}; \mathbf{K}_S, \mathcal{K}_S)_\mathcal{E} \cong \mathrm{Rep}_\xi(\Gamma_\mathcal{E}, \overline{\mathbb{Q}}_\ell)$.

Sketch of proof. We base change the spaces to $\overline{k}$ without changing notation. We can find a finite isogeny $\nu : \widetilde{L}_S \to L_S$ with discrete kernel C, and a character $\chi_C : C \to \overline{\mathbb{Q}}_\ell^\times$ such that the local system $\mathcal{K}_S$ on L_S is of the form

$$\mathcal{K}_S \cong (\nu_*\overline{\mathbb{Q}}_\ell)_{\chi_C} \in \mathrm{CS}_1(L_S).$$

Let $\widetilde{A}_\mathcal{E}$ be the pullback of the cover ν along $\mathrm{ev}_{S,\mathcal{E}} : A_\mathcal{E} \to L_S$. Let $\widetilde{\Gamma}_\mathcal{E} = \pi_0(\widetilde{A}_\mathcal{E})$. Then $\Gamma_\mathcal{E}$ fits into an exact sequence

$$1 \to C \to \widetilde{\Gamma}_\mathcal{E} \to \Gamma_\mathcal{E} \to 1.$$

The pushout of the sequence along $\chi_C : C \to \overline{\mathbb{Q}}_\ell^\times$ gives a 2-cocycle $\xi \in Z^2(\Gamma_\mathcal{E}, \overline{\mathbb{Q}}_\ell^\times)$ (upon choosing a set-theoretic splitting $s : \Gamma_\mathcal{E} \to \widetilde{\Gamma}_\mathcal{E}$, so well-defined up to coboundaries). Let $\mathrm{Rep}^{\chi_C}(\widetilde{\Gamma}_\mathcal{E}, \overline{\mathbb{Q}}_\ell)$

be the category of finite-dimensional $\overline{\mathbb{Q}}_\ell$-representations of $\widetilde{\Gamma}_\alpha$ whose restriction to C is χ_C.. We have an equivalence

$$\mathrm{Rep}^{\chi_C}(\widetilde{\Gamma}_{\mathcal{E}}, \overline{\mathbb{Q}}_\ell) \xrightarrow{\sim} \mathrm{Rep}_\xi(\Gamma_{\mathcal{E}}, \overline{\mathbb{Q}}_\ell)$$

by restriction along s.

The preimage of $\mathcal{E}$ in $\mathrm{Bun}_G(\mathbf{K}_S^+)$ is isomorphic to $L_S/A_{\mathcal{E}} \cong \widetilde{L}_S/\widetilde{A}_{\mathcal{E}}$ where $A_{\mathcal{E}}$ acts on L_S via $\mathrm{ev}_{S,\mathcal{E}}$ and right translation. For $V \in \mathrm{Rep}_\xi(\Gamma_{\mathcal{E}}, \overline{\mathbb{Q}}_\ell)$ viewed as a representation of $\widetilde{\Gamma}_{\mathcal{E}}$ with restriction χ_C to C, we get a $\widetilde{L}_S$-equivariant local system $\mathcal{F}_V$ on $\widetilde{L}_S/\widetilde{A}_{\mathcal{E}} \cong L_S/A_{\mathcal{E}}$ (since $\widetilde{\Gamma}_{\mathcal{E}} = \pi_0(\widetilde{A}_{\mathcal{E}})$). Let $i : L_S/A_{\mathcal{E}} \hookrightarrow \mathrm{Bun}_G(\mathbf{K}_S, \mathcal{K}_S)$, then up to a shift $i_!\mathcal{F}_V$ is the object in $\mathrm{Perv}_c(\overline{k}; \mathbf{K}_S, \mathcal{K}_S)_{\mathcal{E}}$ corresponding to V. □

4.5.3. *General case.* Let

$$\mathrm{Bun}_G(\mathbf{K}_S) = \coprod_{\alpha \in \Omega} \mathrm{Bun}_G(\mathbf{K}_S)_\alpha$$

be the decomposition into connected components. Here $\Omega = \mathbb{X}_*(T)/\mathbb{Z}\Phi^\vee$ is the algebraic π_1 of G. Let $D_c(\overline{k}; \mathbf{K}_S, \mathcal{K}_S)_\alpha \subset D_c(\overline{k}; \mathbf{K}_S, \mathcal{K}_S)$ be the full subcategory of sheaves supported on $\mathrm{Bun}_G(\mathbf{K}_S)_\alpha$. Similarly define the abelian category of perverse sheaves $\mathrm{Perv}_c(\overline{k}; \mathbf{K}_S, \mathcal{K}_S)_\alpha$.

Let $\mathcal{E}_\alpha$ be the unique relevant $\overline{k}$-point in $\mathrm{Bun}_G(\mathbf{K}_S)_\alpha$. Let $A_\alpha = \mathrm{Aut}(\mathcal{E}_\alpha)$ (an algebraic group over $\overline{k}$), and $\Gamma_\alpha = \pi_0(A_\alpha)$ as a discrete group.

Define ${}_\beta\mathfrak{G}_\alpha \to U$ whose fiber over $x \in U$ parametrize isomorphisms of G-torsors $\mathcal{E}_\alpha|_{X-\{x\}} \xrightarrow{\sim} \mathcal{E}_\beta|_{X-\{x\}}$ preserving the $\mathbf{K}_S$-level structures. This is a right torsor under the group ind-scheme ${}_\alpha\mathfrak{G}_\alpha$ and a left torsor under ${}_\beta\mathfrak{G}_\beta$. In particular, A_α acts on ${}_\beta\mathfrak{G}_\alpha$ from the right and A_β acts from the left. Upon choosing trivializations of $\mathcal{E}_\alpha$ and $\mathcal{E}_\beta$ near S, we have the evaluation map

$${}_\alpha\mathrm{ev}_{S,\beta} : {}_\alpha\mathfrak{G}_\beta \to \prod_{x \in S} \mathbf{K}_x \otimes \overline{k} \to L_S \otimes \overline{k}.$$

Choose $(\widetilde{L}_S, \chi_C)$ as in the proof of Lemma 4.5.2. Pulling back the covering $\nu : \widetilde{L}_S \to L_S$ along ${}_\alpha\mathrm{ev}_{S,\beta}$ we get C-torsors ${}_\alpha\widetilde{\mathfrak{G}}_\beta \to {}_\alpha\mathfrak{G}_\beta$ and central extensions $1 \to C \to \widetilde{A}_\alpha \to A_\alpha \to 1$ and $1 \to C \to \widetilde{\Gamma}_\alpha \to \Gamma_\alpha \to 1$.

