GEOMETRY AND ARITHMETIC OF MODULI SPACES OF ABELIAN VARIETIES IN POSITIVE CHARACTERISTIC

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Introduction

This document serves as a set of lecture notes, supporting the course with the same name that I will teach at the Arizona Winter School 2024 on "Abelian Varieties".

Abelian varieties are important objects in arithmetic geometry. When studying their rational points, we can make use of the fact that they are group varieties. That is, the rational points over a fixed field form a group, which provides us with useful extra structure. In this course, we will consider abelian varieties over fields of positieve characteristic p, in particular *supersingular* abelian varieties, and study geometric and arithmetic properties of their moduli spaces.

In the outline below, every section will roughly correspond to one lecture.

- Section 1 provides an introduction to abelian varieties over finite fields of characteristic p.
- Section 2 introduces the moduli space $\mathcal{A}_g$ of g-dimensional principally polarised abelian varieties. For its characteristic p fibre, we will study its geometric structure by means of several stratifications by invariants.
- From Section 3 onwards, we specialise to the supersingular locus $\mathcal{S}_g \subseteq \mathcal{A}_g$. In this section we will study its geometry, explicitly in low dimensions, and generally using flag type quotients and the foliation by central leaves.
- Section 4 treats the arithmetic of $\mathcal{S}_g$, focussing on the endomorphism rings/algebras and automorphism groups of the abelian varieties, using masses and linking these to class number computations for quaternion algebras.

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1. Abelian varieties over finite fields

1.1. **Introduction.**

To get started, in this first section we will collect some useful information about abelian varieties, in particular when they are defined over finite fields. This is a vast topic and many good references on it already exist: see e.g. [45, 55, 71, 83, 85] and this year's PAWS notes of Lassina Dembélé. Here, we have therefore been quite selective and only included notions we will need later in the course. For a more extensive overview, you are encouraged to consult the above-mentioned references.

Subsection 1.2 will be quite general, dealing with abelian varieties over fields of any characteristic (zero or positive). In Subsection 1.3 we will specialise to the situation of characteristic $p > 0$, since this will be the main focus in this course. In characteristic p, interesting behaviour appears that does not occur in characteristic zero; we will exploit this in the next sections.

1.2. **Abelian varieties (in any characteristic).**

Throughout this subsection, we let K be any field of any characteristic. We let $\overline{K}$ denote its algebraic closure.

1.2.1. *First definitions.*

Definition 1.1. An *abelian variety* over a field K is a complete group variety. This means in particular that the K-rational points on X, denoted $X(K)$, have a group structure. The same is true for the rational points over any field extension of K, or more generally over any K-scheme T.

Moreover, every abelian variety is not only complete but also projective. It can be shown that the group structure on an abelian variety is commutative, justifying its name. The group structure is uniquely determined once a zero element has been fixed. Abelian varieties have rigidty properties which also imply that every regular map between abelian varieties is a composition of a homomorphism and a translation. Translation by x is denoted by t_x.

Definition 1.2. Let X be an abelian variety over K and let $\mathrm{Pic}^0(X)$ be the set of isomorphism classes of line bundles $\mathcal{L}$ on X that are translation invariant for all $x \in X(\overline{K})$, i.e. $t_x^*\mathcal{L} \approx \mathcal{L}$. Every abelian variety X over a field K has a *dual variety* $X^\vee = \mathrm{Pic}^0_{X/K}$.

That is, the dual abelian variety is the connected component of the group scheme representing the relative Picard functor $\mathrm{Pic}_{X/K}$, which sends any K-scheme T to the quotient $\{$ line bundles on $X \times T$ $\}/\{$ line bundles pulled back from T $\}$. It is unique up to (K-)isomorphism and satisfies

$$X^\vee(K) = \mathrm{Pic}^0(X).$$

The dual $X^\vee$ of X is again an abelian variety, and we have $\dim(X) = \dim(X^\vee)$ and $(X^\vee)^\vee = X$.

Similarly, we can construct the dual of a homomorphism $f : X \to Y$ of abelian varieties as $f^\vee : Y^\vee \to X^\vee$.

Definition 1.3. A homomorphism $f : X \to Y$ of abelian varieties is an *isogeny* if it is a finite flat surjective morphism of varieties. Equivalently, $\dim(X) = \dim(Y)$ and f is surjective, or equivalently $\dim(X) = \dim(Y)$ and $\ker(f)$ is a finite group scheme.

If we want to emphasise when an isogeny is defined over K, we may call it a K-isogeny. The *degree* of a K-isogeny is the degree of the induced function field extension $[K(X) : K(Y)]$.

If $f : X \to Y$ is an isogeny, its dual $f^\vee : Y^\vee \to X^\vee$ is also an isogeny and $\ker(f)^\vee = \ker(f^\vee)$, where $\ker(f)^\vee$ denotes the Cartier dual of the group scheme $\ker(f)$.

We say X and Y are *isogenous* if there exists an isogeny $f : X \to Y$; we write $X \sim Y$ in this case. By the existence of dual isogenies, being isogenous is an equivalence relation. The equivalence classes are called *isogeny classes*.

Definition 1.4. For a line bundle $\mathcal{L}$ on X let $\varphi_{\mathcal{L}} : X \to X^\vee$ be the homomorphism which on points is given by $x \mapsto [t_x^*\mathcal{L} \otimes \mathcal{L}^{-1}]$, where the square brackets denote the equivalence class of the resulting line bundle.

A *polarisation* of X is an isogeny $\lambda : X \to X^\vee$ such that there exist a finite field extension K' of K and an ample line bundle $\mathcal{L}$ on $X \times K'$ such that $\lambda_{K'} = \varphi_{\mathcal{L}}$.

The *degree* of a polarisation is the degree of the isogeny as in Definition 1.3.

A polarisation is self-dual, i.e. $\lambda^\vee = \lambda$. The degree of a polarisation is always a square. A polarisation which is an isomorphism, i.e. which has degree 1, is a *principal polarisation.*

Any abelian variety X admits a polarisation of some degree. If it admits a principal polarisation λ, we say that (X, λ) (or simply X) is a *principally polarised abelian variety.*

(Principal) polarisations provide geometric rigidifications which are necessary to construct well-defined moduli spaces; we will go into this in Section 2.

Definition 1.5. An abelian variety X is *simple* if its only abelian subvarieties are 0 and X.

Theorem 1.6 (Poincaré reducibility)**.** *Any abelian variety $X(\neq 0)$ over K is K-isogenous to a product*

$$X \sim Y_1^{k_1} \times \ldots \times Y_r^{k_r}, \tag{1}$$

where the Y_i are pairwise non-isogenous simple abelian varieties. Moreover, the varieties Y_i and multiplicities k_i are uniquely determined (up to K-isogeny).

1.2.2. *Endomorphisms.*

Definition 1.7. An *endomorphism* is a homomorphism from an abelian variety to itself. All endomorphisms of a fixed abelian variety form a ring under addition and composition; we denote the *endomorphism ring* of X by $\mathrm{End}(X)$. If we want to emphasise that the endomorphisms are K-endomorphisms, we may write $\mathrm{End}_K(X)$. When we consider the geometric endomorphisms, we will always write $\mathrm{End}_{\overline{K}}(X)$.

