- $\mathcal{H}$ - hermitian symmetric space
- $G/\mathbb{Q}$ - semisimple

$G(\mathbb{R}) \to AUT(\mathcal{H})$

$G(\mathbb{R})/K \simeq \mathcal{H}$

K - maximal compact

e.g. $\mathbb{H}_g = \left\{ Z \in M_g(\mathbb{C}) \text{ sym} \mid \operatorname{im} Z \text{ pos. def.} \right\}$

$G = Sp_{2g}$

$\begin{pmatrix} A & B \\ C & D \end{pmatrix} Z = (AZ+B)(CZ+D)^{-1}$

A_g : $Sp_{2g}(\mathbb{Z}) \backslash \mathcal{H}_g$

"moduli space" of

principally polarized

Abelian varieties dim g.

<u>Def</u>: $x \in \mathcal{H}$ is <u>CM</u>

if $\mathrm{Stab}_{G(\mathbb{R})}(x) \supset T(\mathbb{R})$,

$T_{/\mathbb{Q}} \subset G_{/\mathbb{Q}}$ torus of

maximal rank.

$[x] \in S_\Gamma$ is CM.

- $\Gamma = G(\mathbb{Z})$ or commensurable
$[\Gamma : G(\mathbb{Z}) \cap \Gamma]$, $[G(\mathbb{Z}) : \Gamma \cap G(\mathbb{Z})]$
finite.

- $S = S_\Gamma = \Gamma \backslash \mathcal{H}$

- (THM) S quasi-proj. variety / $\mathbb{C}$

- S has a model over a number field (reflex field).

- $G'_{/\mathbb{Q}} \subset G_{/\mathbb{Q}}$ semisimple
- $x \in \mathcal{H}$ $\quad G'(\mathbb{R}) \cdot x$ is complex analytic.
- $W = [G'(\mathbb{R}) \cdot x] \subset S_\Gamma$ algebraic
- say W is "weakly special"
- W special if it contains a CM pt.

Thm: (André-Oort)

Let $V \subset S_\Gamma$, V contains finitely many maximal special subvarieties.

Galois orbits

- Let $x \in A_g$ CM.

- $A_x \cong \mathbb{C}^g / I$, $[I] \in Cl(K)$

$K/\mathbb{Q}$ CM field of degree $2g$ $\uparrow$ $x \in M_K$.

$$\# M_K \sim \frac{|Cl(K)|}{|Cl(K_0)|} \sim \left|\frac{D_K}{D_{K_0}}\right|^{1/2+o(1)}$$

$Cl(K)$

- $L - K^{Gal}$

$$\phi : Cl(L) \longrightarrow Cl(K)$$

$$\begin{array}{c} \operatorname{im} G_{\mathbb{Q}} \simeq \underline{\operatorname{im} \phi} \leftarrow \text{Large?} \\ \| \\ G_{\mathbb{Q}} \longrightarrow \operatorname{Act}(M_K) \end{array}$$

-

e.g. $L = K$, $\phi = \times m$

$$|\operatorname{im} \phi| = \left| \frac{Cl(K)}{Cl(K)[m]} \right|$$

Conj (ZHANG, DRUMER-SILVERMAN)

$$|Cl(K)[m]| = |D_K|^{o(1)}$$

Learn: All points in M_K are defined over the same #-field.

Upper bounds for Heights

$$\Downarrow$$

Lower bounds for Galois orbits.

Idea (Schmidt)

$$\mu_n^\times = \{x \in \mathbb{C}^\times, \ \mathrm{ord}(x) = n\}$$

$$I \subset \mathbb{C} \times \mathbb{C}^\times : \{(t, e^{2\pi i t}), \ t \in [0,1]\}$$

<u>Note</u>: If $x \in \mu_n^\times$, $H(x) = 1$

(9)

$\therefore H\left(\frac{a}{n}, e^{\frac{2\pi i a}{n}}\right) = H\left(\frac{a}{n}\right) = n$

$\therefore N_{\mathbb{Q}(\mu_n)}(I, n) \geq \phi(n) = n^{1-o(1)}$

(Binyamini) $N_K(X - X^{a/q}, T) \leq T^{o(1)} \cdot [K : \mathbb{Q}]^{O(1)}$

$\therefore [\mathbb{Q}(\mu_n) : \mathbb{Q}] \geq n^{\delta},$

$\delta > 0$

(Binyamini–Schmidt–Yafaev)

Upper bounds for Heights

$\Downarrow$

Lower bounds for Galois orbits.

<u>Pf:</u> (of upper bounds).

(Colmez conj). $x \in A_g$, CM

$h_{Fal}(x) = \sum$ L-values.

<u>Thm</u> (Average Colmez).

"sketch" in $Y(1)$.

$x \in CM$,

$$h(x) = h(j(x))$$

$$= \sum_{z \in Y(1)_{CM,d}}{}' \ln|j(z)|$$

$$\sim \sum_{z \in Y(1)_{CM,d}} 2\pi(\mathrm{Im}\, z)$$