Then by Lemma 4.5.2 we have an equivalence of categories

$$\mathrm{Perv}_c(\overline{k}; \mathbf{K}_S, \mathcal{K}_S)_\alpha \cong \mathrm{Rep}_{\xi_\alpha}(\Gamma_\alpha, \overline{\mathbb{Q}}_\ell) \cong \mathrm{Rep}^{\chi_C}(\widetilde{\Gamma}_\alpha, \overline{\mathbb{Q}}_\ell). \tag{4.7}$$

where the last category consists of finite-dimensional $\overline{\mathbb{Q}}_\ell$-representations of $\widetilde{\Gamma}_\alpha$ whose restriction to C is χ_C.

We have an evaluation map

$${}_\alpha\mathrm{ev}_{U,\beta} : {}_\alpha\mathfrak{G}_\beta \to \left[\frac{L^+G\backslash LG/L^+G}{\mathrm{Aut}^+}\right]$$

recording the modification at $x \in U$. Let ${}_\alpha\mathfrak{G}_\beta^{\leq\lambda}$ be the preimage of $L^+G\backslash\mathrm{Gr}_{\leq\lambda}$ under the above map, then ${}_\alpha\mathfrak{G}_\beta^{\leq\lambda} \neq \varnothing$ only when λ has image $\beta - \alpha \in \Omega$. The pullback ${}_\alpha\mathrm{ev}_{U,\beta}^*\mathrm{IC}_\lambda$ is supported on ${}_\alpha\mathfrak{G}_\beta^{\leq\lambda}$. Let ${}_\alpha\widetilde{\mathrm{ev}}_{U,\beta}$ be the precomposition of ${}_\alpha\mathrm{ev}_{U,\beta}$ with ${}_\alpha\widetilde{\mathfrak{G}}_\beta \to {}_\alpha\mathfrak{G}_\beta$. We also have ${}_\alpha\widetilde{\mathrm{ev}}_{U,\beta}^*\mathrm{IC}_\lambda$ supported on ${}_\alpha\widetilde{\mathfrak{G}}_\beta^{\leq\lambda}$.

Let ${}_\alpha\widetilde{\pi}_\beta^\lambda : {}_\alpha\widetilde{\mathfrak{G}}_\beta \to U$ be the projection. Let Σ_α be the set of irreducible object in $\mathrm{Rep}^{\chi_C}(\widetilde{\Gamma}_\alpha, \overline{\mathbb{Q}}_\ell)$. For $\alpha \in \Omega$, let $\mathcal{F}_\eta \in \mathrm{Perv}_c(\overline{k}; \mathbf{K}_S, \mathcal{K}_S)_\alpha$ corresponding to an irreducible object $\eta \in \Sigma_\alpha$ under (4.7).

4.5.4. **Proposition.** *Let $(\mathbf{K}_S, \mathcal{K}_S)$ be geometrically rigid. Then for a dominant coweight λ with image $\beta - \alpha$ in Ω, we have*

$$\mathbb{T}_V(\mathcal{F}_\eta \boxtimes \overline{\mathbb{Q}}_\ell) \cong \oplus_{\zeta \in \Sigma_\beta} \mathcal{F}_\zeta \boxtimes {}_\zeta E^\lambda_\eta$$

where ${}_\zeta E^\lambda_\eta$ is the local system on U given by

$$(4.8) \qquad {}_\zeta E^\lambda_\eta \cong \mathrm{Hom}_{\widetilde{\Gamma}_\alpha \times \widetilde{\Gamma}_\beta}(\eta \boxtimes \zeta^\vee, {}_\alpha\widetilde{\pi}^\lambda_{\beta !}{}_\alpha\widetilde{\mathrm{ev}}^*_\beta \mathrm{IC}_\lambda)[-1].$$

If $\mathrm{Aut}(\mathcal{E}_\alpha)$ is trivial for all α, there is a unique simple perverse sheaf $\mathcal{F}_\alpha \in D_c(\overline{k}; \mathbf{K}_S, \mathcal{K}_S)_\alpha$. The sum of $\mathcal{F}_\alpha$ is a Hecke eigensheaf whose eigen $\widehat{G}$-local system E is described as follows. For a dominant coweight λ with image $\beta - \alpha$ in Ω, we have

$$E_{V_\lambda} \cong {}_\alpha\pi^\lambda_{\beta !}({}_\alpha\mathrm{ev}^*_{S,\beta}\mathcal{K}_S \otimes {}_\alpha\mathrm{ev}^*_\beta \mathrm{IC}_\lambda)[-1].$$

When λ is minuscule we have

$$(4.9) \qquad E_{V_\lambda} \cong \mathbf{R}^{\langle \lambda, 2\rho \rangle}{}_\alpha\pi^\lambda_{\beta !}\alpha\mathrm{ev}^*_{S,\beta}\mathcal{K}_S.$$

4.6. **Kloosterman local systems.** We apply the method above to the Kloosterman automorphic datum in Example 2.1.5 in the case $G = \mathrm{PGL}_n$. We consider only the case $\lambda = (1, 0, \cdots, 0)$, so V_λ is the standard representation of $\widehat{G} = \mathrm{SL}_n$. So we are looking for an $\widehat{G} = \mathrm{SL}_n$-local system on $U = \mathbb{P}^1 - \{0, \infty\} \cong \mathbb{G}_m$. The answer turns out to be a classical object.

4.6.1. *Calculation of $\mathfrak{G}^{\leq\lambda}$.* The moduli stack $\mathrm{Bun}_{\mathrm{GL}_n}(\mathbf{K}_S)$ classifies equivalence classes of $(\mathcal{E}, F^*, F_*,)$ where

1. $\mathcal{E}$ is a rank n vector bundle over $\mathbb{P}^1$;
2. $F^* : (0 = F^n \subset F^{n-1} \subset \cdots \subset F^0 = \mathcal{E}|_0)$ is a complete flag of the fiber of $\mathcal{E}$ at 0;
3. $F_* : (0 = F_0 \subset F_1 \subset \cdots \subset F_n = \mathcal{E}|_\infty)$ is a complete flag of the fiber of $\mathcal{E}$ at ∞;
4. For $1 \leq i \leq n$, $v_i \in F_i/F_{i-1}$ is a nonzero vector.

Let Pic_∞ be the moduli space of line bundle on $\mathbb{P}^1$ together with a trivialization at ∞. We have a canonical isomorphism $\mathrm{Pic}_\infty \cong \mathbb{Z}$ by taking degrees. Then Pic_∞ acts on $\mathrm{Bun}_{\mathrm{GL}_n}(\mathbf{K}_S)$ by tensoring, and $\mathrm{Bun}_G(\mathbf{K}_S) = \mathrm{Bun}_{\mathrm{GL}_n}(\mathbf{K}_S)/\mathrm{Pic}_\infty$.