Example 1.8. For any X over K and any integer n, we have the endomorphism $[n] : X \to X$ given on points by $x \mapsto nx$. Thus, for any X we see that $\mathbb{Z} \to \mathrm{End}(X)$. The degree of $[n]$ is $n^{2\dim(X)}$ for any n.

If an isogeny $f : X \to Y$ has degree n, then there exists an isogeny $g : Y \to X$ which satisfies $f \circ g = g \circ f = [n]$.

The *endomorphism algebra* of X is $\mathrm{End}^0(X) := \mathrm{End}(X) \otimes_{\mathbb{Z}} \mathbb{Q}$. When X is simple, its endomorphism algebra is a division algebra, since any non-zero element $f \in \mathrm{End}^0(X)$ (of degree n) is invertible (namely, by $\frac{1}{n}g$).

When X is not simple, and admits an isogeny decomposition as in (1), we get that $\mathrm{End}^0(X) = \mathrm{Mat}_{k_1}(\mathrm{End}^0(Y_1)) \times \ldots \times \mathrm{Mat}_{k_r}(\mathrm{End}^0(Y_r))$, since $\mathrm{Hom}(Y_i, Y_i) = 0$ whenever $i \neq j$.

For simple varieties (and hence for general ones) we can say more. Recall that any abelian variety X admits a polarisation λ of some degree. This implies that its endomorphism algebra $\mathrm{End}^0(X)$ has a positive involution $\alpha \mapsto \lambda^{-1} \circ \alpha^\vee \circ \lambda$, called the *Rosati involution.* Such division algebras with positive involutions have been classified as follows.

Theorem 1.9 (Albert's Classification)**.** *The endomorphism algebra $E = \mathrm{End}^0(X)$ of a simple g-dimensional abelian variety X over K is isomorphic to one of the following:*

- (I) *A totally real field of degree dividing g;*
- (II) *A totally indefinite quaternion division algebra over a totally real field (i.e. split at each infinite place);*
- (III) *A totally definite quaternion division algebra over a totally real field (i.e. non-split at each infinite place);*
- (IV) *A central division algebra whose centre is a CM-field, i.e. a totally imaginary quadratic extension of a totally real field.*

Example 1.10. For an elliptic curve (i.e. a one-dimensional abelian variety) E over K we either have $\mathrm{End}_K(E) = \mathbb{Z}$ or $\mathrm{End}_K(E)$ is an order in either a quadratic imaginary field or in a quaternion algebra. In the first case we have $\mathrm{End}^0(E) = \mathbb{Q}$, in the latter cases the endomorphism algebra is either a quadratic imaginary extension of $\mathbb{Q}$ or a quaternion algebra over $\mathbb{Q}$. If $\mathrm{char}(K) = 0$ then the endomorphism algebra is necessarily commutative, so the quaternion case happens only when $\mathrm{char}(K) = p > 0$; the quaternion algebra is then the definite quaternion algebra $Q_{p,\infty}$ ramified at p and infinity.

1.2.3. *Tate modules.*
Now let X be a g-dimensional abelian variety and assume that ℓ is a prime number that is coprime to the characteristic of K. We saw in Example 1.8 that $[\ell^n]$, i.e. multiplication by ℓ^n, is an endomorphism of X for any $n \geq 1$, whose kernel is a finite group scheme of rank $(\ell^n)^{2g}$. This group scheme is étale by our assumption $(\ell, \mathrm{char}(K)) = 1$; in particular, it is determined by its $\overline{K}$-points and the action of $G_K = \mathrm{Gal}(\overline{K}/K)$ on it.

Definition 1.11. Let $X[\ell^n]$ denote the kernel of $[\ell^n]$. The *ℓ-adic Tate module* of X is the inverse limit $T_\ell(X) = \varprojlim_n X[\ell^n](\overline{K})$ where the transition maps are given by multiplication by ℓ:

$$X[\ell^n](\overline{K}) \xleftarrow{\cdot \ell} X[\ell^{n+1}](\overline{K}).$$

It is a free $\mathbb{Z}_\ell$-module of rank $2g$, which inherits a ($\mathbb{Z}_\ell$-linear) G_K-structure. Further, let $V_\ell(X) = T_\ell(X) \otimes_{\mathbb{Z}_\ell} \mathbb{Q}_\ell$; this is a $2g$-dimensional $\mathbb{Q}_\ell$-vector space.

Any isogeny $f : X \to Y$ between abelian varieties, which is surjective with finite kernel by definition, induces an injective map $T_\ell f : T_\ell X \to T_\ell Y$ with finite cokernel, and an isomorphism $V_\ell f : V_\ell(X) \to V_\ell(Y)$. Importantly, this association is injective, as proved by Weil:

Theorem 1.12. *Mapping $f \mapsto T_\ell f$ gives an injection*

$$\mathrm{Hom}(X, Y) \otimes_{\mathbb{Z}} \mathbb{Z}_\ell \hookrightarrow \mathrm{Hom}_{\mathbb{Z}_\ell[G_K]}(T_\ell(X), T_\ell(Y)).$$

In the next subsection, we will see how to modify these constructions when the chosen prime ℓ equals the characteristic p of the field. Moreover, in the setting of finite fields of characteristic $p \neq \ell$, Theorem 1.12 is also surjective, cf. [78].

1.3. **Abelian varieties in characteristic p.**
From now on, we will assume our abelian varieties to be defined over a field of characteristic p for some prime $p > 0$. In particular, we fix the following notation: we let $\mathbb{F}_q$ be a finite field extension of the prime field $\mathbb{F}_p$, and we let $k = \overline{\mathbb{F}}_p$ be their algebraic closure.

1.3.1. *Frobenius and Verschiebung.*
Whenever you are in characteristic p, you can be sure to find Frobeniuses lurking around. There are in fact a couple of different ones to distinguish.

Definition 1.13. For any scheme S in characteristic p (so $p\mathcal{O}_S = 0$), the *absolute Frobenius* $F_S : S \to S$ is the identity on the topological space $|S|$ and acts as the p-power map on the structure sheaf, i.e. $f \mapsto f^p$ for all $f \in \mathcal{O}_S$.

The *relative Frobenius* is defined in the relative setting, i.e. for schemes $g : X \to S$ where S is a scheme of characteristic p. Let $X^{(p)} = X \times_{S,F_S} S$ be the scheme fitting in the Cartesian diagram

$$\begin{array}{ccc} X^{(p)} & \longrightarrow & X \\ \downarrow & & \downarrow g \\ S & \xrightarrow{F_S} & S \end{array}$$

Since $g \circ F_X = F_S \circ g$, this diagram induces a morphism $F_{X/S} : X \to X^{(p)}$; this is the relative Frobenius. It is an S-morphism, while the absolute Frobenius F_X generally is not.

$$\begin{tikzcd}
X \arrow[dr, dashed, "F_{X/S}"] \arrow[drr, bend left, "F_X"] \arrow[ddr, bend right, "g"'] & & \\
& X^{(p)} \arrow[r] \arrow[d] & X \arrow[d, "g"] \\
& S \arrow[r, "F_S"] & S
\end{tikzcd}$$

We could do the same for any power p^n, obtaining the n-th iterate F_X^n acting by $f \mapsto f^{p^n}$ on functions, and $F_{X/S}^n : X \to X^{(p^n)}$.

