# 4. Arithmetic of $\mathcal{S}_g$

$$k = \overline{\mathbb{F}}_p \supseteq \mathbb{F}_q \supseteq \mathbb{F}_p, \quad K$$

① Question: How many ss. ppAV's are there?

Depends how you count!

<u>$g=1$</u> $(\dim(\mathcal{S}_1) = 0)$

<u>Thm</u> (Deuring - Eichler)

$$|\mathcal{S}_1(k)| = \left\lfloor \frac{p-1}{12} \right\rfloor + \begin{cases} 0 & p \equiv 1 \quad (12) \\ 1 & p \equiv 2,3,5,7 \quad (12) \\ 2 & p \equiv 11 \quad (12) \end{cases}$$

and

$$\mathrm{Mass}(\mathcal{S}_1) = \sum_{x \in \mathcal{S}_1} \frac{1}{|\mathrm{Aut}(x)|} = \frac{p-1}{24}$$

# How do we know?

ss.EC $E$ has $\mathrm{End}^\circ(E) = \mathrm{End}(E) \otimes \mathbb{Q} = \mathbb{Q}_{p,\infty}$

quaternion algebra

&

$\mathrm{End}(E) = \mathcal{O} \subseteq \mathbb{Q}_{p,\infty}$ maximal order

Take $E_0$ as before ( $E_0 / \mathbb{F}_{p^2}$, $\pi_{E_0} = -p$ )

For any other $E$, $\mathrm{Hom}_{\bar{k}}(E_0, E)$ is a maximal left $\mathrm{End}(E) = \mathcal{O}$ -ideal.

This is a bijection, so

$$\#\{\text{isomorphism classes of ss.EC's}\} \xleftrightarrow[=\ (\text{Deuring})]{1:1} \#\{\text{class group of } \mathcal{O}\} = h(\mathcal{O})$$

computed by Eichler

## Now $g \geq 2$:

RHS: replace maximal $\mathcal{O}$-ideals with maximal $\mathcal{O}$-lattices in $\mathbb{Q}_{p,\infty}^g$.

Lattices $L_1, L_2$ are <u>equivalent</u> if

$$L_2 = L_1 \alpha,$$

for $\alpha \in GL_g(\mathbb{Q}_{p,\infty}) : \alpha \bar{\alpha}^t = \mathbb{I}_g$

May localise lattices at any prime $\ell$.
Locally, any lattice is equivalent to either principal lattice $(\mathcal{O}_\ell^g, \mathbb{I}_g)$ or non-principal lattice.

Also, non-principal can only happen at $p$.

**Def 4.10** A genus of lattices consists of global equivalence classes of lattices that are everywhere locally equivalent.

Write $\mathcal{L}_g(p,1)$ $\qquad\qquad$ $\mathcal{L}_g(1,p)$

principal at p $\qquad\qquad$ non-principal at p

with cardinalities (class numbers)

$h_g(p,1)$ $\qquad\qquad$ $h_g(1,p)$

and generally,

$$M_g(d_1,d_2) := \mathrm{Mass}(\mathcal{L}_g(d_1,d_2))$$

$$= \sum_{L \in \mathcal{L}_g(d_1,d_2)} \frac{1}{|\mathrm{Aut}(L)|}.$$

# What is known?

- Class numbers only known for small $g$.
- Masses are all known! (Prop 4.17)

(Of all genera in a quaternion Hermitian space of any dimension)

## Example ($g=3$, $\mathbb{Q}_{p,\infty}$)

$$M_3(p,1) = \frac{|\zeta(-1)\zeta(-3)\zeta(-5)|}{2}(p-1)(p^2+1)(p^3-1)$$

$$M_3(1,p) = \frac{|\zeta(-1)\zeta(-3)\zeta(-5)|}{2}(p-1)(p^6-1)$$

$(g \geq 2)$

**Prop 4.21**

$$\{\text{left } \mathcal{O}\text{-lattices in } \mathbb{Q}_{p,\infty}^{g}\} / \sim$$

$\sim$ equivalence

$\mathcal{O} f \bar{f}^{\vee}$

$$\longleftrightarrow \{ f \in M_g(\mathcal{O}) : f = \bar{f}^{\vee} \text{ pos. def.} \} / \simeq$$

$\simeq$ $GL_g(\mathcal{O})$

$f$

$(\lambda_0^{\oplus g})^{-1} \circ \mu$

$$\longleftrightarrow \{ \text{polarisations } \mu \text{ of } E_0^g \} / \simeq$$

$\simeq$ $\mathrm{Aut}(E_0^g)$

$\mu$

Further, fixing kernel of polarisation uniquely determines a genus of $\mathcal{O}$-lattices, & vice-versa.

<u>Def</u> For any $0 \leq c \leq \lfloor \frac{g}{2} \rfloor$, let

$$\Lambda_{g,p^c} := \left\{ \begin{array}{l} g\text{-dim. superspecial } (X,\lambda): \\ \ker(\lambda) \simeq \alpha_p^{2c} \end{array} \right\} / \simeq$$

<u>Recall</u> (Deligne): all ssp. g-dim AV's are isomorphic

$\Rightarrow \Lambda_{g,p^c}$ is a genus.

$$\Lambda_{g,p^0} = \mathcal{L}_g(p,1)$$

$$\Lambda_{g,p^{\lfloor \frac{g}{2} \rfloor}} = \mathcal{L}_g(1,p)$$

<u>Note</u> $\Lambda_{g,p^c}$ is also a central leaf through a polarised ssp. AV.