The components of $\mathrm{Bun}_G(\mathbf{K}_S)$ are parametrized by $\deg \mathcal{E} \mod n \in \mathbb{Z}/n\mathbb{Z}$. Let $\mathcal{E}_i$ be the unique relevant point of degree $i \in \mathbb{Z}_{n\mathbb{Z}}$. Then we have the following description:

1. $\mathcal{E}_0 = \mathcal{O}e_1 \oplus \mathcal{O}e_2 \oplus \cdots \oplus \mathcal{O}e_n$, $F^i = \mathrm{Span}\{e_{i+1}, \cdots, e_n\}$, $F_i = \mathrm{Span}\{e_1, \cdots, e_i\}$, $v_i = e_i \mod F_{i-1}$.
2. $\mathcal{E}_1 = \mathcal{O}e_1 \oplus \mathcal{O}e_2 \oplus \cdots \oplus \mathcal{O}(1)e_n$. Here $\mathcal{O}(1)$ is equipped with a trivialization at ∞, so we identify it with $\mathcal{O}(\{0\})$. Take $F^i = \mathrm{Span}\{e_i, \cdots, e_{n-1}\}$ if $i \geq 1$, $F_i = \mathrm{Span}\{e_1, \cdots, e_i\}$, and $v_i = e_i \mod F_{i-1}$.

A point in $\mathfrak{G}^{\leq\lambda}$ is an injective map of coherent sheaves $f : \mathcal{E}_0 \to \mathcal{E}_1$ preserving F^*, F_* and $\{v_i\}$. We may write f as a matrix (a_{ij}) where $a_{ij} \in k$ if $i < n$, and $a_{nj} \in \Gamma(\mathbb{P}^1, \mathcal{O}(1)) = k \oplus k\tau$. The fact that f preserve F^*, F_* and $\{v_i\}$ imply

1. $(a_{ij})|_{\tau=0}$ is upper triangular with 1 on the diagonal. This implies (a_{ij}) has the form

$$\begin{pmatrix} 1 & * & * & * \\ 0 & \ddots & * & * \\ 0 & 0 & 1 & * \\ a'_{n1}\tau & \cdots & a'_{n,n-1}\tau & 1 + a'_{nn}\tau \end{pmatrix}$$

(2) Let $a'_{ij} = a_{ij}$ if $i < n$ and let a'_{nj} be as above. Then (a'_{ij}) sends the flag e_i to $\mathrm{Span}\{e_{i-1}, \cdots, e_{n-1}\}$ for $i = 2, \cdots, n$. This implies (a_{ij}) has the form

$$\begin{pmatrix} 1 & a_1 & 0 & 0 \\ 0 & \ddots & * & 0 \\ 0 & 0 & 1 & a_{n-1} \\ a_0\tau & \cdots & 0 & 1 \end{pmatrix}$$

Moreover, $a_0, \cdots, a_{n-1}$ are nonzero since they are the action of f on the associated graded of the filtrations F^* at 0.

We get an isomorphism $\mathfrak{G}^{\leq\lambda} \cong \mathbb{G}_m^n$ with coordinates $(a_0, \cdots, a_{n-1})$. The map $\pi : \mathfrak{G}^{\leq\lambda} \to U$ sends f to the support of the cokernel of f. Since $\det(f) = 1 + (-1)^{n-1}a_0 \cdots a_{n-1}t^{-1} \in \Gamma(\mathbb{P}^1, \mathcal{O}(1))$, we see that $\pi(f) = (-1)^n a_0 a_1 \cdots a_{n-1}$.

For simplicity assume $\mathcal{K}_0$ is trivial. By (4.9), we get that ($\mathrm{St} = V_\lambda$ is the standard representation of $\widehat{G} = \mathrm{SL}_n$)

$$E_{\mathrm{St}} \cong \mathbf{R}^{n-1}\pi_!\phi^*\mathrm{AS}_\psi \tag{4.10}$$

where we use ϕ to denote the composition $\mathfrak{G}^{\leq\lambda} \cong \mathbb{G}_m^n \xrightarrow{\mathrm{ev}_S} \prod_{i=0}^{n-1} U_{\alpha_i} \xrightarrow{\varphi} \mathbb{G}_a$; it is linear in each coordinate a_i with nonzero coefficient.

4.6.2. *The classical Kloosterman sheaf.* We first recall the definition of Kloosterman sums. Let p be a prime number. Fix a nontrivial additive character $\psi : \mathbb{F}_p \to \overline{\mathbb{Q}}_\ell^\times$. Let $n \geq 2$ be an integer. Then the n-variable Kloosterman sum over $\mathbb{F}_p$ is a function on $\mathbb{F}_p^\times$ whose value at $a \in \mathbb{F}_p^\times$ is

$$\mathrm{Kl}_n(p; a) = \sum_{x_1, \cdots, x_n \in \mathbb{F}_p^\times; x_1 x_2 \cdots x_n = a} \psi(x_1 + \cdots + x_n).$$

These exponential sums arise naturally in the study of automorphic forms for GL_n.

Deligne [6] gave a geometric interpretation of the Kloosterman sum. He considered the following diagram of schemes over $\mathbb{F}_p$

$$\begin{tikzcd} & \mathbb{G}_m^n \arrow[dl, "\pi"'] \arrow[dr, "\varphi"] & \\ \mathbb{G}_m & & \mathbb{A}^1 \end{tikzcd}$$

Here π is the morphism of taking the product and φ is the morphism of taking the sum. He defines the *Kloosterman sheaf* to be

$$\mathrm{Kl}_n := \mathbf{R}^{n-1}\pi_!\varphi^*\mathrm{AS}_\psi,$$

over $\mathbb{G}_m = \mathbb{P}^1_{\mathbb{F}_p} - \{0, \infty\}$. Up to a change of coordinates, this is essentially (4.10). The relationship between the local system Kl_n and the Kloosterman sum $\mathrm{Kl}_n(p; a)$ is explained by the following identity

$$\mathrm{Kl}_n(p; a) = (-1)^{n-1}\mathrm{Tr}(\mathrm{Frob}_a, (\mathrm{Kl}_n)_a).$$

Here Frob_a is the geometric Frobenius operator acting on the geometric stalk $(\mathrm{Kl}_n)_a$ of Kl_n at $a \in \mathbb{G}_m(\mathbb{F}_p) = \mathbb{F}_p^\times$.

4.6.3. *Properties of Kloosterman local systems.* The following properties of Kl_n were proved by Deligne.

(0) Kl_n is a local system of rank n.
(1) Kl_n is tamely ramified at 0, and the monodromy is unipotent with a single Jordan block.
(2) Kl_n is totally wild at ∞ (i.e., the wild inertia at ∞ has no nonzero fixed vector on the stalk of Kl_n), and the Swan conductor $\mathrm{Sw}_\infty(\mathrm{Kl}_n) = 1$.