We may apply the above either to an abelian scheme $X \to S$, i.e. a smooth proper S-group scheme whose fibres are all abelian varieties, or to an abelian variety X when S is the spectrum of a field of characteristic p. If $S = \mathrm{Spec}(\mathbb{F}_q)$ with $q = p^n$, then $F_X^n = F_{X/S}^n : X \to X$; this map is also denoted by π_X and called the *geometric Frobenius* or *Frobenius endomorphism* of X. By extension of scalars, we obtain geometric Frobeniuses over any field extension of $\mathbb{F}_q$ as well. Later on, we will mostly take $S = \mathrm{Spec}(k)$.

Finally, there is also an *arithmetic Frobenius* σ_{p^n} for any $n \geq 1$; this is the topological generator of the absolute Galois group $G_{\mathbb{F}_{p^n}} = \mathrm{Gal}(k/\mathbb{F}_{p^n})$ of $\mathbb{F}_{p^n}$.

Dually, since an abelian scheme over a scheme S of characteristic p is commutative and flat, then there exists a *Verschiebung* morphism $V_{X/S} : X^{(p)} \to X$ such that $V_{X/S} \circ F_{X/S} = [p]_X$ and $F_{X/S} \circ V_{X/S} = [p]_{X^{(p)}}$. For an abelian variety X over a field K of characteristic p, both $F_{X/K}$ and $V_{X/K}$ are isogenies of degree $p^{\dim(X)}$. We can similarly iterate the Verschiebung to obtain $V_{X/S}^n$; then $V_{X/S}^n \circ F_{X/S}^n = [p^n]_X$.

1.3.2. *Characteristic polynomial of Frobenius.*
We will now study the Frobenius endomorphism π_X of X in more detail. For ease of notation, we will write π instead of π_X when the variety X is clear from context.

Recall that π, being an isogeny from X to itself, induces maps $T_\ell \pi : T_\ell(X) \hookrightarrow T_\ell(X)$ and $V_\ell \pi : V_\ell(X) \to V_\ell(X)$ for any $\ell \neq p$; both maps are also denoted by π_ℓ. The latter has a characteristic polynomial $h_\pi(x) = \det(x \cdot \mathrm{id} - V_\ell \pi)$. It turns out that this characteristic polynomial has coefficients in $\mathbb{Z}$ and is independent of the prime ℓ.

Definition 1.14. We say $h_\pi(x) \in \mathbb{Z}[x]$ is the *characteristic polynomial of Frobenius* π on X. It is also called the *Weil polynomial* of X.

The above construction yields characteristic polynomials for any endomorphism of X. That for π however has special properties and significance. First we list some properties.

Theorem 1.15. *Let X be a g-dimensional abelian variety over $K = \mathbb{F}_q$ with Frobenius $\pi = \pi_X$.*

(1) *The characteristic polynomial $h_\pi(x)$ has degree $2g$.*
(2) *All complex roots of $h_\pi(x)$ have absolute value $\sqrt{q}$. They are called* (q-)Weil numbers.
(3) *The roots come in pairs: if α is a root then so is $\overline{\alpha} = q/\alpha$. The root $\sqrt{q}$ appears with even multiplicity.*

The significance of h_π is twofold. First of all, there is a direct relation to point counting on X. The main realisation for this is that $\mathbb{F}_{q^m}$-rational points on X are fixed by π_X^m.

For any variety over $\mathbb{F}_q$, not necessarily an abelian variety, its point counts over extensions are encoded in its zeta function.

Definition 1.16. The *zeta function* of a variety V over $\mathbb{F}_q$ is

$$Z(V, x) = \exp\left(\sum_{m \geq 1} N_m \frac{x^m}{m}\right) \in \mathbb{Q}[[x]], \qquad \text{where } N_m = |V(\mathbb{F}_{q^m})| \text{ for any } m \geq 1.$$

Theorem 1.17. *We use the same notation as in Thereom 1.15 and choose a factorisation $h_\pi(x) = \prod_{i=1}^{2g}(x - \alpha_i)$ over $\overline{\mathbb{Z}}$. For any $m \geq 1$, we have*

$$N_m = |X(\mathbb{F}_{q^m})| = \prod_{i=1}^{2g}(1 - \alpha_i^m).$$

Furthermore,

$$Z(X, x) = \frac{P_1(x) \cdots P_{2g-1}(x)}{P_0(x)P_2(x) \cdots P_{2g-2}(x)P_{2g}(x)},$$

where for any $0 \leq r \leq 2g$, we take

$$P_r(x) = \prod_{1 \leq j_1 < \ldots < j_r \leq 2g} (1 - (\alpha_{j_1} \cdots \alpha_{j_r})x) \in \mathbb{Z}[x].$$

Secondly, we may equivalently use the Frobenius endomorphism, Weil polynomials and Weil numbers to determine abelian varieties up to isogeny.

We say that two q-Weil numbers π, π' are *conjugate*, denoted $\pi \sim \pi'$, if they have the same minimal polynomial over $\mathbb{Q}$.

Theorem 1.18. *Let X, Y be two abelian varieties over $\mathbb{F}_q$. As mentioned below Theorem 1.12, we have an isomorphism*

$$\mathrm{Hom}(X, Y) \otimes_{\mathbb{Z}} \mathbb{Z}_\ell \xrightarrow{\sim} \mathrm{Hom}_{\mathbb{Z}_\ell[G_K]}(T_\ell(X), T_\ell(Y)). \tag{2}$$

From this, it can be shown that two simple abelian varieties X, Y with respective Frobenius endomorphisms π_X, π_Y are isogenous if and only if $h_{\pi_X} = h_{\pi_Y}$, if and only if $Z(X, x) = Z(Y, x)$. Moreover, for every Weil number there exists a simple abelian variety with this Weil number. That is, mapping an abelian variety X to its Frobenius endomorphism π_X yields a bijection

$$\{ \text{ simple abelian varieties over } \mathbb{F}_q \ \}/\sim \ \Leftrightarrow \ \{ \ q\text{-Weil numbers } \}/\sim . \tag{3}$$

Theorem 1.18 is often called the Honda-Tate theorem; injectivity in Equation (3) was proven by Tate [79] and surjectivity by Honda [27].

1.3.3. *p-torsion in characteristic p.*
In Definition 1.11 we considered the ℓ^n-torsion group schemes $X[\ell^n]$ when ℓ is coprime to the characteristic of the field, which is étale and of rank $(\ell^n)^{2\dim(X)}$. By contrast, the p^n-torsion group scheme $X[p^n]$ in characteristic p is not étale. As a consequence, the rank of its étale part is smaller, and at most $p^{n\dim(X)}$.

Definition 1.19. Let X be an abelian variety over a field of characteristic p with algebraic closure k. The *p-rank* of X, denoted $f(X)$, is the integer f such that

$$|X[p](k)| = p^f.$$

When $\dim(X) = g$, we have $0 \leq f \leq g$.

Definition 1.20. Assume we are in the same setting as Definition 1.19. When $f(X) = g$, the variety is called *ordinary*.

Ordinary varieties are called this way because generically the p-rank is as large as it can be.

