Arithmetic Equidistribution

late 1990s
- Szpiro-Ullmo-Zhang — ab. var/$\overline{\mathbb{Q}}$
- Bilu — $(\overline{\mathbb{Q}}^*)^n$
- Rumely — $\mathbb{A}^1/\overline{\mathbb{Q}}$

⋮

- Yuan-Zhang, 2021 preprint

Theorem

X = quasi-projective smooth alg var. $/K$ number field

$N = \dim X$.

Assume $h : X(\overline{\mathbb{Q}}) \longrightarrow \mathbb{R}$

is a height with lots of good properties.

If $\{x_n\} \subset X(\bar{\mathbb{Q}})$ is generic (no subseq. contained in propr subv.)

and

$$h(x_n) \xrightarrow[n\to\infty]{} h(X) \quad \text{minimal value}$$

$$h(X) = \underbrace{\bar{\mathcal{L}} \cdots \bar{\mathcal{L}}}_{N+1}$$

then

$$\underbrace{\frac{1}{\#} \sum_{x \in \mathrm{Gal}(\bar{K}/K)\cdot x_n} \delta_x}_{\text{prob msr.}} \xrightarrow{\text{weak}^*} \mu_h$$

$X(\mathbb{C})$

Good examples

Standard log. Weil height on $\mathbb{P}^N(\overline{\mathbb{Q}})$. $h \geq 0$

Dynamical canonical heights $f : \mathbb{P}^N \circlearrowleft \;/\overline{\mathbb{Q}}$

$\hat{h}_f$

μ_f = can. measure.

$h(\mathbb{P}^N) = 0$ $\qquad$ $\mathcal{L} = \mathcal{O}(1)$

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h is height for $\overline{\mathcal{L}}$

family of metrics $\{|\cdot|_v\}_{v \in M_K}$

$$\begin{array}{c} \mathcal{L} \\ \downarrow \\ X \end{array} \quad s$$

$\alpha \in X(K)$

$$h(\alpha) = \sum_{v \in M_K} n_v \left(-\log |s(\alpha)|_v\right)$$

$X(\mathbb{C})$ ④'

$$c_1(\bar{\mathcal{L}}) = dd^c(-\log|s(z)|)$$

in local coord. on $X(\mathbb{C})$

≥ 0

in sense of dist.

pos. (1,1)-current..

Ex. Dyn. on $\mathbb{P}^N$

$$c_1(\bar{\mathcal{L}}) = T_f$$

$$\mu_h := \underbrace{c_1(\bar{\mathcal{L}}) \wedge \cdots \wedge c_1(\bar{\mathcal{L}})}_{N}$$

in $X(\mathbb{C})$

$h(\mathbb{P}) = 0$

In dynamical case, over $\mathbb{C}$, equidistribution results were already known

$$\frac{1}{\#} \sum_{x \in \mathrm{Per}_n} \delta_x \longrightarrow \mu_f$$

all points $f^n(x) = x$

Briend-Duval

$f : \mathbb{P}^N \to \mathbb{P}^N / \overline{\mathbb{Q}}$.

Example Problem

Bogomolov-Tschinkel 2007

Conj in B-Fu-T preprint 2017

$$\begin{tikzcd} E_1 \arrow[dr, "\pi_1"', "2"] & & E_2 \arrow[dl, "\pi_2", "2"'] \\ & \mathbb{P}^1 & \end{tikzcd}$$

ell. curves $/\mathbb{C}$

$p \sim -p$

$\exists$? uniform Bounds on $\pi_1(E_1^{tors}) \cap \pi_2(E_2^{tors})$

unless $\left(\begin{tikzcd} E_1 \arrow[r, "\exists\varphi", "\sim"'] \arrow[dr, "2"'] & E_2 \arrow[d, "2"] \\ & \mathbb{P}^1 \end{tikzcd} \quad \pi_2 \circ \varphi = \pi_1 \right)$

$\dim_{\mathbb{C}}$ space $\{(E_1, \pi_1), (E_2, \pi_2)\} = 5$

Theorem – Uniform bound exists.