Applying a special case of Theorem 4.3.3 to the Kloosterman automorphic data, we get $\widehat{G}$-local systems $\mathrm{Kl}_{\widehat{G}}(\chi, \varphi)$. In [18], we show that $\mathrm{Kl}_{\widehat{G}}(\chi, \varphi)$ enjoy analogous properties as Kl_n. These properties were predicted by Gross [14], Frenkel–Gross [10]. For example, when $\chi = 1$ we prove:

(1) $\mathrm{Kl}_{\widehat{G}}(1, \varphi)$ is tame at 0, and a generator of the tame inertia maps to a regular unipotent element in $\widehat{G}$.
(2) The local monodromy of $\mathrm{Kl}_{\widehat{G}}(1, \varphi)$ at ∞ is a *simple wild parameter* in the sense of Gross and Reeder [16, §5].

5. Rigidity for local systems; Applications

This section is likely not to be covered in the lectures. We compare the notion of rigidity for automorphic data and for local systems. We also mention some applications of the rigidity automorphic datum and some open problems.

5.1. Rigid $\widehat{G}$-local systems.

We shall review the notion of rigidity for $\widehat{G}$-local systems, introduced by Katz [22] for $\widehat{G} = \mathrm{GL}_n$. We assume the base field k to be **algebraically closed** in this subsection. Let X be a complete smooth connected algebraic curve over k. Fix an open subset $U \subset X$ with finite complement S.

5.1.1. *Physical rigidity.*

5.1.2. **Definition** (extending Katz [22, §1.0.3])**.** Let E be an $\widehat{G}$-local system on U. Then E is called *physically rigid* if, for any other $\widehat{G}$-local system E', $E'|_{\mathrm{Spec} F_x} \cong E|_{\mathrm{Spec} F_x}$ for all $x \in S$ implies $E' \cong E$.

Although the definition uses U as an input, the notion of physical rigidity is in fact independent of the open subset U: for any nonempty open subset $V \subset U$, ρ is rigid over U if and only if $E|_V$ is rigid over V. Therefore, physical rigidity is a property of the Galois representation $\rho_E : \mathrm{Gal}(F^{\mathrm{sep}}/F) \to \widehat{G}(\overline{\mathbb{Q}}_\ell)$ obtained by restricting E to a geometric generic point η of the X.

5.1.3. *Cohomological rigidity.* Next we introduce cohomological rigidity. Let $\widehat{\mathfrak{g}}$ be the Lie algebra of $\widehat{G}$, and let $\mathrm{Ad}(E)$ be local system $E_{\widehat{\mathfrak{g}}}$, viewing $\widehat{\mathfrak{g}}$ as the adjoint representation of $\widehat{G}$. Let $j : U \hookrightarrow X$ be the open embedding and $j_{!*}\mathrm{Ad}(E)$ be the non-derived direct image of $\mathrm{Ad}(E)$ along j. Concretely, the stalk of $j_{!*}\mathrm{Ad}(E)$ at $x \in S$ is the $\mathcal{I}_x$-invariants on $\widehat{\mathfrak{g}}$.

5.1.4. **Definition** (extending Katz [22, §5.0.1])**.** A $\widehat{G}$-local system E on U is called *cohomogically rigid*, if

$$\tau(E) := \mathrm{H}^1(X, j_{!*}\mathrm{Ad}(E)) = 0.$$

The vector space $\tau(E)$ does not change if we shrink U to a smaller open subset. Therefore cohomological rigidity is also a property of the Galois representation $\rho_E : \mathrm{Gal}(F^{\mathrm{sep}}/F) \to \widehat{G}(\overline{\mathbb{Q}}_\ell)$.

5.1.5. **Remark.** When we work over the base field $\mathbb{C}$ and view U as a topological surface, one can define a moduli stack $\mathcal{M}$ of $\widehat{G}$-local systems over U with prescribed local monodromy around the punctures S. Then $\tau(E)$ is the Zariski tangent space of $\mathcal{M}$ at E. The condition $\tau(E) = 0$ in this topological setting says that E does not admit infinitesimal deformations with prescribed local monodromy around S. This interpretation is the motivation for Definition 5.1.4.

5.1.6. **Remark.** An alternative approach to define the notion of rigidity for a local system E over U over a finite field k is by requiring that the adjoint L-function of E to be trivial (constant function 1). This is the approach taken by Gross in [15]. When $\mathrm{H}^0(U_{\overline{k}}, \mathrm{Ad}(E)) = 0$, triviality of the adjoint L-function of E is equivalent to cohomological rigidity of E.

Using the Grothendieck-Ogg-Shafarevich formula, it is easy to give the following numerical criterion for cohomological rigidity: E is cohomologically rigid if and only if

$$\frac{1}{2}\sum_{x\in S} a_x(\mathrm{Ad}(E)) = (1-g_X)\dim \widehat{G} - \dim \mathrm{H}^0(U, \mathrm{Ad}(E)). \tag{5.1}$$

Here $a_x(\mathrm{Ad}(E))$ is the Artin conductor of $\mathrm{Ad}(E)$ at x, see §3.2.3, and g_X is the genus of X.

From (5.1) we see that cohomologically rigid $\widehat{G}$-local systems exist only when $g_X \le 1$. When $g_X = 1$, rigid examples are very limited (see [22, §1.4]).

Most examples of rigid local systems are over open subsets of $X = \mathbb{P}^1$. In this case, if we further assume $\mathrm{H}^0(U, \mathrm{Ad}(E)) = 0$, then

$$\frac{1}{2}\sum_{x\in S} a_x(\mathrm{Ad}(E)) = \dim \widehat{G}.$$

Compare with (3.1), it is natural to expect that if E comes as the Hecke eigen local system of a weakly rigid geometric automorphic datum $(\mathbf{K}_x, \mathcal{K}_x)$, then

$$\frac{1}{2}a_x(\mathrm{Ad}(E)) \overset{?}{=} [\mathfrak{g}(\mathcal{O}_x) : \mathfrak{k}_x], \quad \forall x \in S.$$

When $\widehat{G} = \mathrm{SL}_n$ and the local system is irreducible, the two notions of rigidity are equivalent.

5.1.7. **Theorem.** *Let E be an irreducible rank n $\overline{\mathbb{Q}}_\ell$-local system on $U = X - S$ with trivial determinant, viewed as an* SL_n*-local system.*

(1) *(Katz* [22, Theorem 5.0.2]*) If E is cohomological rigid, then it is physical rigid.*
(2) *(L.Fu* [11, Theorem 0.1]*) If E is physical rigid, then it is cohomological rigid.*

5.2. **Applications of rigid automorphic datum.** We shall give three applications of the rigid objects in the Langlands correspondence.