Example 1.21. Suppose that $g = \dim(X) = 1$, so X is an elliptic curve. Then $0 \leq f(X) \leq 1$, so the p-rank of X is either 0 or 1. If $f(X) = 1 = g$, the elliptic curve is ordinary. If $f(X) = 0$, then $X[p](k) = \{0\}$, i.e. the elliptic curve has no p-torsion points. In this case it is called a *supersingular elliptic curve*.

Example 1.21 allows us to give the following definition.

Definition 1.22. Again let X be a g-dimensional abelian variety over a field of characteristic p with algebraic closure k. Then X is *supersingular* if, over k, it is isogenous to a product of g supersingular elliptic curves:

$$X \sim_k E_1 \times \ldots \times E_g,$$

with $E_i[p](k) = \{0\}$ for all $1 \leq i \leq g$, and *superspecial* if, over k, it is moreover isomorphic to such a product:

$$X \simeq_k E_1 \times \ldots \times E_g.$$

Supersingular abelian varieties will be the main players in the second half of this course. They are called supersingular not because they are singular, but because they are much rarer than ordinary varieties.

The following result shows that all superspecial abelian varieties of the same dimension ≥ 2 are k-isomorphic, and hence that all supersingular abelian varieties of the same dimension ≥ 2 are k-isogenous.

Proposition 1.23. (Deligne, [59, Theorem 6.2], [75, Theorem 3.5]) *Let $n \geq 2$ and let $E_1, \ldots, E_{2n}$ be supersingular elliptic curves over k. Then $E_1 \times \ldots \times E_n \simeq E_{n+1} \times \ldots \times E_{2n}$.*

Remark 1.24. There is a number of equivalent definitions of supersingularity. One such is the following: an abelian variety X is supersingular if all of its Weil numbers α satisfy that $\alpha/\sqrt{q}$ is a complex root of unity. In Subsection 2.3 we will see that two other definitions are that its p-divisible group is k-isogenous to $G_{1,1}^{\oplus \dim(X)}$, or that its Newton polygon is a line segment of unique slope $1/2$.

From the above, it may seem that we could have equivalently defined a supersingular variety to have p-rank zero. While supersingular varieties will always have p-rank zero, the other implication holds only in dimensions 1 and 2.

We also saw in Definition 1.11 how to construct ℓ-adic Tate modules of X over K for any prime ℓ coprime to $\mathrm{char}(K)$. This construction thus also works well when working over fields of characteristic p, as long as $\ell \neq p$. The analogous p-adic Tate module of a g-dimensional abelian variety X would have rank $f \leq g$ (instead of $2g$), so we lose some information with this construction. Instead, we therefore work with the p-divisible group.

Definition 1.25. Let X be an abelian variety over a field of characteristic p. Its *p-divisible group* is the direct limit of group schemes

$$X[p^\infty] = \varinjlim_n X[p^n]$$

with respect to the natural inclusions $X[p^n] \hookrightarrow X[p^{n+1}]$. The rank of $X[p]$ as a group scheme is $p^{2\dim X}$, hence the *height* of $X[p^\infty]$ is defined to be $2\dim(X)$.

A notion closely related to the p-divisible group of an abelian variety is its Dieudonné module. We first define Dieudonné modules in general, cf. [45, §5.2].

Definition 1.26. Let K be a perfect field of characteristic p (e.g. $K = \mathbb{F}_q$ or $K = k = \overline{\mathbb{F}}_p$). Let $W = W(K)$ be the ring of infinite Witt vectors over K with an automorphism σ induced from the automorphism $x \mapsto x^p$ on K. Next, let F, V be indeterminates satisfying $FV = VF = p$ with commutation rules $wV = V\sigma(w)$ and $Fw = \sigma(w)F$ for all $w \in W$. Define $A = \varprojlim_n W[F, V]/p^n W[F, V]$. Then a *Dieudonné module* over W module is a left A-module which is finitely generated as a W-module.

There is an anti-equivalence $G \mapsto M(G)$ between finite commutative group schemes G over K of p-power rank (p^n) and left A-modules $M(G)$ of finite W-length (n). We now use this to determine the Dieudonné module of an abelian variety through its p-divisible group.

Definition 1.27. Let X be an abelian variety over a field K of characteristic p. Its (contravariant) *Dieudonné module* is

$$M(X) = M(X[p^\infty]) = \varprojlim_n M(X[p^n]),$$

where for each n,
$$M(X[p^n]) := \varinjlim_m \mathrm{Hom}_K(X[p^n], W_m),$$
where W_m is the m-th Witt group scheme so the scheme $\varinjlim_m W_m = \mathcal{W}$ satisfies $\mathcal{W}(K) \simeq W(K)$, cf. [45, §5.1]. Then $M(X[p^n])$ is a free W/p^nW-module of rank $2\dim(X)$ for every n, and the Dieudonné module of X is free of rank $2\dim(X)$ over W.

The Frobenius and Verschiebung maps on abelian varieties translate into semi-linear operators on their Dieudonné modules. While their definition might seem a bit cumbersome, the structure of these modules is well understood and explicit, making Dieudonné modules great tools with which to study abelian varieties.

In fact, many important results about abelian varieties (about their moduli spaces, deformations, etc.), some of which are contained in the later sections of these notes, were proved by first proving the corresponding result for Dieudonné modules. To make these notes as self-contained and computation-light as possible, I have omitted these proofs, referring to the reference instead.

For now, the main thing to take away is that Dieudonné modules are really the "right" objects to study, since the analogue of Tate's theorem (Theorem 1.12, Equation (2)) now holds:

Theorem 1.28. *If X, Y are two abelian varieties over a finite field $K = \mathbb{F}_q$, then there is an isomorphism*
$$\mathrm{Hom}(X, Y) \otimes_{\mathbb{Z}} \mathbb{Z}_p \xrightarrow{\sim} \mathrm{Hom}_A(M(Y), M(X)). \tag{4}$$

Note that the order on the right-hand side of Equation (4) is the opposite of that in (2), by contravariance of the Dieudonné module.

Remark 1.29. The p-torsion and p-divisible group of an abelian variety gives rise to other invariants of the variety, such as the Newton polygon and the Ekedahl-Oort type. We will define and study these in detail in the next section.

1.3.4. *The a-number.*
To conclude this section we introduce the a-number, another important invariant of abelian varieties which we will use many times in the next sections. We first define the group scheme α_p appearing in its definition.

Definition 1.30. Let α_p denote the finite group scheme representing $R \mapsto \mathrm{Spec}(R[x]/(x^p))$ for any ring R of characteristic p. In other words, it is the kernel of the Frobenius morphism on the additive group $\mathbb{G}_a$.

It can be shown that α_p is one of the three non-isomorphic group schemes over k of rank p, the others being μ_p and $\mathbb{Z}/p\mathbb{Z}$; the latter are each other's Cartier dual, while α_p is self-dual.

Definition 1.31. Again let X be an abelian variety over a field K of characteristic p. Its *a-number* is
$$a(X) := \dim_K \mathrm{Hom}(\alpha_p, X).$$
The a-number does not depend on the ground field, so we could replace K with any extension here, and we will later often use $k = \overline{\mathbb{F}}_p$ instead.

The a-number of a Dieudonné module M is $\dim_K M/(F, V)M$, where F, V respectively denote the semi-linear Frobenius and Verschiebung operators. Then $a(X) = a(M(X))$.