Poineau 2022
Kühne 2021
Gao-Ge-Kühne 2021
D.-Krieger-Ye

w/ Mavraki, another proof
M-Schmidt based on ideas
\+ from AWS lecture

Why is $I = \pi_1(E_1^{tors}) \cap \pi_2(E_2^{tors})$

Manin-Mumford (Thm of Raynaud):

$$\begin{array}{ccccc} A = E_1 \times E_2 & \supset & \pi^{-1}(\Delta) & = & C \\ \pi \downarrow (\pi_1, \pi_2) & & \downarrow & & \\ \mathbb{P}^1 \times \mathbb{P}^1 & \supset & \Delta & = & \text{diagonal} \end{array}$$

$I \longleftrightarrow$ torsion points of A in C

finite unless C is special

Riemann-Hurwitz

$g(\tilde{C}) \geq 2$ unless Branch pts (π_1) = Branch pts (π_2)

normalization

$\Longleftrightarrow E_1 \cong E_2$ $/\mathbb{C}$

Pf of finiteness via equidistribution: want $|I| < \infty$

Suppose first $E_1, E_2 / \overline{\mathbb{Q}}$

Points of I are preperiodic for Lattès maps f_1, f_2.

$$\hat{h}_{f_1}(\alpha) = \hat{h}_{f_2}(\alpha) = 0 \quad \forall \alpha \in I.$$

If $|I| = \infty$, then

$\exists$ generic seq. $\{x_n\} \subset I$

Equid, Gal. orbits

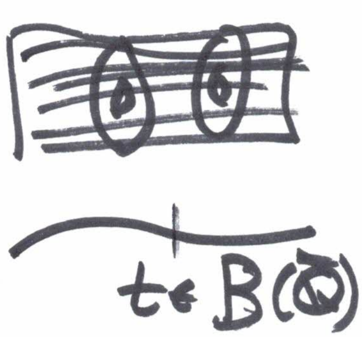
equid. w.r.t. μ_{f_1}

and μ_{f_2}. i.e., $\mu_{f_1} = \mu_{f_2}$

$\Rightarrow Br(\pi_1) = Br(\pi_2)$

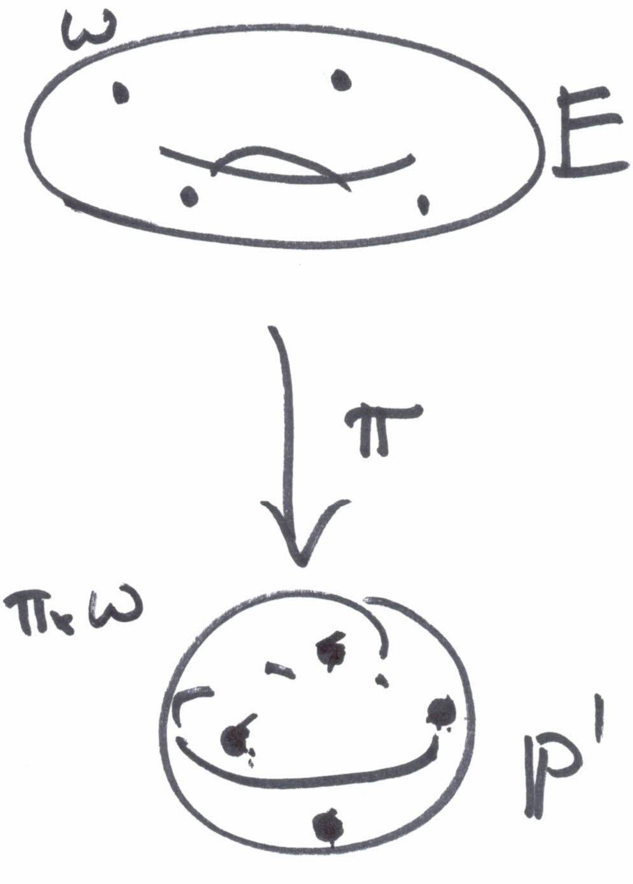
$E_1(\mathbb{C}) \cong E_2(\mathbb{C})$

For $E_1, E_2 / \mathbb{C}$



$/ \overline{\mathbb{Q}}$

$\mu = \pi_* \omega$



In fact, via cx-dynamical argument (via stability)

$$\left| \pi_1(E_1^{tors}) \cap \pi_2(E_2^{tors}) \right| \geq 5$$

for a Z. dense set in

$$\mathcal{S} = \{ (E_1, \pi_1), (E_2, \pi_2) \}$$

<u>Question</u> Is this optimal/sharp ~~when~~ on Z. open subset of $\mathcal{S}$?