MODEL THEORY PROBLEM SET 5

Throughout, p is either a prime number or $p = 0$.

Beginner problems

Question 1: Show that ACF_p has no finite models.

Question 2: Let $K \models \mathrm{ACF}_p$. Show that every definable subset of K is either finite or cofinite.

Question 3: Let T be an $\mathcal{L}$-theory with quantifier elimination. Let $\mathcal{L}_c = \mathcal{L} \cup \{c\}$, where c is a new constant symbol. Let T_c be any $\mathcal{L}_c$-theory which extends T. Show that T_c has quantifier elimination.

Question 4: Let $K \subset L$ be algebraically closed fields, and let $V, W \subseteq L^n$ be Zariski closed sets defined over K. Suppose there is a bijective polynomial map $f : V \to W$ defined over L. Show that there is a bijective polynomial map $g : V \cap K^n \to W \cap K^n$ defined over K.

Intermediate problems

Question 5: Recall that Ax's theorem says that every polynomial map $f : \mathbb{C}^n \to \mathbb{C}^n$ which is injective must also be surjective. However, f may be surjective without being injective (think of $f(x) = x^2$ for example). Where does the proof seen in lectures break if you try to show that f being surjective implies injectivity?

Question 6: Let $K \models \mathrm{ACF}_0$. Let $V \subseteq K^n$ be a Zariski closed set and suppose that $f : V \to V$ is a polynomial map. Show that if f is injective, then it is also surjective.

Question 7: We say that an $\mathcal{L}$-theory T **eliminates** $\exists^\infty$ if for any $\mathcal{L}$-formula $\phi(x, \bar{y})$, there is a natural number n_ϕ such that for all models $\mathcal{M}$ of T and all $\bar{a}$ in M, the set $\{x \in M : \phi(x, \bar{a})\}$ is infinite if and only if it has more then n_ϕ elements. Prove that ACF_p eliminates $\exists^\infty$. Hint: Suppose not, as witnessed by $\phi(x, \bar{y})$. Show by compactness that there is a model $K \models \mathrm{ACF}_p$ and parameters $\bar{a}$ in K such that the set $\{x \in K : \phi(x, \bar{a})\}$ is neither finite nor cofinite. Then use Question 2.

Question 8: Suppose that $K \models ACF$, and let $I \subset K[x_1, \ldots, x_n]$ be a maximal ideal. Prove that $I = (x_1 - a_1, \ldots, x_n - a_n)$ for some $a_1, \ldots, a_n \in K$. (Hint: use the Weak Hilbert's Nullstellensatz.)

Advanced problems

Question 9: Let $\mathcal{L}_c = \{+, -, \times, 0, 1, c\}$ extend the language of rings by a new constant symbol c. Let $\mathrm{ACF}_p(c)$ be the theory ACF_p, but viewed as an $\mathcal{L}_c$-theory (we don't add any axioms describing how c is interpreted). Of course, $\mathrm{ACF}_p(c)$ is not complete; for example, we may interpret c to be 0, or 1, or $\sqrt{2}$. What are the completions of $\mathrm{ACF}_p(c)$? That is, describe all of the complete $\mathcal{L}_c$-theories which extend $\mathrm{ACF}_p(c)$. Hint: By Question 3, we know that $\mathrm{ACF}_p(c)$ has quantifier elimination, so we can use Proposition 5.11.

Question 10: Let K be a countable model of ACF_0, and let $\Sigma(x)$ be a finitely satisfiable set of formulas with parameters from K. Show that there is $t \in \mathbb{C}$ satisfying all the formulas in $\Sigma(x)$.