MODEL THEORY PROBLEM SET 6

Beginner problems

Question 1: Let $F \models RCF$ and $A \subseteq F^n$ be semialgebraic. Show that the closure (in the Euclidean topology) of F is semialgebraic.

Question 2: Let x and y be algebraically independent over $\mathbb{R}$. Show that $\mathbb{R}(x, y)$ is formally real and that we can find orderings $<_1$ and $<_2$ of $\mathbb{R}(x, y)$ such that $x <_1 y$ and $y <_2 x$.

Question 3: Let F be a field. Show that F is formally real if and only if for all $a_1, \ldots, a_n \in F$ we have that

$$a_1^2 + \cdots + a_n^2 = 0 \implies a_1 = \cdots = a_n = 0.$$

Question 4: Let $\mathcal{L} = \{0, 1, +, -, \times, <, f\}$, where f is a function symbol.

(a) Consider $\mathbb{R}$ as an $\mathcal{L}$-strucutre, where f is interpreted as the sine function on $\mathbb{R}$: $\sin(x)$. Is the theory of $\mathbb{R}$ in this language o-minimal?

(b) Consider $\mathbb{R}$ as an $\mathcal{L}$-strucutre, but now interpret f as the complex exponential function on $\mathbb{C}$ (so now f is a function of two variables). Is the theory of $\mathbb{R}$ in this language o-minimal?[1]

Question 5: Let $F \models RCF$ and $S \subseteq F^{m+n}$ be semialgebraic. For $\bar{a} \in F^m$, let $S_{\bar{a}} := \{\bar{b} \in F^n : (\bar{a}, \bar{b}) \in S\}$. Show that the set

$$\{\bar{a} \in F^m : S_{\bar{a}} \text{ is open}\}$$

is semialgebraic.

Intermediate problems

Question 6: Let $F \models RCF$, $X \subseteq F^n$ closed and bounded, and $f : F^n \to F$ semialgebraic (i.e., the graph of f is semialgbraic). Show that $f(X)$ is closed and bounded. (Hint: notice that this is true in $\mathbb{R}$, and transfer the result over to any other real closed field.)

Question 7: Let $F \models RCF$ and $f(\bar{X}) \in F(X_1, \ldots, X_n)$ be a rational function. We say that f is *positive semidefinite* if $f(\bar{a}) \geqslant 0$ for all $\bar{a} \in F^n$. Show that if f is positive semidefinite, then f is a sum of squares of rational functions.

Question 8: Let $\mathcal{L}$ be a language that contains the the symbol $<$. Let T be an o-minimal theory on this language and let $\mathcal{M}$ be a model of T. Suppose that $\phi(x)$ is an $\mathcal{L}$-formula which has only finitely many realizations in M. Show that for every $m \in M$ realizing $\phi(x)$ there is another $\mathcal{L}$-formula $\psi(x)$ such that the only realization of $\psi(x)$ is m.

Question 9: Let $\mathcal{L} = \{e, *, <\}$ and let G be an ordered group (i.e. G is a group and for all $x, y, z \in G$ we have that $x < y \implies x * z < y * z$). Considering G as an $\mathcal{L}$-structure, let T be the complete first-order theory of G in the language $\mathcal{L}$. Suppose that T is o-minimal.

(a) Let $X \subseteq G$ be a definable set (defined with parameters from G). Show that if X is a subgroup of G, then either $X = \{0\}$ or $X = G$.
Hint: First show that if $X \neq \{0\}$, then there is $h \in G$ such that $(-h, h) \subseteq X$.

[1] A very famous theorem of Wilkie shows that if we interpret f as the real exponential function, then the theory of $\mathbb{R}$ is o-minimal.

(b) Show that G is abelian.
Hint: Given $h \in G$, consider the definable (with parameter h) subgroup $C(h) := \{g \in G : g * h = h * g\}$.

(c) Show that G is divisible, i.e. for every $g \in G$ and every positive integer n, there exists $h \in G$ such that $nh = g$.

Question 10: Let $\mathcal{L}$ be a language that contains the symbol $<$ and let $\mathcal{M}$ be an o-minimal $\mathcal{L}$-structure. Assume that the underlying ordered set of $\mathcal{M}$ is densely ordered. Show that for $a < b \in M$, if $f : [a.b] \to M$ is a definable continuous function, then f assumes all values between $f(a)$ and $f(b)$.

Advanced problems

Question 11: Let $\mathcal{L} = \{0, 1, +, -, \times, \{f\}_{i \in I}\}$, where $\{f\}_{i \in I}$ denotes a set of function symbols. Suppose we interpret the f_i on $\mathbb{R}$ in such a way that the theory of $\mathbb{R}$ in this language is o-minimal.

(a) Let $g : \mathbb{R} \to \mathbb{R}$ be a definable function and assume that $g^{-1}(x)$ is a finite set for all $x \in \mathbb{R}$. Show that there is a positive integer N such that for all $x \in R$, $g^{-1}(x)$ has at most N elements.

(b) Let $g : \mathbb{R}^{n+1} \to \mathbb{R}^n$ be a definable function and assume that $g^{-1}(\overline{x})$ is a finite set for all $\overline{x} \in \mathbb{R}^n$. Show that there is a positive integer N such that for all $\overline{x} \in R^n$, $g^{-1}(\overline{x})$ has at most N elements.
Hint: Use cell-decomposition and induction.

Question 12: Let $\mathcal{L}$ be the language of ordered rings. Let R be an ordered ring such that the complete first-order theory T of R in the language $\mathcal{L}$ is o-minimal. Show that R is a real closed field.