Goal

Methods from complex dynamical systems

Intersections - both unlikely & likely

- currents, plurisubharmonic functions (height functions)
- equidistribution results.

Ex~~ample~~ 0

$$f : \mathbb{P}^N \longrightarrow \mathbb{P}^N \big/ \mathbb{C}$$

$$f = (f_0 : f_1 : \cdots : f_N)$$

homog. poly of deg d

$$\{ f_0 = \cdots = f_N = 0 \} = \emptyset$$

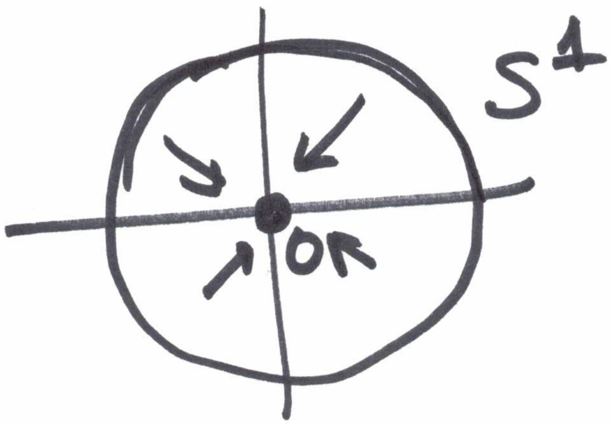
Assume $d > 1$.

Study $\underline{f^n} = \underbrace{f \circ \cdots \circ f}_{n \text{ times}}$ as $n \to \infty$

<u>Orbit</u> of $z_0 \in \mathbb{P}^N$
$= \{ f^n(z_0) \}_{n \geq 0}$

Example 0

$N=1$ $\quad f(z)=z^2$ $\quad f:\mathbb{P}^1\to\mathbb{P}^1$



$|z|<1 \Rightarrow$
$f^n(z)\to 0$

$|z|>1 \Rightarrow$
$f^n(z)\to\infty$

In $\mathbb{C}^*$ a point z_0 has finite orbit $\Longleftrightarrow$ z_0 is a root of unity. (torsion in $\mathbb{C}^*$)

$f|_{S^1}$ is chaotic $\quad f(e^{2\pi i\theta})=e^{2\pi i(2\theta)}$

Example 1 – Lattès maps

E = elliptic curve $/\mathbb{C}$

e.g. $E_t = \{y^2 = x(x-1)(x-t)\}$

$t \in \mathbb{C} \setminus \{0,1\}$

Take $\varphi : E \to E$

$\varphi(P) = P+P = 2P$

$$\begin{tikzcd} E \arrow[r, "\varphi"] \arrow[d, "\pi"'] & E \arrow[d, "\pi"] \\ \mathbb{P}^1 \arrow[r, "f"'] & \mathbb{P}^1 \end{tikzcd}$$

$\varphi(P) = 2P$

$\deg \pi = 2$

$\pi(P) = \pi(-P)$

$\deg f = 4$

Example – E_t $\quad \pi(x,y) = x$

$$f_t(x) = \frac{(x^2 - t)^2}{4\,x(x-1)(x-t)}$$

Called <u>Lattès</u> examples

<u>Observation</u> A point $P \in E$ has finite orbit for $\varphi \iff P$ is torsion.

$$P,\ 2P,\ 4P,\ 8P,\ \ldots$$

$$2^n P = 2^m P$$

$\iff \pi(P) \in \mathbb{P}^1$ has finite orbit for f.

<u>Preperiodic</u> = finite orbit

Lattès observed (1918)

Preperiodic points for this f are dense

Julia set of f is all of $\mathbb{P}^1$.

$\Updownarrow$

chaotic set.

$f : \mathbb{P}^N \to \mathbb{P}^N$ is Lattès if $\exists$ abelian variety A

$$\begin{tikzcd} A \arrow[r, "\varphi"] \arrow[d, "\pi"'] & A \arrow[d, "\pi"] \\ \mathbb{P}^N \arrow[r, "f"] & \mathbb{P}^N \cong A/G \end{tikzcd} \qquad G \subset \operatorname{Aut} A$$

For $N>1$, these examples are __rare__.

__Remark__ (Fakhruddin)

$$A \xrightarrow{\varphi} A \qquad \varphi(P) = 2P$$

$$\exists\, A \hookrightarrow \mathbb{P}^M$$

so φ extends to a morphism $f: \mathbb{P}^M \circlearrowleft$.

