Tame covering spaces

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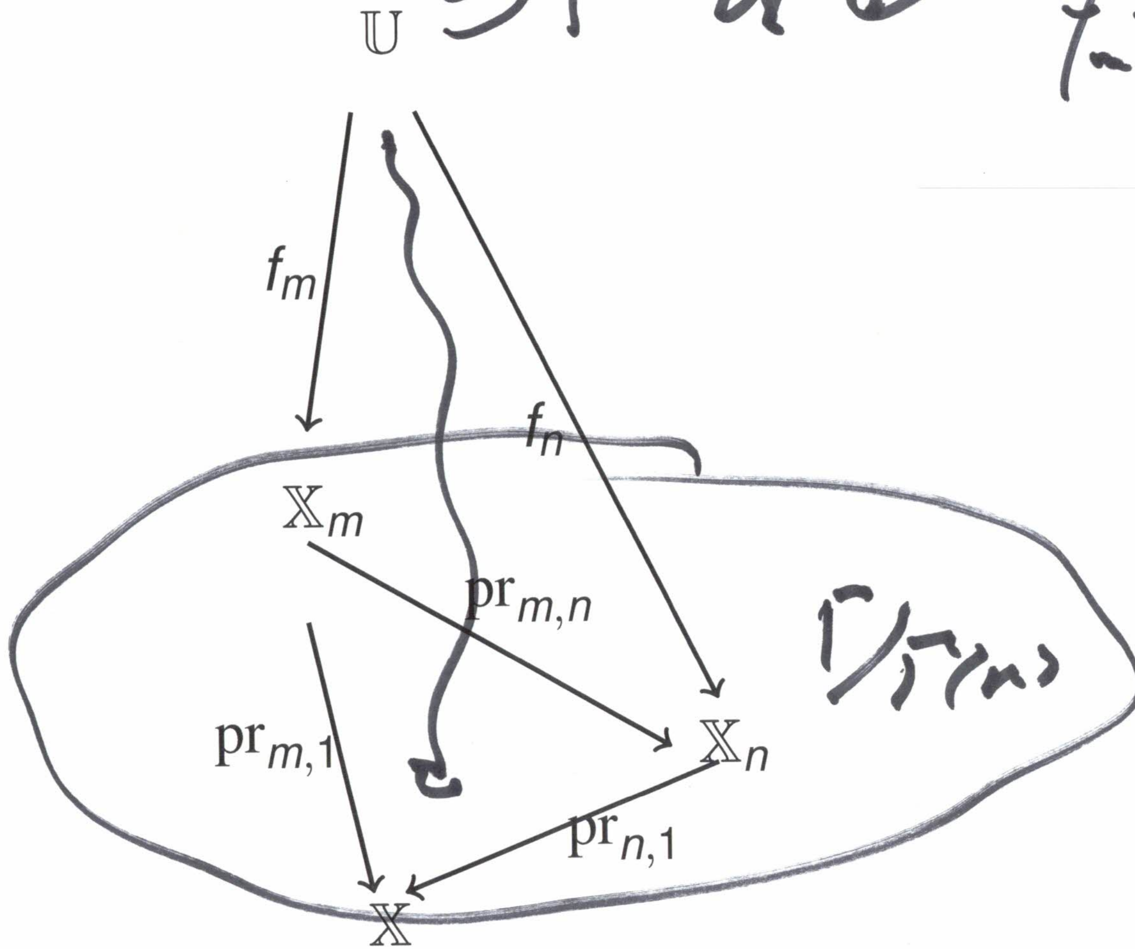
The setting



$\mathbb{U}$
f_m
f_n
$\mathbb{X}_m$
$\mathbb{X}_n$
$\mathrm{pr}_{m,n}$
$\mathrm{pr}_{m,1}$
$\mathrm{pr}_{n,1}$
$\mathbb{X}$

$L_{\omega_1,\omega}$-axiomatisation

In order to understand the $L_{\omega_1,\omega}$-theory of a structure such as $\mathbb{C}_{\exp}$ one needs first to work out a complete $L_{\omega_1,\omega}$-axiomatisation of the respective cover structure.

