3. Geometry of $\mathcal{S}_g$

$k = \overline{\mathbb{F}_p} \supseteq \mathbb{F}_q \supseteq \mathbb{F}_p$, K

<u>Recall</u> E/K is supersingular if $E[p](k) = \{0\}$

X/K is supersingular if $X \sim_k E^g$, E ss EC

X/K is superspecial if $X \simeq_k E^g$, E ss EC

<u>Thm 1.23</u> (Deligne)

Any two products $(n \geqslant 2)$

$E_1 \times \ldots \times E_n$, $E_{n+1} \times \ldots \times E_{2n}$

of ss EC's are isomorphic over k

as abelian varieties.

From now on, we study

$$\mathcal{S}_g = \{ (X,\lambda) \in \mathcal{A}_g : X \text{ is supersingular} \}$$

- This is a coarse moduli space of g-dim supersingular ppAV's.

- $\mathcal{S}_g = W_\sigma \subseteq \mathcal{A}_g$ is a Newton stratum.

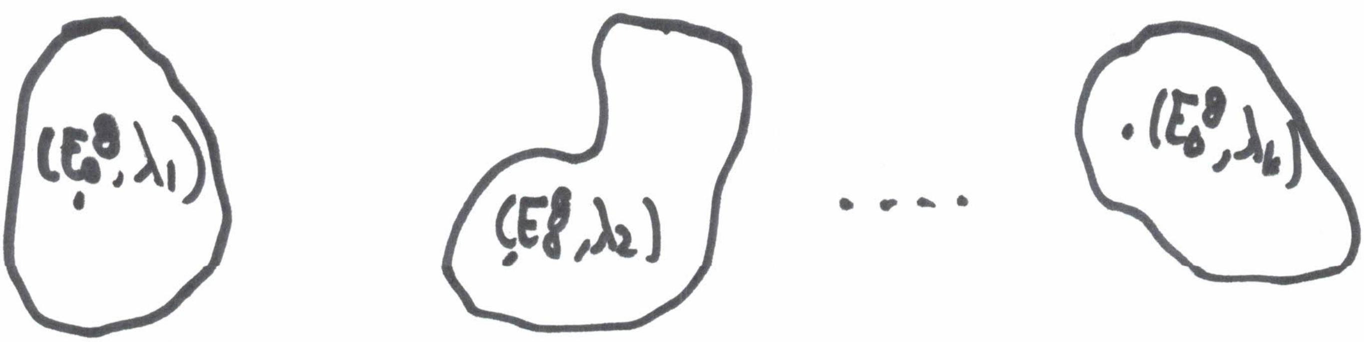
- When $g=1$, $\mathcal{S}_g$ has dim 0
  (ss points on j-line)

Notation 3.2 Let E_0 be a ssEC defined over $\mathbb{F}_{p^2}$, with $\pi_{E_0} = -p$.

Fix g-dimensional superspecial (ssp) AV to be E_0^g / k.

Idea Any (ss)ppAV may be related to a polarised (ssp)AV through some isogeny.

Every irr component of $\mathcal{S}_g$ is determined by a choice of polarised ssp AV.



<u>$g=2$</u> Any irr. component of $\mathcal{S}_2$ is the image of a "Moret-Bailly family" $q:(\mathcal{X},D)\to\mathbb{P}^1$

$$\pi^*M = p_1^*\mathcal{L}$$

$$\begin{tikzcd}
1 \arrow[r] & \alpha_p\times\mathbb{P}^1_k \arrow[r] & & A\times\mathbb{P}^1_k \arrow[rr, "\pi"] \arrow[dl, "p_1"'] \arrow[dr, "p_2"] & & M,\ \mathcal{X},\ D \arrow[r] \arrow[dl, "q"] & 1 \\
& & \mathcal{L},\ A \arrow[dr] & & \mathbb{P}^1_k \arrow[dl] & & \\
& & & \mathrm{Spec}(k) & & &
\end{tikzcd}$$

where A is a ssp surface, $\mathcal{L}$ line bundle

$\leftrightarrow \lambda$ polarisation kernel $\alpha_p\times\alpha_p$

$q:\ (\mathcal{X},D)\to\mathbb{P}^1_k$

So:

