$f : \mathbb{P}^N \to \mathbb{P}^N \,/\, \cancel{\mathbb{Q}}\ \overline{\mathbb{Q}}$

$f = (f_0 : \cdots : f_N)$ $\quad$ deg $d > 1$

Call-Silverman (1994)

Canonical height

$h : \mathbb{P}^N(\overline{\mathbb{Q}}) \longrightarrow \mathbb{R}_{\geq 0}$

(absolute) log. Weil height

$$\hat{h}_f(\alpha) = \lim_{n \to \infty} \frac{h(f^n(\alpha))}{d^n}$$

$\exists\ C = C(f)$ so

$$|h - \hat{h}_f| \leq C \quad \forall \alpha \in \mathbb{P}^N(\overline{\mathbb{Q}})$$

Fact $\hat{h}_f(\alpha)=0 \iff \alpha$ is preperiodic (has finite orbit)

α preperiodic $\underset{\text{clear}}{\Longrightarrow} \hat{h}_f(\alpha)=0$

If $\hat{h}_f(\alpha)=0$

$$\hat{h}_f(f(\beta)) = d\cdot \hat{h}_f(\beta)$$

$\{f^n(\alpha)\}_{n\geq 0}$ bounded height.

Northcott (1950) For any ~~bounded~~ $H, D>0$

$$\#\left\{\alpha\in \mathbb{P}^N(\bar{\mathbb{Q}}) : \begin{array}{l} h(\alpha)\leq H \\ \deg\alpha \leq D \end{array}\right\}<\infty$$

Example $f(z) = z^2$ on $\mathbb{P}^1$

$$\hat{h}_f = h$$

Lattès examples on $\mathbb{P}^1$

$$\begin{tikzcd} E \arrow[r, "\varphi"] \arrow[d] & E \arrow[d, "\pi"] \\ \mathbb{P}^1 \arrow[r, "f"'] & \mathbb{P}^1 \end{tikzcd}$$

$$\hat{h}_f(\pi(P)) = \underset{\deg \pi}{(\text{const})} \cdot \hat{h}_{NT}(P)$$

Néron-Tate

Local heights

$\alpha \in K = \#$ field

$$h(\alpha) = \frac{1}{[K:\mathbb{Q}]} \sum_{v \in M_K} n_v \circledast$$

$$* \quad \underline{\log \max\{1, |\alpha|_v\}}$$

$v = \infty$

$$V(z) = \log^+|z| = \max\{\log, 0\}$$

Continuous and subharmonic

$$V(z_0) \le \frac{1}{2\pi}\int_0^{2\pi} V(z_0 + re^{i\theta})\, d\theta$$

$\forall r$

Equivalently

$$\Delta V \geqslant 0$$

in sense of distributions

$$\Delta = \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2}$$

i.e.
$$\int V \cdot \Delta \varphi \, dx \wedge dy \geqslant 0$$

$\forall$ smooth $\varphi : \mathbb{C} \longrightarrow \mathbb{R}_{\geqslant 0}$
with
compact support

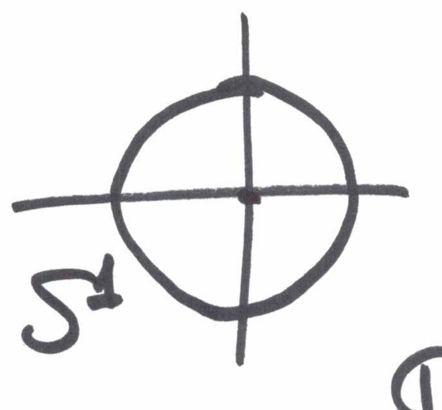
$$V(z) = \log^+|z|.$$

$$\frac{1}{2\pi}\,\underline{\Delta V\,dx\wedge dy} = m_{S^1}$$

(normalized) Lebesgue measure (Haar) on S^1

$$h = \hat{h}_f \qquad f(z) = z^2$$

$$\mu_f = m_{S^1}$$



Not a coincidence.

Last time:

$$f : \mathbb{P}^N \to \mathbb{P}^N \ / \ \mathbb{C}$$

deg $d > 1$

$\exists!$ prob. measure

μ_f s.t. $\mu_f(V) = 0$

$\forall$ subvarieties

& $$\frac{1}{d^N} f^* \mu_f = \mu_f$$

How is μ_f constructed?

$$\begin{array}{c} F \circlearrowright \mathbb{C}^{N+1} \setminus \{0\} \\ \Big\downarrow \tau \\ f \circlearrowright \mathbb{P}^N \end{array}$$

τ = tautological proj.

$F = (f_0, \ldots, f_N)$ ← poly of deg d

$F(0) = 0$

$F^{-1}(0) = 0$

$$G_F(z) = \lim_{n \to \infty} \frac{1}{d^n} \log \| F^n(z) \|$$

$z \in \mathbb{C}^{N+1}$

$$\|(z_0, \ldots, z_N)\| = \max\{|z_0|, \ldots, |z_N|\}.$$

Continuous & <u>plurisubharmonic</u> on $\mathbb{C}^{N+1}$.

A function $V : \Omega \to \mathbb{R}\cup\{-\infty\}$

open $\cap$ ~~open~~

$\mathbb{C}^M$

is plurisubharmonic (psh)
if u.s.c. & subharmonic
on every $\Omega \cap L$
for complex lines L in
$\mathbb{C}^M$.

Equivalently,

$$dd^c V \geq 0$$ in sense of distributions

$$d = \partial + \bar{\partial}$$

$$\partial g = \sum_{j=1}^{M} \frac{\partial g}{\partial z_j} dz_j$$

$$\bar{\partial} g = \sum_{j=1}^{M} \frac{\partial g}{\partial \bar{z}_j} d\bar{z}_j$$

$$d^c = \frac{1}{2\pi i}(\partial - \bar{\partial})$$

$$V(z) = \log^+ |z|$$

prob. measure

$$\underbrace{\frac{1}{2\pi} \Delta V \, dx \wedge dy}_{\substack{\| \\ dd^c V}} = m_{S^1}$$

on $\mathbb{C}$

$dd^c V \geqslant 0$

positive (1,1)-current

$f : \mathbb{P}^N \to \mathbb{P}^N$

$F : \mathbb{C}^{N+1} \to \mathbb{C}^{N+1}$

$G_F \rightsquigarrow \underbrace{dd^c G_F}$

$\mathbb{C}^{N+1} \setminus \{0\}$ $\qquad = \tau^* T_f$

$\downarrow \tau$

$\underline{\mathbb{P}^N}(\mathbb{C})$ $\quad T_f = \text{pos.}(1,1)\text{-current}$

$N=1 \quad T_f = \mu_f$

$N \quad \mu_f = \underbrace{T_f \wedge \dots \wedge T_f}_{N \text{ times}}$

$f / \cancel{\mathbb{Q}} \; K$

$\hat{h}_f(\alpha) = \frac{1}{d} \sum_v n_v G_{F,v}(\alpha)$