Some history

Theorem(s) *The* **natural** $L_{\omega_1,\omega}$*-theory of covers in basic cases (* exp, $\mathfrak{p}$, j, ...*) is categorical in uncountable cardinals.*

Proofs require Shelah's theory of AEC (with some important extensions) plus some strong **arithmetic theorems**:

1. A version of the Mumford-Tate conjecture
2. an extension of Kummer theory
3. Galois action on torsion (special) points

(Z., Kirby, Gavrilovich, Bays, Harris, Daw, Hart, Haykazian, Hyttinen, Eterovich,...)

Theorem(s). *In the above, the required arithmetic facts are sufficient and necessary:*

Assuming that the natural theory is categorical, the arithmetic facts follow.

Where does all this lead to?

A. How general might the phenomenon of the categoricity of covers be?

B. What the impact of model theory on arithmetic geometry can be?

Conjecture. *For any smooth complex algebriac variety* $\mathbb{X}$ *there is an* $L_{\omega_1,\omega}$*-axioms* Σ_X *of the universal cover of* $\mathbb{X}$ *which is categorical in all uncountable cardinals.*

Where does all this lead to?

A categorical descrition Σ_X of the universal cover of a variety $\mathbb{X}$ is a formulation of a complete *formal invariant* of $\mathbb{X}$.

By its very nature such an $L_{\omega_1,\omega}$-invariant is of "algebraic/descrte type" and the conjecture states that it is equivalent to a notion given in topological/analytic terms:

$$\text{algebraic/descrete} \equiv \text{topological/analytic}$$

This indicates a possibility of connection to certain key conjectures of algebraic geometry such as the Hodge and Mumford-Tate conjectures.

A weak form the “categoricity of covers” conjecture

AEC-$\mathbb{X}$-Conjecture (Partially proved 2022). *For any smooth complex algebriac variety $\mathbb{X}$ there is an “abstract elementary” axiomatisation $\Sigma^?_X$ of the universal cover of $\mathbb{X}$ which is categorical in all uncountable cardinals.*

This is a theorem when $\mathbb{X}$ is a projective curve.

Also holds for many more general cases for cardinal = $\aleph_1$.

AEC-$\mathbb{X}$-Conjecture: The scheme of proof

1. One can interpret the cover structure in $\mathbb{R}_{\mathbb{X}}$, an appropriate o-minimal expansion of the reals (using $L_{\omega_1,\omega}$ formulas too).

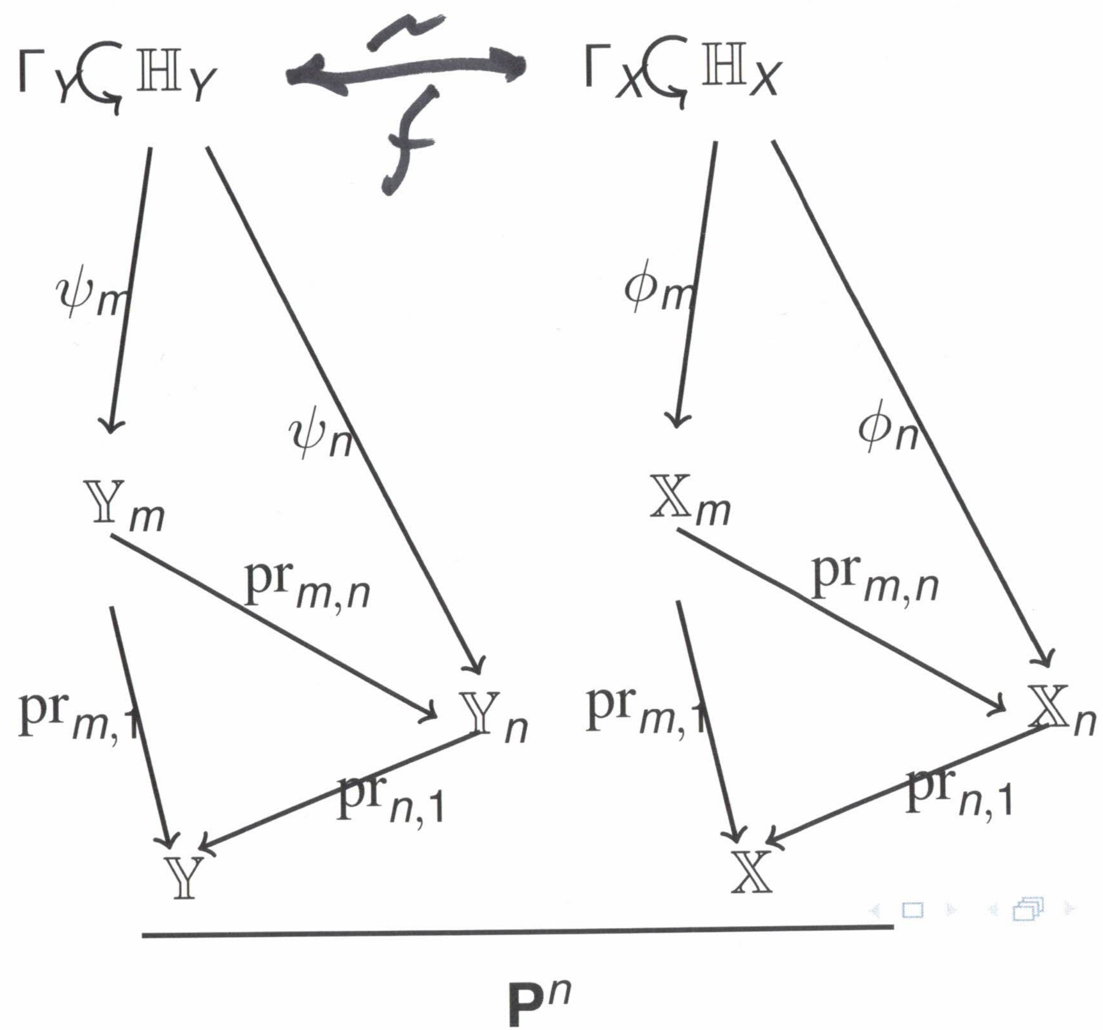
2. Consider models $\mathbb{U}(\mathrm{R}_{\mathbb{X}})$ of the interpretation for arbitrary models $\mathrm{R}_{\mathbb{X}}$ of the o-minimal theory.