Remark 1.32. When $g = \dim(X)$, we have $0 \leq a(X) \leq g$, and even $0 \leq a(X) + f(X) \leq g$. Generically, the a-number of a non-ordinary abelian variety (with $f(X) < g$) is 1.

Superspecial abelian varieties have maximal a-number $g = \dim(X)$ by definition (Definition 1.22). In fact the converse holds too, cf. [62, Theorem 2].

2. The moduli space $\mathcal{A}_g$ of principally polarised abelian varieties

2.1. Introduction.

In the previous section, we collected some useful facts to study abelian varieties over finite fields and their arithmetic properties. Now, rather than considering individual varieties, we will develop the tools that are needed to consider *families* of abelian varieties. This will enable us to study the variation in arithmetic properties of the varieties.

To this end, we will first introduce the concept of a *moduli space* in Subsection 2.2. Very roughly speaking, the points of a moduli space correspond to isomorphism classes of varieties. The main advantage of working with moduli spaces is that these are (at least in favourable cases like $\mathcal{A}_g$) themselves schemes, whose geometry we can study.

We will define the moduli space $\mathcal{A}_g = \mathcal{A}_{g,1}$ of principally polarised g-dimensional abelian varieties, which was first constructed by Mumford in [53]. This moduli space is defined over $\mathbb{Z}$, but we will mostly be interested in the characteristic p fibre $\mathcal{A}_g \otimes \mathbb{F}_p$, which for ease of notation will again be denoted $\mathcal{A}_g$ (Notation 2.10).

Example 2.1. You might already be familiar with the moduli space $\mathcal{A}_1 \otimes \mathbb{C}$ of elliptic curves over the complex numbers. By complex uniformisation, for any elliptic curve $E/\mathbb{C}$ we have a description $E(\mathbb{C}) \simeq \mathbb{C}/\Lambda$ as a complex torus with some lattice Λ. Two such complex tori are isomorphic if and only if the corresponding lattices are homothetic, i.e. they differ by a complex scalar. Every homothety class of lattices has a representative $\mathbb{Z} \oplus \mathbb{Z}\tau$ for some τ in the complex upper-half plane $\mathfrak{H} = \{z \in \mathbb{C} : \mathrm{im}(z) > 0\}$. Now $\mathfrak{H}$ carries an action of $\mathrm{SL}_2(\mathbb{Z})$ through linear fractional transformations - or rather of $\Gamma = \mathrm{SL}_2(\mathbb{Z})/\{\pm\mathrm{Id}\}$, since $\pm\mathrm{Id}$ both act trivially. The upshot is that points of the quotient space $\Gamma \setminus \mathfrak{H}$ correspond to isomorphism classes of complex elliptic curves, so we can think of this space as the moduli space of complex elliptic curves.

In Subsections 2.3 and 2.4 we will introduce four different *stratifications* on $\mathcal{A}_g$, which are defined using different (isogeny or isomorphism) invariants of the varieties corresponding to the moduli points. Again roughly speaking, a stratification is a certain way in which to break up a space into disjoint locally closed subsets. Often, it is easier to study individual strata than to study them all at the same time.

First, in Subsection 2.3, we will treat the p-rank and Newton (polygon) stratifications, which are determined by isogeny invariants of respectively the p-rank and the p-divisible group of the abelian varieties. The Newton stratification is a refinement of the p-rank stratification, i.e. the p-rank is constant on each Newton stratum.

Second, in Subsection 2.4 we construct the a-number and Ekedahl-Oort stratifications, which are defined in terms of isomorphism invariants, namely the a-number and the canonical filtration of the p-torsion subscheme of the abelian varieties, respectively. The Ekedahl-Oort stratification is a refinement of the a-number stratification, as well as of the p-rank stratification (since if p-torsion subschemes are isomorphic, their sets of k-rational points have the same cardinality).

2.2. The moduli space $\mathcal{A}_g$.

A moduli space (or moduli scheme) gives a way of classifying, or parametrising, a set of objects. In algebraic geometry, these objects are typically algebraic varieties; for us, the objects will be abelian varieties. There are two flavours of moduli spaces, which we define as follows.

Definition 2.2. Let $F : \{\text{Scheme}\} \to \{\text{Set}\}$ be a contravariant functor that sends any scheme S to the set of isomorphism classes of objects over S.

(1) A *coarse moduli space* is a scheme $\mathcal{F}$ with a natural transformation $F \to \mathrm{Hom}(-, \mathcal{F})$, such that over an algebraically closed field k, the k-rational points $\mathcal{F}(k)$ are in bijection with the set $F(k)$. Moreover, we require that for any other scheme $\mathcal{F}'$ with this property, the natural transformation $F \to \mathrm{Hom}(-, \mathcal{F}')$ factors uniquely through $F \to \mathrm{Hom}(-, \mathcal{F})$.

(2) A *fine moduli space* is a scheme $\mathfrak{F}$ representing F, i.e. for each scheme S we have an isomorphism $F(S) = \mathrm{Hom}(S, \mathfrak{F})$. There is a universal family (namely, the unique element of $F(\mathfrak{F}) = \mathrm{Hom}(\mathfrak{F}, \mathfrak{F})$ corresponding to the identity map), which has the property that any family of objects over S is uniquely a pullback of it.

In other words, while the fine moduli space actually represents the functor F if it exists, the coarse moduli space does not have a universal family, but comes as close as possible to representing F. The existence of a universal family (and hence of a fine moduli space) can be obstructed by the existence of non-trivial automorphisms of the objects. An alternative solution to this is to work with (moduli) stacks; however, we will not use this terminology in this course.

The following functor was first introduced in this way by Mumford [56] in the 1960's.

Definition 2.3. For integers $g, d, n \geq 1$, consider the functor

$$\mathcal{A}_{g,d,n} : S \mapsto \{(X, \lambda, \sigma)\}$$

where for any locally noetherian base scheme S on which n is invertible, the image is the set of isomorphism classes of triples with X/S a g-dimensional abelian scheme, λ a polarisation on X of degree d^2, and $\sigma : (\mathbb{Z}/n\mathbb{Z})^{2g} \xrightarrow{\sim} X[n]$ a level-n structure on X/S.

We will mostly be interested in the case where $S = \operatorname{Spec}(K)$ for a field K. Note that in Definition 2.3 we allow $n = 1$; we write $\mathcal{A}_{g,d} = \mathcal{A}_{g,d,1}$. Further setting $d = 1$ means that we are restricting ourselves to principally polarised abelian varieties; we write $\mathcal{A}_g = \mathcal{A}_{g,1}$.