- every irr component of $\mathcal{S}_2$ is a rational curve
  $\Rightarrow \dim \mathcal{S}_2 = 1$

- $\#$ cpts $=$ $\#$ pol^{n}s λ with kernel $\alpha_p\times\alpha_p$
  $\Rightarrow$ count tomorrow

Let $F = F_{X/k}$, $V = V_{X/k}$. Write $X[F] = \mathrm{Ker}(F)$ on X

For general g, over k:

<u>Def 3.8</u> A polarised flag type quotient (PFTQ)
w.r.t. polarisation μ on E_0^g
s.t. $\mathrm{Ker}(\mu) = \begin{cases} E_0^g[F] & g \text{ even} \\ 0 & g \text{ odd} \end{cases}$

is a chain of isogenies

$$(Y_{g-1}, \lambda_{g-1}) \xrightarrow{\rho_{g-1}} (Y_{g-2}, \lambda_{g-2}) \to \dots \xrightarrow{\rho_2} (Y_1, \lambda_1) \xrightarrow{\rho_1} (Y_0, \lambda_0)$$

s.t.

- $(Y_{g-1}, \lambda_{g-1}) = (E_0^g, p^{\lfloor \frac{g-1}{2} \rfloor} \mu)$
- $\mathrm{Ker}(\rho_i) \simeq \alpha_p^i \quad \forall\ 1 \leq i \leq g-1$
- $\mathrm{Ker}(\lambda_i) \subseteq Y_i[V^j \circ F^{i-j}] \quad \forall\ 0 \leq i \leq g-1, \ 0 \leq j \leq \lfloor \frac{i}{2} \rfloor$

$\Rightarrow (Y_0, \lambda_0)$ is a ss pp AV

Back to $g=2$

PFTQ is

$$(E_0^2, \mu) \longrightarrow (E_0^2/\alpha_p, \lambda_0) = (Y_0, \lambda_0)$$

μ: $\ker(\mu) = E_0^2[F] = \alpha_p \times \alpha_p$

λ_0: principal

This is determined by

$$\alpha_p \hookrightarrow \alpha_p \times \alpha_p \hookrightarrow E_0 \times E_0$$

$\mathrm{End}(\alpha_p) = k$, so may view this as

$$(a:b) \in \mathbb{P}^1_k$$

So again $\mathbb{P}^1$-family!

<u>Def 3.11</u> g-dim PFTQ's w.r.t. μ have a (fine!) moduli space $\mathcal{P}_{g,\mu}$ defined over $\mathbb{F}_{p^2}$.

It is geom. irreducible, quasi-projective of dim $\lfloor \frac{g^2}{4} \rfloor$.

Projection to last member gives

$$\coprod_{\mu} \mathcal{S}_{g,\mu} \longrightarrow \mathcal{S}_g$$

surjective & generically finite

(a-number 1 $\Rightarrow \exists!$ PFTQ above it)

So $\dim(\mathcal{S}_g) = \lfloor \frac{g^2}{4} \rfloor$.

