1

CIT $\Longrightarrow$ Lang's Problem

$X \subseteq \mathbb{G}_m^g$

$$\overline{X(\mathbb{C}) \cap (\mathbb{C}^*)^g_{tor}}^{Zar}$$

= finite union of torsion translates of subgroups

- reduction step

we may assume

$X \not\subseteq T < G_m^g$

T proper algebraic subgp.

$X \neq G_m^g$

- $\left(G_m^g\right)_{tor} = \bigcup_{n=1}^{\infty} \mu_n^g$

$\mu_n^g \leq G_m^g$

$$X \subsetneq \mathbb{G}_m^g$$

$$a \in X \cap \mu_n^g$$

$$\dim\{a\} = 0 > \dim X + 0 - g$$

$$\therefore \quad (\mathbb{G}_m^g)_{tor} \cap X \subseteq X^{atyp}$$

not Zariski dense

CIT $\Longrightarrow$ Laurent's Theorem

$\Gamma \leq (\mathbb{C}^*)^g$

Γ finite generated

$X \subseteq \mathbb{G}_m^g$

$$\overline{X(\mathbb{C}) \cap \Gamma}^{Zar}$$

is a finite union of translates of alg. subgps.

As before, we reduce to the case

$$X \notin \gamma T < \mathbb{G}_m^g$$

$\gamma \in \Gamma$.

need only show

$$\overline{X \cap \Gamma} \neq X$$

$$\Gamma = \langle \gamma_1, \ldots, \gamma_r \rangle$$

$$\Gamma^g \cap X \ni a = (a_1 \dots a_g)$$

$$\begin{array}{l|l}
a_1 = \gamma_1^{l_{1,1}} \cdots \gamma_r^{l_{1r}} & x_1 = y_1^{l_{11}} \cdots y_r^{l_{1r}} \\
\vdots & \vdots \\
a_g = \gamma_1^{l_{g1}} \cdots \gamma_r^{l_{gr}} & x_g = y_1^{l_{g1}} \cdots y_r^{l_{gr}}
\end{array}$$

$$(l_{ij}) \in M_{g\times r}(\mathbb{Z})$$

$$\tilde{X} = X \times \cancel{G_m^r}\{(\gamma_1 \dots \gamma_r)\}$$

$$(a, \gamma_1 \dots \gamma_r) \in \tilde{X} \cap T$$

$\dim T = r$

$\dim \tilde{X} = \dim X + 0$

$\dim \tilde{X} + \dim \overline{T}$

$< g + r$

~~Fix the compactification!!~~

$\therefore$ the intersection

is atypical.

CIT $\Longrightarrow$

uniform version

of itself:

$X \subseteq G_m^a \times B$

then $\exists\ T_1, \ldots, T_s < G_m^a$

s.t. $\forall b\ \exists\ c_1(b), \ldots, c_s(b) \in G_m^a$

s.t. $(X_b)^{\text{atyp}} = \bigcup_{i=1}^{s} X_b \cap c_i(b) T_i$

We may take

$$B \subseteq \mathbb{G}_m^n$$

$$\cancel{\tilde{X} = X \times \mathbb{G}_m^n}$$

$$(\tilde{X}^{\mathrm{aty}}) \subseteq \bigcup_{i=1}^{s} S_i \cap X$$

$$S_i < \mathbb{G}_m^{g+n}$$

take $b \in B$ generic

If $T < \mathbb{G}_m^g$ alg. subgp

C_b an atypical component

of $T \cap \tilde{X}_b = T \cap X_b$

C a component of

$(T \times \mathbb{G}_m^n) \cap X$

$\dim C = \dim C_b + \dim B$

$\dim X = \dim X_b + \dim B$

$\therefore$ $\dim C$

$= \dim C_b + \dim B$

$> \dim X_b + \dim T - g + \dim B$

$= \dim X + (\dim(T \times \mathbb{G}_m^n)) - n - g$

$\therefore$ C is atypical.

$\therefore$ $C \subseteq \bigcup S_i$.

$\therefore$ $\cancel{\mathcal{H}_b}$

$$C_b \subseteq \underbrace{S_i \cap G_m^g \times \{b\}}$$

translate of

T_i $\quad S_i \cap G_m^g \times \{1\}$,, Prop CIT

$\Rightarrow$ SCOK may be expressed by a first-order theory.

SCOK′

$X \subseteq \mathbb{G}_a^g \times \mathbb{G}_m^g$

$\pi : \mathbb{G}_a^g \times \mathbb{G}_m^g \longrightarrow \mathbb{G}_m^g$

X is defined over

$k = \mathbb{Q}(\ker E)$

$\dim X < g$

$(a, Ea) \in X$

$$Ea \in X^{atyp}$$

$\Longrightarrow$

or $\dim X_{Ea} > \dim X - \dim \overline{\pi X}$

Let $X = \mathrm{loc}(\alpha, E\alpha / k)$

$\cancel{= Z}$

k-Zariski closure of $(\alpha, E\alpha)$

By SCOK'

either $\dim X_{E\alpha} > \dim X - \dim \pi X$

or $E\alpha \in (\cancel{X_{E\alpha}})(\pi X)^{\mathrm{alg}}$

$$SCOK \Longleftrightarrow SCOK'$$

<u>Pf</u> $\Longleftarrow$

$\alpha = (\alpha_1 \dots \alpha_g) \in \hat{T}_a^g$

must show: if

$\operatorname{tr}\deg_k h(\alpha, E\alpha) < g$,

then $\operatorname{rk} \langle E\alpha_1, \dots E\alpha_g \rangle < g$.

$\therefore \exists\, T < \mathbb{G}_m^g$

$\Sigma\alpha \in T$ ✓

$\Longrightarrow X \subseteq \mathbb{G}_a^g \times \mathbb{G}_m^g$

irreducible variety $/ b$

$\dim X < g$

$(\underline{a}, \underline{\Sigma a}) \in X$

We may assume

$\dim X_{\Sigma a} = \dim X - \dim \pi X$

Let $T \leq \mathbb{G}_m^g$

be the smallest

alg subgp w/ $Ea \in \overline{T}$

C be a component of

$T \cap \overline{\pi X}$ $\quad Ea \in C$.

$\dim C \geq \operatorname{trdeg}_k k(Ea)$

$= \underbrace{\operatorname{trdeg}_k k(a, Ea)}_{\geq \dim T} - \operatorname{trdeg}_{k(Ea)} k(Ea, a)$

$$\geq \dim T + \dim X_{E_a}$$

$$= \dim T + \dim \pi X - \dim X$$

$$> \dim T + \dim \pi X - g$$

$\therefore$ C is atypical

$\therefore$ $E_a \in (\pi X)^{\mathrm{atyp}}$. //

ELA /

SK' ~ theory of arithmetic

EAC

SCOK

$X \subseteq \mathbb{G}_m^2$

$\leadsto X^{\text{atyp}}$

Question Is the 1^{st}-order theory of $\mathbb{C}(+)$ decidable?