## What is known?

- $|\Lambda_{g,p^c}|$ only known for small $g$ and $c$.
- $\mathrm{Mass}(\Lambda_{g,p^c}) = \sum_{(X',\lambda')\in\Lambda_{g,p^c}} \frac{1}{|\mathrm{Aut}(X',\lambda')|}$

  is known for all $g \geq 1$ and $0 \leq c \leq \lfloor \frac{g}{2} \rfloor$.

  (Thm 4.23)

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**Recall:** # irr. cpts of $\mathcal{S}_2$

$=$ # polarisations $\mu$ on $E_0^2$

with kernel $\alpha_p^2$

$= h_2(1,p)$

**Thm 3.16** # irr. cpts of $\mathcal{S}_g$ is

$$\begin{cases} h_g(p,1) & g \text{ odd} \\ h_g(1,p) & g \text{ even} \end{cases}$$

**Thm 3.17** (Ibukiyama-K-Yu)

$S_g$ is geom. irreducible iff one of the following holds:

- $g=1$, $p \in \{2,3,5,7,13\}$ (Deuring-Eichler)
- $g=2$, $p \in \{2,3,5,7,11\}$ (Katsura-Oort)
- $(g,p) = (3,2)$ or $(4,2)$

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What about $|\mathcal{C}(x)|$ or $\mathrm{Mass}(\mathcal{C}(x))$ for non-ssp. ss. ppAV's?

**Lemma 4.2g** $\forall x \in \mathcal{S}_g(k)$, there exists a non-canonical surjection

$$\mathcal{C}(x) \twoheadrightarrow \Lambda_{g,p^c}$$

for some $0 \leq c \leq \lfloor \frac{g}{2} \rfloor$

How do we find $\mathcal{C}(x) \twoheadrightarrow \Lambda_{g,p^c}$?

One strategy:

<u>Def 4.27</u> For any $(X,\lambda) \in \mathcal{S}_g(\bar{k})$,
there exists a polarised ssp. AV $(\tilde{X},\tilde{\lambda})$
and an isogeny $\varphi : (\tilde{X},\tilde{\lambda}) \to (X,\lambda)$
such that any other isogeny
$(\tilde{X}',\tilde{\lambda}') \longrightarrow (X,\lambda)$ factors:

$$\begin{tikzcd}
 & (\tilde{X}',\tilde{\lambda}') \arrow[dr] & & (X,\lambda) \\
\text{ssp} \arrow[ur] & & (\tilde{X},\tilde{\lambda}) \arrow[ur, "\varphi"'] &
\end{tikzcd}$$

Call $\varphi$ the <u>minimal isogeny</u>.

If $\mathcal{C}(x) \twoheadrightarrow \Lambda_{g,p^c}$ is realised through the minimal isogeny of $x$, then:

__Prop 4.31__ The minimal isogeny

$$\varphi : (\tilde{X}, \tilde{\lambda}) \to (X, \lambda) = x$$

induces

$$\varphi^* : \mathrm{End}(X[p^\infty]) \hookrightarrow \mathrm{End}(\tilde{X}[p^\infty])$$

and

$$\mathrm{Mass}(\mathcal{C}(x)) = \mathrm{Mass}(\Lambda_{g,p^c}) \times$$

$$\underbrace{[\mathrm{Aut}((\tilde{X},\tilde{\lambda})[p^\infty]) : \mathrm{Aut}((X,\lambda)[p^\infty])]}_{\text{comparison factor}}$$

When $g \leq 3$, $\mathcal{C}(x) \twoheadrightarrow \Lambda_{g,p^c}$ can always be realised by minimal isogeny.

Moreover, the minimal isogeny is realised via the PFTQ!

<u>Example</u> $(g=3)$ $\quad x = (X, \lambda)$

$$y = (Y_2, \lambda_2) \xrightarrow{\rho_2} (Y_1, \lambda_1) \xrightarrow{\rho_1} (Y_0, \lambda_0) = (X, \lambda)$$

$a=3$

$a=2$

$a=1$

$a(X)=3$: minimal isogeny $=$ identity

$\Rightarrow \mathcal{C}(x) \twoheadrightarrow \Lambda_{3,1}$

$a(X)=2$: min. isogeny $\rho_1 : (Y_1, \lambda_1) \longrightarrow (X, \lambda)$

kernel $\alpha_p^2$

$\Rightarrow \mathcal{C}(x) \twoheadrightarrow \Lambda_{3,p}$

$a(X)=1$: min. isogeny $\rho_1 \circ \rho_2$, $\lambda_2$ principal

$\Rightarrow \mathcal{C}(x) \twoheadrightarrow \Lambda_{3,1}$

Comparison factor in Prop 4.31 depends on parameters $y = (t,u) \in \mathcal{P}_{3,\mu}(k)$

Eg.

<u>Thm 4.35</u> (K.-Yobuko-Yu)

Choose $y = (t,u) \in \mathcal{P}_{3,\mu}(k)$ s.t. $t \in C(\mathbb{F}_{p^2})$
so $a(y) \geq 2$

Then

$$\mathrm{Mass}(\mathcal{E}(x)) = \frac{L_p}{2^{10} \cdot 3^4 \cdot 5 \cdot 7}$$

where

$$L_p = \begin{cases} (p-1)(p^2+1)(p^3-1) & \text{if } u \in \mathbb{P}^1_t(\mathbb{F}_{p^2}) \\ (p-1)(p^3+1)(p^3-1)(p^4-p^2) & u \in \mathbb{P}^1_t(\mathbb{F}_{p^4}) \setminus \mathbb{P}^1_t(\mathbb{F}_{p^2}) \\ 2^{-e(p)}(p-1)(p^3+1)(p^3-1)p^2(p^4-1) & u \notin \mathbb{P}^1_t(\mathbb{F}_{p^4}) \end{cases}$$

$$e(p) = \begin{cases} 0 & p = 2 \\ 1 & p > 2 \end{cases}$$