- can also see this from $\dim(W_\sigma) = |\Delta(\sigma)|$ -

and # irr cpts of $\mathcal{S}_g$ =

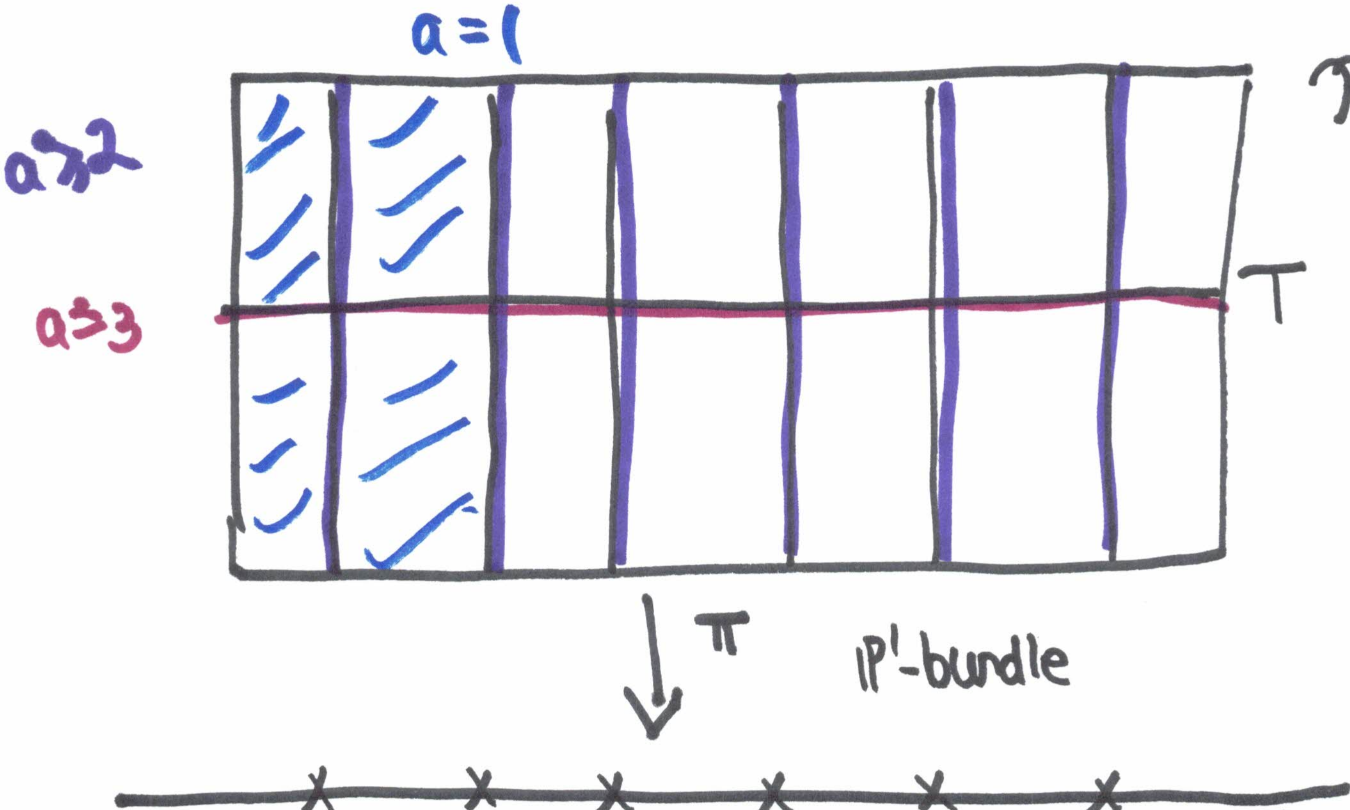
suitable polarisations μ

$\Rightarrow$ count tomorrow

EX 3.13 13.18 $(g=3)$

$\mathcal{P}_{3,\mu}$ has dim $\left\lfloor \frac{9}{4} \right\rfloor = 2$, structure indep. of μ

$(Y_2, \lambda_2) \to (Y_1, \lambda_1) \to (Y_0, \lambda_0) = y$



$C(\mathbb{F}_{p^2})$ ~~(Y_2,λ_2)~~ $(Y_2, \lambda_2) \to (Y_1, \lambda_1)$

$C: X_1^{p+1} + X_2^{p+1} + X_3^{p+1} = 0$
$\subseteq \mathbb{P}^2$

Ex (ctd)

- Away from T,
  $y \mapsto (Y_0, \lambda_0)$ is generically finite.