3. Any $\mathbb{U}(\mathrm{R}_{\mathbb{X}})$ in its natural "pseudo-analytic" language allows elimination of quantifiers and is "ω-homogeneous over submodels".

4. The above implies:
   A: In case $\dim \mathbb{X} = 1$: any two structures $\mathbb{U}(\mathrm{R}_{\mathbb{X}})$ of the same uncountable cardinality are isomorphic. Moreover, the class can be $L_{\omega_1,\omega}(Q)$-axiomatised. ;
   B: In the general case: any two structures $\mathbb{U}(\mathrm{R}_{\mathbb{X}})$ of cardinality $\aleph_1$ are isomorphic.

What is the full tame analytic structure on $\mathbb{H}$?

"Non-commensurable" curves: *Hrushovski fusion*

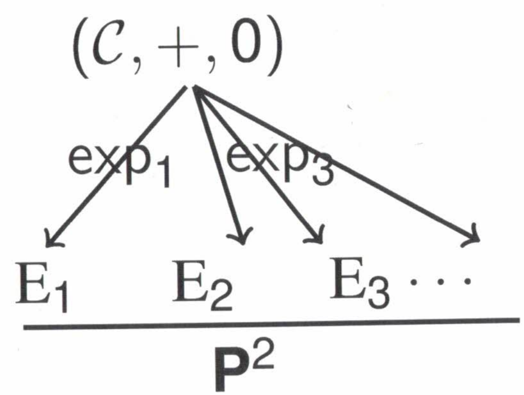


Our aim in this case is to construct a structure $\mathbb{H}/\mathrm{k}$, the formal analog of $\mathbb{H}$ as a cover of all non-singular projective complex curves of genus > 1 defined over k,

or in the case of curves of genus ≤ 1, a structure $\mathbb{C}/\mathrm{k}$, the formal cover of curves of genus ≤ 1.

The result for genus ≤ 1

For any number field k there is a categorical AEC whose models are universal covers of $\mathbb{G}_m$ and all the elliptic curves E_τ defined over k;



The structure on $\mathcal{C}$ is quasiminimal, definable sets in $\mathcal{C}$ are classifiable.

Assuming Grothendieck - André period conjecture for 1-motives, along with respective Nullstellensatz, $\mathbb{C}$ is is a model of $\mathcal{C}/\mathrm{k}$,

The same can be claimed for $\mathbb{H}/\mathrm{k}$, but stronger assumptions.