Theorem 2.4. (cf. [56, Theorems 7.9 and 7.10])

(1) *For $n \geq 3$, the functor $\mathcal{A}_{g,d,n}$ is represented by a fine moduli scheme, denoted $A_{g,d,n}$, which is defined over* $\operatorname{Spec}(\mathbb{Z}[1/n])$ *and quasi-projective.*
(2) *For any $g, d, n \geq 1$, this functor has a coarse moduli space, often again denoted $\mathcal{A}_{g,d,n}$, which is defined over* $\operatorname{Spec}(\mathbb{Z}[1/n])$ *and quasi-projective.*

Corollary 2.5. *The coarse moduli space $\mathcal{A}_g$ of principally polarised abelian varieties (with level-1 structure) exists over* $\operatorname{Spec}(\mathbb{Z})$ *and is quasi-projective.*

Theorem 2.6. (cf. [28, pp. 106-107] and [60, Theorem 2.4.1]) *For any d and n (including $d = n = 1$), the moduli space $\mathcal{A}_{g,d,n} \to \operatorname{Spec}(\mathbb{Z}[1/n])$ has relative dimension $g(g+1)/2$, and is smooth over* $\operatorname{Spec}(\mathbb{Z}[1/dn])$ *if $n \geq 3$.*

Example 2.7. We saw in Example 2.1 how to construct a coarse moduli space $\Gamma \setminus \mathfrak{H}$, with $\Gamma = \mathrm{SL}_2(\mathbb{Z})/\{\pm \mathrm{Id}\}$ and $\mathfrak{H} = \{z \in \mathbb{C} : \operatorname{im}(z) > 0\}$, of elliptic curves over $\mathbb{C}$ by viewing $E(\mathbb{C}) \simeq \mathbb{C}/\Lambda$ as a complex torus. For higher-dimensional principally polarised abelian varieties X over $\mathbb{C}$, say of dimension g, we can similarly identify $X(\mathbb{C}) \simeq \mathbb{C}^g/\Lambda$ for some lattice $\Lambda \subseteq \mathbb{C}^g$. Again similarly, we find a coarse moduli space $\mathcal{A}_g \otimes \mathbb{C} \simeq \Gamma_g \setminus \mathfrak{H}_g$, where $\Gamma_g = \mathrm{Sp}_{2g}(\mathbb{Z})$ (and where again $\{\pm \mathrm{Id}\}$ act trivially) and where $\mathfrak{H}_g = \{M \in \mathrm{Mat}_g(\mathbb{C}) : \operatorname{im}(M) > 0, M = M^t\}$ is the Siegel upper-half plane. Considering abelian varieties with level-n structure (with respect to a choice of primitive n-th root of unity) comes down to considering the quotient $\Gamma_g(n) \setminus \mathfrak{H}_g$ where $\Gamma_g(n) = \{A \in \mathrm{Sp}_{2g}(\mathbb{Z}) : A \equiv \mathrm{Id}_{2g} \bmod n\}$.

Example 2.8. While Example 2.1 treated elliptic curves over $\mathbb{C}$, we can consider elliptic curves and their moduli space over any field K. If $K = \overline{K}$ is algebraically closed, the j-invariant of an elliptic curve effectively encodes its isomorphism class over K. Thus a coarse moduli space for elliptic curves is obtained by mapping a curve E to its j-invariant $j(E)$ on the affine line $\mathbb{A}^1$ ("the j-line"). A fine moduli space generally does not exist because elliptic curves may have non-trivial automorphisms.

Remark 2.9. The moduli space $\mathcal{A}_g$, and more generally $\mathcal{A}_{g,n,d}$, have been studied in detail by many mathematicians after Mumford. A detailed discussion is beyond the scope of these notes; here we only mention some facts (cf. also [12, §3, pp. 4-9]).

- Chai and Faltings proved that $\mathcal{A}_g \otimes K$ is irreducible for any field K, cf. [13].
- A result attributed to Freitag, Tai and Mumford states that $\mathcal{A}_g \otimes \mathbb{C}$ is of general type for $g \geq 7$, cf. [54].
- The space $\mathcal{A}_g$ is not compact; over the years several different compactifications of $\mathcal{A}_g$ have been constructed by Satake [74], Baily-Borel [2], Chai and Faltings [13], and Alexeev [1].

Notation 2.10. In this course, we will only work in characteristic p (with $p > 0$). Thus, we will only consider the fibre $\mathcal{A}_g \otimes \mathbb{F}_p$. To ease notation, we will denote this again by $\mathcal{A}_g$. Moreover, we will sometimes further ease the notation by identifying $\mathcal{A}_g$ with $\mathcal{A}_g(k)$, where $k = \overline{\mathbb{F}}_p$, e.g. when writing "$(X, \lambda) \in \mathcal{A}_g$" to mean the principally polarised abelian variety (X, λ) over k.

Later in this section, we will be concerned with various stratifications of $\mathcal{A}_g$:

Definition 2.11. A *stratification* of a scheme X is a partition of X into a disjoint union of finitely many locally closed subsets. A *good stratification* satisfies the extra property that the Zariski closure of each stratum is a union of the stratum itself and lower-dimensional strata.

2.3. **The p-rank and Newton stratifications.**

We introduce two stratifications on $\mathcal{A}_g$, which are respectively determined by the p-rank of the abelian variety and the isogeny type of the p-divisible group of the variety; the latter is combinatorially encoded in the Newton polygon of the variety.

Both stratifications are therefore isogeny invariants, meaning that in each, two isogenous varieties will lie in the same stratum. Furthermore, the stratification by Newton polygon is a refinement of the stratification by p-rank, since truncations of isogenous p-divisible groups will yield isogenous p-torsion schemes.

Below, we will state the main facts about p-rank strata, and spend most of our time on the Newton stratification.

2.3.1. *The p-rank stratification.*

Recall from Definition 1.19 that the p-rank $f(X)$ of X/k is the integer f such that $|X[p](k)| = p^f$. The p-rank is an isogeny invariant, and $0 \leq f(X) \leq g = \dim(X)$. Below, we first give an alternative definition. Then we define the p-rank strata V_f and study some of their properties.

Definition 2.12. Let X be an abelian variety over a field of characteristic p with algebraic closure k. We may equivalently define the p-rank of X as the stable rank of its Hasse-Witt matrix. This matrix is a representation of the action of the induced Frobenius map F^* on the Čech cohomology, i.e. $F^* : H^1(X, \mathcal{O}_X) \to H^1(X, \mathcal{O}_X)$. Its stable, or semi-simple, rank is the dimension of the semisimple part of $H^1(X, \mathcal{O}_X)$ under this action, i.e.

$$f = \dim H^1(X, \mathcal{O}_X)_{\mathrm{ss}} = \dim(\cap_{n=1}^{\infty} \mathrm{im}((F^*)^n)).$$

Definition 2.13. For any $0 \leq f \leq g$, consider the subset

$$V_f = \{x = (X, \lambda) \in \mathcal{A}_g(k) : f(X) \leq f\}.$$

These form closed subschemes of $\mathcal{A}_g$ by [61, Corollary 1.5]. We call such an V_f a *p-rank stratum.* We see that $V_f \subseteq V_{f+1}$ for any $f \leq g - 1$.

One of the first results on the V_f was the following, originally stated by Oort for not necessarily algebraically closed fields k.

Lemma 2.14. (cf. [61, Lemmas 1.4 and 1.6]) *Let S be an irreducible k-scheme and $X \to S$ an abelian scheme. Let f be the p-rank of the generic fibre and let $W \subseteq S$ be the closed subset over which the fibre has p-rank at most $f - 1$. Then either $W = \emptyset$ or every component of W has codimension 1 in S.*

Denoting an irreducible component of V_f by W_f, this lemma says that if $W_{f-1} \subseteq W_f$ and $W_{f-1} \neq W_f$, then $\dim(W_f) - \dim(W_{f-1}) = 1$. For any $f < g$, it follows inductively that the codimension of any W_f in $\mathcal{A}_g$ is at most $g - f$.