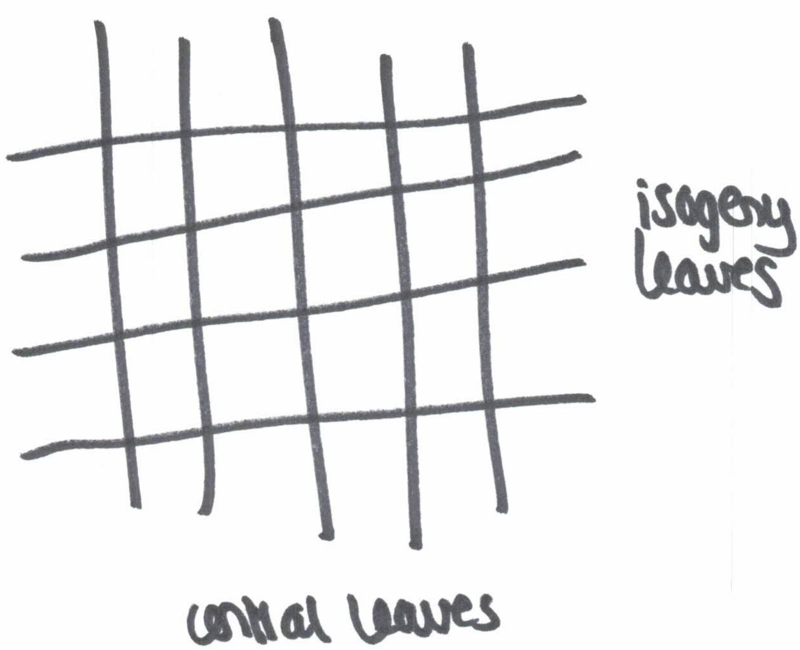
- y is determined by 2 parameters:
  $t = \pi(y) \in C(k)$
  $u \in \pi^{-1}(t) = \mathbb{P}^1_t(k)$

- $y \in T \Rightarrow a(y) := a(Y_0) = 3$
  $t \in C(\mathbb{F}_{p^2}) \Rightarrow a(y) \geq 2$
  $a(y) = 3 \Leftrightarrow u \in \mathbb{P}^1_t(\mathbb{F}_{p^2})$

  $a(y) = 1 \Leftrightarrow y \notin T,\ t \notin C(\mathbb{F}_{p^2})$

Foliation structure

Thm 3.26 [COOA] For any irr cpt V of $\mathcal{S}_g$,
$\exists$ finite surjective $\overline{\mathbb{F}}_p$-morphism
$$\Phi : D \times J \to V$$
s.t. any $\Phi(D \times \{j\})$ is a <u>central leaf</u>
and any $\Phi(\{d\} \times J)$ is an <u>isogeny leaf</u>
(& all leaves are found this way)



"almost-product structure"

Def 3.20 The <u>central leaf</u> through

$$x = (X_0, \lambda_0) \in \mathcal{A}_g(k) \text{ is}$$

$$\mathcal{C}(x) = \{ (X,\lambda) \in \mathcal{A}_g(k) : (X,\lambda)[p^\infty] \simeq (X_0,\lambda_0)[p^\infty] \}$$

(Recall: $(X,\lambda)[p^\infty] = \varinjlim_n (X,\lambda)[p^n]$.)

This is a closed subset of $\mathcal{S}_g$ of dim 0.

Fixed under degree-ℓ isogenies ($\ell \neq p$).

Roughly:

Def 3.25 An <u>isogeny leaf</u> through $(X_0, \lambda_0) \in \mathcal{A}_g(\bar{k})$ contains all $(Y_0, \mu_0) \in \mathcal{A}_g(\bar{k})$ isogenous to (X_0, λ_0) via an iterated α_p-isogeny.

This is a closed integral subscheme, for $(X_0, \lambda_0) \in \mathcal{S}_g(\bar{k})$ of dim $\lfloor \frac{g^2}{4} \rfloor$.

<u>Remark</u> The theorem holds for any irr cpt $V \subseteq W^0_\xi \subseteq A_g$ of an open Newton stratum.

dim central leaf $= c_\xi$

(> 0 whenever $\xi \neq \sigma$)
[Chai]

irreducible whenever $\xi \neq \sigma$.

dim isogeny leaf $= \underbrace{|\triangle(\xi)|}_{\dim W^0_\xi} - c_\xi = i_\xi$

Thm 3.25 (Ibukiyama - K - Yu)

For $x \in \mathcal{S}_g(\bar{k})$, $\quad x \leftrightarrow (X, \lambda)$

$\# \mathcal{C}(x) = 1 \iff$ one of the following holds:

- $g = 1$, $p \in \{2, 3, 5, 7, 13\}$
- $g = 2$, $p \in \{2, 3\}$
- $g = 3$, $p = 2$, $a(X) \geq 2$.

(In these cases, V = isogeny leaf,
and x is determined by its p-divisible group.)