5.2.1. *Local systems with exceptional monodromy groups.* Katz [21] has constructed an example of a local system over $\mathbb{P}^1_{\mathbb{F}_p} - \{0, \infty\}$ whose geometric monodromy lies in the exceptional group G_2 and is Zariski dense there. This G_2-local system comes from a rank 7 rigid local system which is an example of Katz's hypergeometric sheaves.

The work [18], inspired by the work of Gross [14] and Frenkel–Gross [10], gives the first examples of local systems (coming from geometry) with Zariski dense monodromy in other exceptional groups F_4, E_7 and E_8 in a uniform way. The Zariski closure of the geometric monodromy of $\mathrm{Kl}_{\widehat{G}}(1, \varphi)$ is a connected simple subgroup of $\widehat{G}$ of types given by the following table (assuming $p \neq 2, 3$)

$\widehat{G}$	geometric monodromy
A_{2n}	A_{2n}
A_{2n-1}, C_n	C_n
B_n, D_{n+1} $(n \geq 4)$	B_n
E_7	E_7
E_8	E_8
E_6, F_4	F_4
B_3, D_4, G_2	G_2

5.2.2. *Motives over number fields with exceptional motivic Galois groups.* In early 1990s, Serre asked the following question [30]: Is there a motive over a number field whose motivic Galois group is of exceptional type such as G_2 or E_8?

A motive M over a number field K is, roughly speaking, part of the cohomology $\mathrm{H}^i(X)$ for some (smooth projective) algebraic variety X over K and some integer i, which is cut out by geometric operations (such as group actions). For each prime ℓ, the motive M has the associated ℓ-adic cohomology $\mathrm{H}_\ell(M) \subset \mathrm{H}^i(X_{\overline{K}}, \overline{\mathbb{Q}}_\ell)$, which admits a Galois action:

$$\rho_{M,\ell} : \mathrm{Gal}(\overline{K}/K) \to \mathrm{GL}(\mathrm{H}_\ell(M))$$

The *ℓ-adic motivic Galois group* $G_{M,\ell}$ of M is the Zariski closure of the image of $\rho_{M,\ell}$. This is an algebraic group over $\overline{\mathbb{Q}}_\ell$. Since the motivic Galois groups that appear in the original question of Serre are only well-defined assuming the standard conjectures in algebraic geometry, we shall use the ℓ-adic motivic Galois group as a working substitute for the actual motivic Galois group (conjecturally they should be isomorphic to each other). Classical groups appear as motivic Galois groups of abelian varieties. This is why Serre raised the question for exceptional groups only.

The G_2 case was answered affirmatively by Dettweiler and Reiter [8].

In [34], we construct motivic local systems on $\mathbb{P}^1_{\mathbb{Q}} - \{0, 1, \infty\}$ with Zariski dense monodromy in exceptional groups E_7, E_8 and G_2 in a uniform way. They arise as eigen $\widehat{G}$-local systems from the geometric automorphic datum in Example 3.3.5. As a consequence of this construction, we give an affirmative answer to the ℓ-adic version of Serre's question for E_7, E_8 and G_2. With a bit more work, one can also realize F_4 as a motivic Galois group over $\mathbb{Q}$ (unpublished).

Recently, E_6 has also been realized as a motivic Galois group by Boxer–Calegari–Emerton–Levin–Madapusi-Pera–Patrikis [4].

We remark that in the case of E_8, no connections are known between E_8-motives and Shimura varieties. The example given by the rigidity method in [34] seems to be the only approach, even if one assumes knowledge about cohomology of Shimura varieties.

5.2.3. *Inverse Galois Problem.* The inverse Galois problem over $\mathbb{Q}$ asks whether every finite group can be realized as the Galois group of some Galois extension $K/\mathbb{Q}$. The problem is still open for many finite simple groups, especially those of Lie type.

The same rigid local systems over $\mathbb{P}^1_{\mathbb{Q}} - \{0, 1, \infty\}$ constructed to answer Serre's question can be used to solve new cases of the inverse Galois problem. We show in [34] that for sufficiently large primes ℓ, the finite simple groups $G_2(\mathbb{F}_\ell)$ and $E_8(\mathbb{F}_\ell)$ can be realized as Galois groups over $\mathbb{Q}$. With a bit more work, one can show that $F_4(\mathbb{F}_\ell)$ is also a Galois group over $\mathbb{Q}$.

In inverse Galois theory, people use the "rigidity method" to prove certain finite groups H are Galois groups over $\mathbb{Q}$. This will be reviewed in the next subsection. In particular, the case of $G_2(\mathbb{F}_\ell)$ for all primes $\ell \geq 5$ was known by the work of Thompson [32] and Feit and Fong [9]. However, the case of $E_8(\mathbb{F}_\ell)$ was known previously only for primes ℓ satisfying a certain congruence condition modulo 31 (see the book of Malle and Matzat [27, II.10] for a summary of what was known before).

The E_8-local system contructed in [34] also sheds some light to the rigidity method in inverse Galois theory. In fact, in [34, Conjecture 5.16] we suggest a rigid triple for $E_8(\mathbb{F}_\ell)$, which was subsequently proved by Guralnick and Malle [17]. Their result shows that $E_8(\mathbb{F}_\ell)$ is a Galois group over $\mathbb{Q}$ for all primes $\ell > 7$.

5.3. **Rigidity in the inverse Galois theory.** It is instructive to compare the notion of rigidity for local systems with the notion of a rigid tuple in inverse Galois theory. We give a quick review following [31, Chapter 8]. Let H be a *finite* group with trivial center.

5.3.1. **Definition.** A tuple of conjugacy classes $(C_1, C_2, \cdots, C_n)$ in H is called *(strictly) rigid*, if

- The equation
$$g_1 g_2 \cdots g_n = 1 \tag{5.2}$$
has a solution with $g_i \in C_i$, and the solution is unique up to simultaneous H-conjugacy;
- For any solutions $(g_1, \cdots, g_n)$ of (5.2), $\{g_i\}_{i=1,\cdots,n}$ generate H.

The connection between rigid tuples and local systems is given by the following theorem. Let $S = \{P_1, \cdots, P_n\} \subset \mathbb{P}^1(\mathbb{Q})$, and let $U = \mathbb{P}^1_{\mathbb{Q}} - S$.

5.3.2. **Theorem** (Belyi, Fried, Matzat, Shih, and Thompson)**.** *Let $(C_1, \cdots, C_n)$ be a rigid tuple in H. Then up to isomorphism there is a unique connected unramified Galois H-cover $\pi : Y \to U \otimes_{\mathbb{Q}} \overline{\mathbb{Q}}$ such that a topological generator of the (tame) inertia group at P_i acts on Y as an element in C_i.*

Furthermore, if each C_i is rational (i.e., C_i takes rational values for all irreducible characters of H), then the H-cover $Y \to U \otimes_{\mathbb{Q}} \overline{\mathbb{Q}}$ is defined over $\mathbb{Q}$.