To prove the following result, Koblitz [40] establishes the reverse inequality, by computing the codimension of the Zariski tangent space to any V_f via local deformations of the abelian varieties.

Theorem 2.15. (cf. [40, Theorem 7.(1)]) *For any $0 \leq f \leq g$, we have*

$$\mathrm{codim}(W_f) = g - f.$$

In the same theorem, Koblitz establishes that V_f is smooth at those varieties whose Hasse-Witt matrix has (full) rank $g-1$.

Remark 2.16. It follows from Theorem 2.15 that each irreducible component W_f contains an open dense set of points with p-rank f; otherwise W_f would be an irreducible component of V_{f-1} and then $\operatorname{codim}(W_f) \leq g-f+1$, contradiction. Hence, the p-rank strata form a good stratification of $\mathcal{A}_g$.

Example 2.17. (cf. [40, § 11, p. 193]) Let $g=2$, so $\dim(\mathcal{A}_2)=3$ and $0 \leq f \leq 2$, so $V_2 = \mathcal{A}_2$. Koblitz states there are four isomorphism types of Hasse-Witt matrices, with the following representatives:

$$\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, \quad \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}, \quad \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}, \quad \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}.$$

- The abelian varieties X with p-rank $f(X)=2$ have Hasse-Witt matrix $\left(\begin{smallmatrix} 1 & 0 \\ 0 & 1 \end{smallmatrix}\right)$.
- The varieties with p-rank $f(X)=1$ have Hasse-Witt matrix $\left(\begin{smallmatrix} 1 & 0 \\ 0 & 0 \end{smallmatrix}\right)$. Theorem 2.15 yields that $\operatorname{codim}(V_1) = 2-1 = 1$, so $\dim(V_1)=2$.
- The varieties with p-rank zero have Hasse-Witt matrix either $\left(\begin{smallmatrix} 0 & 1 \\ 0 & 0 \end{smallmatrix}\right)$ or $\left(\begin{smallmatrix} 0 & 0 \\ 0 & 0 \end{smallmatrix}\right)$; these correspond to (non-superspecial) supersingular and superspecial surfaces, respectively. Theorem 2.15 yields that $\dim(V_0)=1$. (In fact $V_0 = \mathcal{S}_2$ is precisely the supersingular locus, which indeed has dimension $\lfloor 2^2/4 \rfloor = 1$.)
- V_1 and V_0 are both singular precisely at the varieties with Hasse-Witt matrix $\left(\begin{smallmatrix} 0 & 0 \\ 0 & 0 \end{smallmatrix}\right)$; in both cases these are isolated points, conic (A_1) singularities in V_1 and ordinary $(p+1)$-points in V_0. These points correspond precisely to superspecial abelian varieties.

Five years after Koblitz' results, Norman and Oort [57] generalise Theorem 2.15 to abelian varieties that are polarised but not principally polarised, i.e. to the moduli space $\mathcal{A}_{g,d}$. Rather than directly studying deformations of abelian varieties, Norman-Oort prove facts about deformation spaces of the corresponding Dieudonné modules. Their result can be stated as follows.

Theorem 2.18. (cf. [57, Theorems 3.1 and 4.1])

(1) *Let V_f be the closed subscheme of $\cup_{d=1}^{\infty} \mathcal{A}_{g,d}$ of abelian varieties with p-rank at most f. Any irreducible component W_f of V_f has codimension $g-f$. Its generic point has a-number* 1.

(2) *The generic point of any component of $\mathcal{A}_{g,d}$ is an ordinary variety (with maximal p-rank $f(X)=g$) and the dimension of each component is $\frac{g(g+1)}{2}$.*

2.3.2. *The Newton polygon stratification.*

To any abelian variety X we can associate a Newton polygon, which is an isogeny invariant that depends on the canonical decomposition of its p-divisible group. We therefore first provide a general decomposition result for p-divisible groups up to k-isogeny due to Manin (Theorem 2.19), then give its form for p-divisible groups of abelian varieties (Theorem 2.21), and describe how to attach a Newton polygon to this data (Definition 2.23). Then we define the Newton (polygon) strata and study some of their properties.

Recall the definition of the p-divisible group $X[p^\infty]$ of an abelian variety X over k from Definition 1.25. The following result gives a decomposition result for any p-divisible group (not necessarily coming from an abelian variety) up to k-isogeny.

Theorem 2.19. (cf. [48, §II.4], see also [6, §IV.4]) *Any p-divisible group Y is k-isogenous to a finite direct product*

$$Y \sim_k \prod_i G_{m_i,n_i},$$

where for any pair of coprime integers (m,n), $G_{m,n}$ is the unique (up to isogeny) isosimple p-divisible group whose dimension is m, whose height is $m+n$, and whose dual has dimension n.

Remark 2.20. When $(m, n) = (1, 0)$, $G_{m,n}$ is the formal group of $\mathbb{G}_m$; when $(m, n) = (0, 1)$ it is $\mathbb{Q}_p/\mathbb{Z}_p$, and otherwise it is a local-local group scheme. We see that an ordinary elliptic curve has p-divisible group $G_{1,0} \oplus G_{0,1}$ while a supersingular elliptic curve has p-divisible group $G_{1,1}$.

Since any abelian variety admits a polarisation (of some degree), its Dieudonné module admits a quasi-polarisation; this is equivalent to a symmetry condition on its p-divisible group. So for p-divisible groups of abelian varieties, Theorem 2.19 specialises to the following statement.

Theorem 2.21. (cf. [48, §IV.3, Theorem 4.1], see also [45, §1.4]) *Any p-divisible group $X[p^\infty]$ of an abelian variety X is k-isogenous to a direct product*

$$X[p^\infty] \sim_k \prod_i (G_{m_i,n_i} \oplus G_{n_i,m_i}) \bigoplus G_{1,1}^{\oplus s} \bigoplus (G_{1,0} \oplus G_{0,1})^{\oplus f}, \tag{5}$$

for $m_i, n_i \in \mathbb{Z}_{>0}$ coprime, and $0 \leq s, f$ such that $s + f \leq g$. This decomposition is also called the formal isogeny type *of X.*

Remark 2.22. We see from Theorem 2.21 that X is supersingular if $X[p^\infty] \sim_k G_{1,1}^{\oplus \dim(X)}$; this is in fact an equivalent definition of supersingularity. We also see that f is the p-rank of X and in particular that X is ordinary if $X[p^\infty] \sim_k (G_{1,0} \oplus G_{0,1})^{\oplus \dim(X)}$.

Using the formal isogeny type of an abelian variety X, we now construct its Newton polygon. This procedure generalises to any p-divisible group.

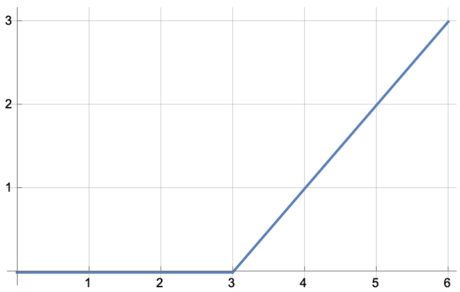
Definition 2.23. (cf. [64, §1.6]) Let X be a g-dimensional abelian variety over k with formal isogeny type given by (5). To every $G_{m,n}$ we associate a *slope* $\lambda = \frac{m}{m+n}$ and a *multiplicity* $m+n$. Arrange the slopes in non-decreasing order. This determines a ("g-dimensional") Newton polygon starting at $(0, 0)$ and ending at $(2g, g)$, by joining line segments of the prescribed slopes λ with length equal to their respective multiplicities. We denote it by $\mathcal{N}(X)$.