Canonical measures

Theorem (Lyubich, Mañé $\leftarrow N=1$, Briend-Duval)

Given $f : \mathbb{P}^N \to \mathbb{P}^N / \mathbb{C}$ deg $d > 1$

$\exists!$ probability measure μ_f on $\mathbb{P}^N(\mathbb{C})$ s.t.

- $\mu_f(V) = 0$ for proper subvariety
- $\frac{1}{d^N} f^* \mu_f = \mu_f$.

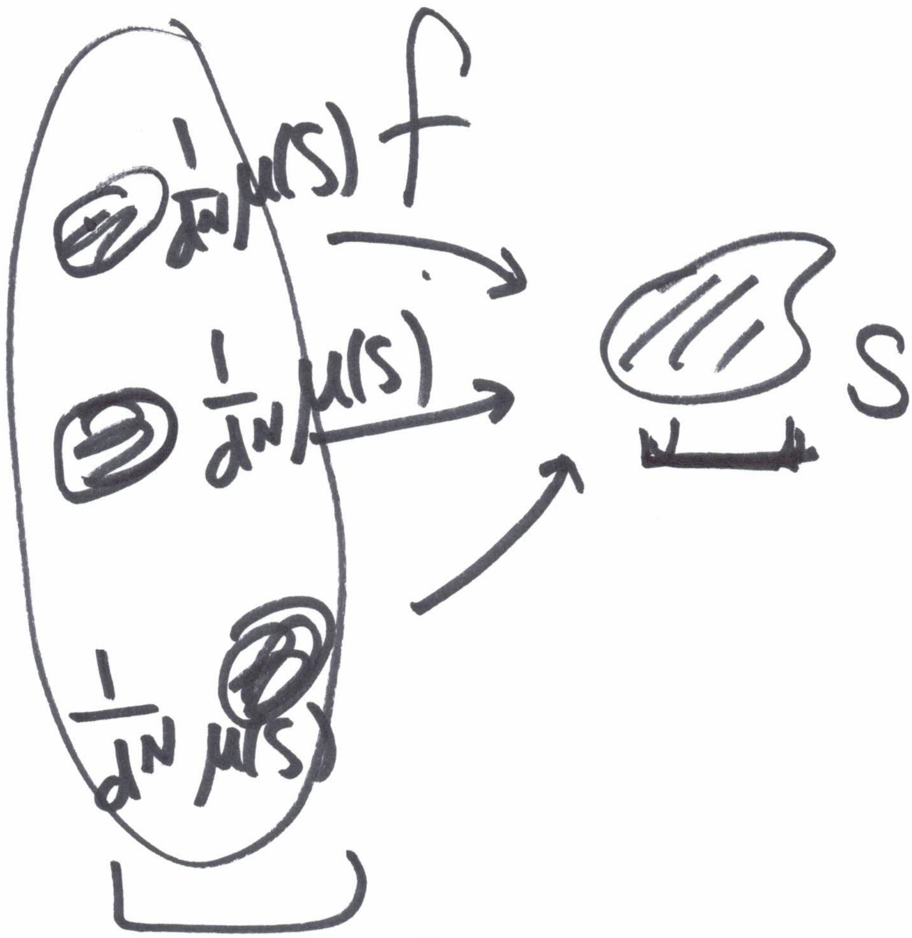
μ_f is unique measure of maximal entropy

Def $f_* \mu_f (S) := \mu_f(f^{-1}(S))$

$g : \mathbb{P}^N(\mathbb{C}) \to \mathbb{R}$ cont.

Def $\int g \, d(f^* \mu) := \int \left(\sum_{f(x)=y} g(x) \right) d\mu(y)$

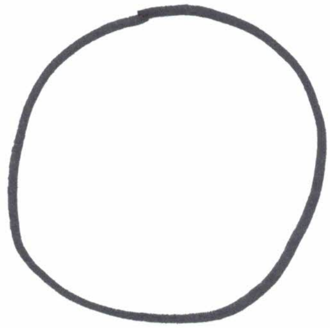
d^N preimages



$f_* \mu_f = \mu_f$

Example 0

$f(z) = z^2$



S^1

$\mu_f =$ uniform on S^1,
Haar

Example Lattès

$$\begin{tikzcd} E \arrow[d, "\pi"] \\ \mathbb{P}^1 \end{tikzcd} \qquad \mu_H$$

$$\pi_* \mu_H = \mu_f$$