From the above theorem we see that the notion of a rigid tuple is an analog of physical rigidity for H-local systems when the algebraic group H is a finite group.

Rigid tuples combined with the Hilbert irreducibility theorem solves the inverse Galois problem for H.

5.3.3. **Corollary.** *Suppose there exists a rational rigid tuple in H, then H can be realized as* $\mathrm{Gal}(K/\mathbb{Q})$ *for some Galois number field $K/\mathbb{Q}$.*

For a comprehensive survey of finite simple groups that are realized as Galois groups over $\mathbb{Q}$ using rigidity tuples, we refer the readers to the book [27] by Malle and Matzat.

6. Projects (by Konstantin Jakob and Zhiwei Yun)

6.1. **Project 1: a rigid local system with icosahedral monodromy.** The rotational symmetry group of an icosahedron is the alternating group A_5. As such it can be realized as a subgroup of $\mathrm{SO}(3) \cong \mathrm{PGL}(2)$. It is well-known that this alternating group is isomorphic to the group
$$G = \langle a, b, c \mid a^5 = b^2 = c^3 = abc = 1 \rangle.$$

It is the goal of this project to construct a rigid local system whose projective monodromy group is A_5. This local system will be a special case of Katz's hypergeometric sheaves. Using the above description in terms of generators and relations it is natural to look for a local system of rank 2 on $\mathbb{P}^1 \setminus \{0, 1, \infty\}$ whose local monodromy at the punctures corresponds to the generators a, b, c.

Let $S = \{0, 1, \infty\}$ and consider the moduli stack $\mathrm{Bun}_2(\mathbf{I}_S)$ of vector bundles on $X = \mathbb{P}^1$ with Iwahori level structure at every $x \in S$. We will also call these parabolic vector bundles. More concretely $\mathrm{Bun}_2(\mathbf{I}_S)$ classifies tuples $(\mathcal{E}, \ell_S)$ where $\mathcal{E}$ is a vector bundle of rank 2 on X and ℓ_S is a set of lines ℓ_x in the fiber $\mathcal{E}_x$ for $x \in S$. Let $\chi_x : \mathbf{I}_x \to \overline{\mathbb{Q}}_\ell^\times$ be three characters that are trivial on the pro-unipotent radical $\mathbf{I}_x^+$ of orders $2, 3$ and 5. Note that each χ_x factors through the torus $T \cong \mathbf{I}_x/\mathbf{I}_x^+$. Assuming that each $\mathbf{I}_x$ is the standard upper-triangular Iwahori subgroup we may write $\chi_x = (\chi_x^{(1)}, \chi_x^{(2)})$. We assume that $\prod_{x \in S} \chi_x^{(1)} \chi_x^{(2)} = 1$.

Up to the fact that we are working with GL_2 rather than SL_2, the automorphic datum considered here is essentially a special case of Example 2.1.4.

Step 1. There's an evaluation map $\mathrm{ev}_x : \mathrm{Aut}(\mathcal{E}, \ell_S) \to \mathbb{G}_m^2$ which sends an automorphism to the scalars through which it acts on ℓ_x and $\mathcal{E}_x/\ell_x$. A tuple $(\mathcal{E}, \ell_S)$ is *relevant* if and only if the character $e_x^* \chi_S$ is trivial on $\mathrm{Aut}(\mathcal{E}, \ell_S)^\circ$ (where $\chi_S = \prod_{x \in S} \chi_x$). The first step is to analyze tuples $(\mathcal{E}, \ell_S)$ to describe the relevant points in $\mathrm{Bun}_2(\mathbf{I}_S)$.

A parabolic bundle $(\mathcal{E}, \ell_S)$ is decomposable if $\mathcal{E} = \mathcal{E}_1 \oplus \mathcal{E}_2$ and for each $x \in S$ the line ℓ_x is contained in the fiber of one the of line bundles $\mathcal{E}_1$ or $\mathcal{E}_2$. In this situation formulate a condition on χ_S such that the character χ_S is non-trivial on $\mathrm{Aut}(\mathcal{E}, \ell_S)$.

Use the Grothendieck-Birkhoff decomposition for vector bundles to analyze the indecomposable case. The connected components of $\mathrm{Bun}_2(\mathbf{I}_S)$ are parametrized by the degree of $\mathcal{E}$. Conclude that there is a unique relevant point on each connected component of $\mathrm{Bun}_2(\mathbf{I}_S)$.

Step 2. Since there is a unique relevant point on each connected component, there is a rigid Hecke eigensheaf $\mathcal{A}$ on $\mathrm{Bun}_2(\mathbf{I}_S^+)$. We wish to compute the eigenvalue for the standard representation $V = \mathrm{St}$. It is enough to consider the restriction of $\mathcal{A}$ to the component $\mathrm{Bun}_2^0(\mathbf{I}_S^+)$ of parabolic bundles of degree 0. In our situation the local system E_V has the following geometric description.

Let $\mathcal{E} = \mathcal{O}e_1 \oplus \mathcal{O}e_2$ be the trivial bundle equipped with lines $\ell_0 = \langle e_1 \rangle$, $\ell_1 = \langle e_1 + e_2 \rangle$ and $\ell_\infty = \langle e_2 \rangle$ and $\mathcal{E}' = \mathcal{O}(1)e_1' \oplus \mathcal{O}_2' e_2'$ and lines ℓ_x' in the same way as before. These are the relevant bundles on $\mathrm{Bun}_2^0(\mathbf{I}_S^+)$ and $\mathrm{Bun}_2^1(\mathbf{I}_S^+)$. The support $\mathfrak{G}_V$ of the intersection cohomology sheaf IC_V on the relevant part of the Hecke stack classifies embeddings $\varphi : \mathcal{E} \hookrightarrow \mathcal{E}'$ which are isomorphisms at $x \in S$ and which send ℓ_x to ℓ_x' together with a point $y \in \mathbb{P}^1 \setminus \{0, 1, \infty\}$ where φ has a zero. The character χ_S defines a character sheaf $\mathcal{K}_S$ on T^3 which descends to the quotient $T^3/\Delta(\mathbb{G}_m)$ and the local system E_V is given by a push-pull procedure through the diagram

$$\begin{tikzcd} & \mathfrak{X}_V/Z(\mathrm{GL}_2) \arrow[dl, "\mathrm{ev}"'] \arrow[dr, "\pi"] & \\ T^3/\Delta(\mathbb{G}_m) & & \mathbb{P}^1 \setminus \{0, 1, \infty\} \end{tikzcd}$$

where π is projection to $\mathbb{P}^1 \setminus \{0, 1, \infty\}$ and ev is the evaluation map. Explicitly we have

$$E_V \cong \pi_! \mathrm{ev}^* \mathcal{K}_S.$$

Describe the space $\mathfrak{G}_V/Z(\mathrm{GL}_2)$ and the maps ev and π explicitly. Using this description, compute the Frobenius trace of the eigenvalue. Conclude that the eigenvalue is a hypergeometric sheaf in the sense of Katz with projective monodromy group A_5.

6.2. **Project 2: Deeper rigid situations.** In this project the focus is proving rigidity for a more complicated situation. Using Fourier transform one can show that there exists a rigid local system of rank $n + 1$ which is unramified outside 0 and ∞, whose local monodromy at 0 is unipotent with Jordan type $(2,1,\dots,1)$ and whose local monodromy at ∞ is irreducible and wildly ramified of slope $n/(n+1)$.

Starting from this, we want to guess the correct choice of level structure and prove rigidity. Unipotent monodromy at 0 suggests to have a parahoric level group at 0. We will take a standard parahoric corresponding to the Jordan type $(n-1,n)$ and we denote it by $\mathbf{Q}_0$. Note that this is the conjugate partition to $(2,1,\dots,1)$.

Step 1. Guess the level structure to construct the above rigid local system. At the tame point 0 the choice is given above. In contrast to the Kloosterman sheaf example, slope $n/(n+1)$ suggests to look at a deeper step of the Moy-Prasad filtration of the Iwahori subgroup to find the level group at ∞. Choose a generic linear functional ψ on $\mathbf{I}_{\frac{n}{n+1}}/\mathbf{I}_1$ and extend it to a larger group for which the numerical criterion for rigidity holds. For that recall that $[\mathbf{I}_r,\mathbf{I}_s]\subset\mathbf{I}_{r+s}$. Let $\mathfrak{i}=\mathrm{Lie}\mathbf{I}$. Using the identification

$$(\mathfrak{i}_{\frac{n}{n+1}}/\mathfrak{i}_1)^*\cong\mathfrak{i}_{-\frac{n}{n+1}}/\mathfrak{i}_{-\frac{n-1}{n+1}}\cong\mathfrak{i}_{\frac{1}{n+1}}/\mathfrak{i}_{\frac{2}{n+1}}$$

the functional ψ can be thought of as a regular semisimple element in $\mathfrak{i}=\mathrm{Lie}\mathbf{I}$. Describe its stabilizer and use it to define the correct level group $\mathbf{K}_\infty$ inside the pro-uniptent radical of a standard Iwahori subgroup at ∞. What difficulty do you encounter for even n?

Step 2. Let $G=\mathrm{GL}_n$. We want to prove rigidity by studying the space of automorphic forms with local behaviour prescribed by our above choice of level structure. Let F be the function field of $X=\mathbb{P}^1$, identified with $F=k(\!(t)\!)$ where t is a local coordinate at ∞, denote by $\mathcal{O}_x$ (resp. F_x) the local ring (resp local field) at x and let $\mathbb{A}_F$ be the ring of adles. We consider the space

$$\mathcal{F}=\mathrm{Fun}(G(F)\backslash G(\mathbb{A}_F)/(\mathbf{Q}_0\times\prod_{x\neq 0,\infty}G(\mathcal{O}_x)),\overline{\mathbb{Q}}_\ell)^{(\mathbf{K}_\infty,\psi)}$$

of $\overline{\mathbb{Q}}_\ell$-valued functions on which $\mathbf{K}_\infty$ acts through the character ψ. It is our goal to prove that $\mathcal{F}$ decomposes into a direct sum of 1-dimensional spaces parametetrized by $\Omega=X^*(ZG)$ (this group parametrizes the connected components of the moduli stack of vector bundles). Two key ingredients for this are 1-point uniformization and the Birkhoff decomposition. We let $\Gamma_0=\mathbf{Q}_0\cap G(k[t,t^{-1}])$ and identify $\mathcal{F}=\mathrm{Fun}(\Gamma_0\backslash G(F_\infty))^{(\mathbf{K}_\infty,\psi)}$ and decompose

$$G(F_\infty)=\coprod_{\tilde{w}\in W_{\mathbf{Q}}\backslash\widetilde{W}}\Gamma_0\tilde{w}\mathbf{I}^+$$

where $\widetilde{W}=W_{\mathrm{aff}}\ltimes\Omega$ is the extended affine Weyl group of G and $\mathbf{I}^+$ denotes the pro-unipotent radical of $\mathbf{I}$. Prove that any $f\in\mathcal{F}$ is uniquely determined by its values on $\omega\in\Omega$. Proceed in two steps. First prove that f has to vanish on the coset $\Gamma_0\tilde{w}\mathbf{I}^+$ for any $\tilde{w}\notin\Omega$ using the generic functional ψ. Then prove that for every $\omega\in\Omega$ and every $g\in\mathbf{I}^+\backslash\mathbf{K}_\infty$ there is an $x\in\Gamma_0$ such that $\mathrm{Ad}_{(\omega g)^{-1}}x\in\mathbf{K}_\infty$ and $\psi(\mathrm{Ad}_{(\omega g)^{-1}}x)\neq 0$, i.e. $\mathrm{Ad}_{(\omega g)^{-1}}x$ has a non-trivial root entry in $\mathbf{I}_{\frac{n}{n+1}}/\mathbf{I}_1$. Note that it is necessary that $x\in\Gamma_0\cap\mathbf{I}^+$. In that case

$$f(\omega g)=f(x\omega g)=f(\omega g\mathrm{Ad}_{(\omega g)^{-1}}x)=\psi(\mathrm{Ad}_{(\omega g)^{-1}}x)f(\omega g)$$

and we need to have $f(\omega g)=0$. Therefore f is non-zero only on $\Gamma_0\omega\mathbf{K}_\infty$ and there it is determined by the functional ψ and its value on ω.

Question. Let k and n be positive integers. Iterating Fourier transform, we can generalize the rigid local system described above to a rigid local system of rank $kn+1$ that has unipotent monodromy at 0 with Jordan type $(k+1, k, ..., k)$ (the k appearing $n-1$ times) and whose local monodromy at ∞ is irreducible and wildly ramified of slope $n/(kn+1)$. Can you guess the level structure in this more general situation?

6.3. **Project 3: Generalized Kloosterman sheaves for symplectic groups.** Let G be a geometrically almost simple reductive group over the finite field k and F be the completion at ∞ of the function field of $X = \mathbb{P}^1_k$. This project studies generalized Kloosterman sheaves constructed from an admissible parahoric subgroup $\mathbf{P} \subset G(F)$ and a functional ϕ on $V_{\mathbf{P}} = \mathbf{P}^+/\mathbf{P}^{++}$ that is stable for the action of the Levi quotient $L_{\mathbf{P}}$ on $V_{\mathbf{P}}$. Here $\mathbf{P}^+$ is the pro-unipotent radical of $\mathbf{P}$ and $\mathbf{P}^{++}$ ist the next step of the Moy-Prasad filtration. Types of admissible parahoric subgroups are parametrized by regular elliptic numbers for the Weyl group of G. These are orders of regular elliptic elements.

For a representation V of the dual group $\widehat{G}$ the generalized Kloosterman sheaf $\mathrm{Kl}^V_{\widehat{G},\mathbf{P}}(\phi)$ is a local system on $\mathbb{P}^1 \setminus \{0, \infty\}$ which is tamely ramified with unipotent monodromy at 0 and wildly ramified at ∞. In this project we want to compute $\mathrm{Kl}^V_{\widehat{G},\mathbf{P}}(\phi)$ for symplectic groups where V is the standard representation of the classical group in question.

We will take the rigidity and the existence of a family of Hecke eigensheaves over the subspace of stable functionals $V_{\mathbf{P}}^{*,\mathrm{st}}$ with eigenvalue $\mathrm{Kl}_{\widehat{G},\mathbf{P}}$ for granted. The sheaf $\mathrm{Kl}_{\widehat{G},\mathbf{P}}$ is a local system on $\mathbb{G}_m \times V_{\mathbf{P}}^{*,\mathrm{st}}$ and its restriction to the point of $V_{\mathbf{P}}^{*,\mathrm{st}}$ given by ϕ is the local system $\mathrm{Kl}^V_{\widehat{G},\mathbf{P}}(\phi)$ on $\mathbb{G}_m$. We'll focus on the geometry needed to describe $\mathrm{Kl}^V_{\widehat{G},\mathbf{P}}(\phi)$ explicitly.

Step 1. Let Bun be the moduli stack of bundles with $\mathbf{P}^{\mathrm{opp}}$-level structure at 0 and $\mathbf{P}^+$-level structure at ∞ and for Bun^+ change the level at ∞ to $\mathbf{P}^{++}$. The Hecke eigensheaf on $\mathrm{Bun}^+ \times V_{\mathbf{P}}^{*,\mathrm{st}}$ can be described as

$$\mathcal{A} = (j \times \mathrm{id}_{V_{\mathbf{P}}^{*,\mathrm{st}}})_! \mathrm{AS}$$

where $j : V_{\mathbf{P}} \hookrightarrow \mathrm{Bun}^+$ is the open embedding defined by the unique relevant point $\mathcal{E}$ (with trivial automorphism group) and AS is the pullback along the canonical pairing $\langle -, - \rangle : V_{\mathbf{P}} \times V_{\mathbf{P}}^{*,\mathrm{st}} \to \mathbb{G}_a$ of the Artin-Schreier sheaf on $\mathbb{G}_a$ corresponding to a fixed character of $\psi : k \to \overline{\mathbb{Q}}_\ell$.

Let $\mathfrak{G}$ be the group of automorphisms of $\mathcal{E}|_{X\setminus\{1\}}$ that preserve the level structures. Let λ be a dominant coweight of G and V_λ the irreducible representation of $\widehat{G}$ of heighest weight λ. On the subscheme $\mathfrak{G}_{\leq\lambda}$ of automorphisms with relative position close to 1 bounded by λ we have an intersection complex IC_λ pulled back from the affine Grassmannian. Using the evaluation map $\mathrm{ev}_\infty : \mathfrak{G}_{\leq\lambda} \to \mathbf{P}^+$ we obtain a map $f : \mathfrak{G}_{\leq\lambda} \to \mathbf{P}^+ \to V_{\mathbf{P}}$ by composing with the canonical projection.

Prove the following statement. Let $\mathrm{Four} : D^b_c(V_{\mathbf{P}}) \to D^b_c(V^*_{\mathbf{P}})$ be Fourier-Deligne transform. Then

$$\mathrm{Kl}^{V_\lambda}_{\widehat{G},\mathbf{P}} \cong \mathrm{Four}(f_! \mathrm{IC}_\lambda)|_{V^*_{\mathbf{P}}}.$$

Step 2. We want to apply the first step in the situation where G is a symplectic group. For that we can describe an admissible parahoric subgroup in terms of lattice chains. Let (W, ω) be a symplectic vector space of dimension $2n$ over k and let $G = \mathrm{Sp}(W \otimes F, \omega)$ with ω extended linearly. The admissible parahoric subgroups can in this case be described as follows. Let d be a divisor of n. Then $m = 2n/d$ is a regular elliptic number for G. We fix a decomposition

$$W = W_1 \oplus \cdots \oplus W_{n/d} \oplus W_{n/d+1} \ldots W_m$$

such that $\dim W_i = d$ and such that $\omega(W_i, W_j) \neq 0$ only if $i + j = m + 1$. Define a parahoric $\mathbf{P}_m \subset G(F)$ to be the stabilizer of the lattice chain $\Lambda_m \supset \Lambda_{m-1} \cdots \supset \Lambda_1$ where $\Lambda_i = \sum_{1\leq j\leq i} W_j \otimes \mathcal{O}_F + \sum_{i<j\leq m} W_j \otimes \varpi\mathcal{O}_F$ and ϖ is a uniformizer for $\mathcal{O}_F$.

Describe $V_m = V_{\mathbf{P}_m}$ in terms of a cyclic quiver whose nodes are the spaces $W_1, \dots, W_m$ and describe the stable locus in V_m^* in terms of this quiver. Let λ be the dominant short coroot corresponding to the standard representation of $\widehat{G} = \mathrm{SO}_{2n+1}$. Let $\mathrm{Sym}^2(W)_{\leq 1} \subset \mathrm{Sym}^2(W)$ be the subscheme of symmetric pure 2-tensors. Show that the scheme $\mathfrak{G}_{\leq\lambda}$ can be embedded as an open subscheme of $\mathrm{Sym}^2(W)_{\leq 1}$ whose complement is an explicit divisor. Finally, using Step 1 show that there's an explicit map $f_\phi : \mathbb{G}_m \times \mathfrak{G}_{\leq\lambda} \to \mathbb{A}^1$ such that for the projection $\pi : \mathbb{G}_m \times \mathfrak{G}_\lambda \to \mathbb{G}_m$ we have

$$\mathrm{Kl}^{\mathrm{St}}_{\widehat{G},\mathbf{P}}(\phi) \cong \pi_! f_\phi^* \mathrm{AS}_\psi[2n-1](\frac{2n-1}{2}).$$

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