The Newton polygon is lower convex and has its breakpoints at integral coordinates, since every slope appears with a multiplicity that is a multiple of its denominator. By symmetry of (5), the Newton polygon is also symmetric, in the sense that any slope λ appears with the same multiplicity as the slope $1 - \lambda$.

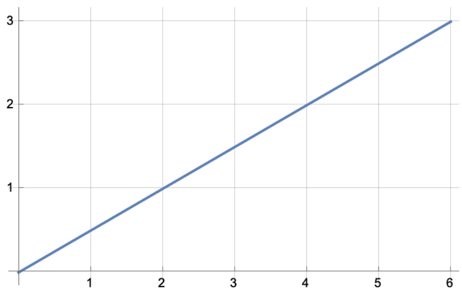
Notation 2.24. The ordinary Newton polygon is often denoted ρ and the supersingular one σ.

Example 2.25. The slopes of a g-dimensional ordinary abelian variety are 0 and 1, each with multiplicity g (since $G_{1,0}$ has slope $1/(1 + 0) = 1$ and $G_{0,1}$ has slope $0/(0 + 1) = 0$); those of a supersingular abelian variety are $1/2$ everywhere (since $G_{1,1}$ has slope $1/(1 + 1) = 1/2$).

Below we have drawn the Newton polygon of an ordinary threefold and that of a supersingular threefold (so $g = 3$).



ρ



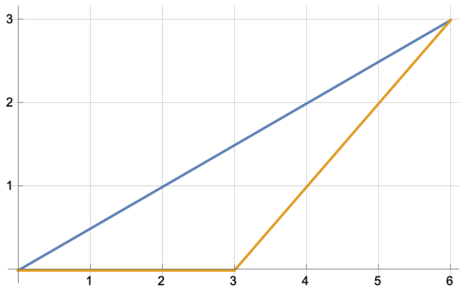
σ

Manin conjectured in [48, §IV.5, Conjecture 2, p. 76] that the converse of his Theorem 2.21 also holds. That is, he conjectured that every formal isogeny type of the form (5) (or equivalently, every symmetric Newton polygon as in Definition 2.23) occurs as $\mathcal{N}(X)$ for some abelian variety X in any positive characteristic. This was first proved independently by Honda and

Serre, cf. [79, p. 98]. It was later reproved by Oort using deformation theory, cf. [64, §5]. The latter methods were also used to prove strong results on Newton polygon strata (see Theorem 2.34), as we will explain below.

Definition 2.26. Consider the set of g-dimensional symmetric Newton polygons. We put a partial ordering on this set, by defining that $\alpha \prec \beta$ for two polygons α and β if no point of α lies strictly below β. We say "α lies above β".

Example 2.27. We see from Example 2.25 that $\sigma \prec \rho$. In fact $\sigma \prec \xi \prec \rho$ for any other symmetric Newton polygon ξ, so ξ will lie strictly between ρ and σ.



ρ in orange, σ in blue

Definition 2.28. For any g-dimensional symmetric Newton polygon ξ, we define the subsets

$$W_\xi := \{(X,\lambda) \in \mathcal{A}_g : \mathcal{N}(X) \prec \xi\};$$
$$W_\xi^0 := \{(X,\lambda) \in \mathcal{A}_g : \mathcal{N}(X) = \xi\}.$$

It was proved by Katz (cf. [39, Theorem 2.3.1, Corollary 2.3.2]) that the W_ξ are closed, hence the W_ξ^0 are locally closed. Both are called Newton polygon strata; often the W_ξ are *closed* strata while the W_ξ^0 are *open* strata. The stratification by $\{W_\xi^0\}_\xi$ is a good stratification of $\mathcal{A}_g$.

Remark 2.29. In Definition 1.19 we gave two equivalent definitions of the p-rank of an abelian variety X over k. A third equivalent definition is that the p-rank of X equals the number of zero slopes in the Newton polygon of X. The lowest Newton polygon with prescribed p-rank f is $\alpha = f(1,0) + (g-f-1,1) + (1,g-f-1) + f(0,1)$, according to [63, Remark 3.3]. That means that $W_\alpha = V_f$, i.e. the p-rank f stratum coincides with the Newton stratum of α.

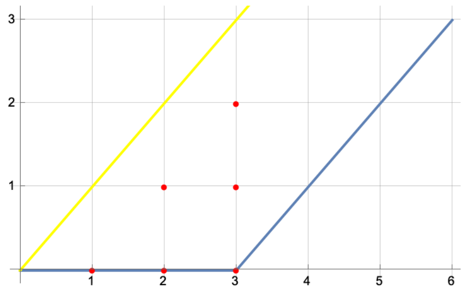
Definition 2.30. (cf. [65, §1.9]) For any g-dimensional symmetric Newton polygon ξ, define

$$\triangle(\xi) := \{(x,y) \in \mathbb{Z} \times \mathbb{Z} : y < x \leq g, (x,y) \prec \xi\};$$
$$\mathrm{sdim}(\xi) := |\triangle(\xi)|.$$

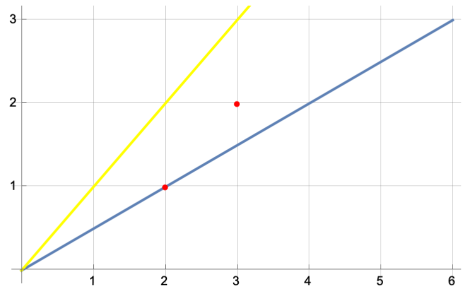
That is, $\triangle(\xi)$ contains all the integral lattice points strictly within the $g \times g$ region lying above ξ, and $\mathrm{sdim}(\xi)$ gives the number of such lattice points.

Remark 2.31. For general g, you may convince yourself that $\mathrm{sdim}(\rho) = \frac{g(g+1)}{2}$ and $\mathrm{sdim}(\sigma) = \lfloor \frac{g^2}{4} \rfloor$. These numbers have an important geometric meaning: we have already seen that $\dim(\mathcal{A}_g) = \frac{g(g+1)}{2}$ and we will see in Theorem 3.15 that the dimension of the supersingular locus $\mathcal{S}_g$ equals $\lfloor \frac{g^2}{4} \rfloor$. This is no coincidence: we will see in Theorem 2.34.(3) that $\mathrm{sdim}(\xi) = \dim(W_\xi)$ for any symmetric Newton polygon ξ. By definition $W_\sigma = \mathcal{S}_g$, explaining the second result; for the first, we note that the ordinary locus in $\mathcal{A}_g$ is open and dense.

Example 2.32. For ρ and σ as given in Example 2.25, we determine $\triangle(\xi)$ in the images below. The elements of $\triangle$ are marked by red points; the yellow line is the line $y = x$.
We see that $\mathrm{sdim}(\rho) = 6$ and $\mathrm{sdim}(\sigma) = 2$ when $g = 3$. We also see that the longest chain of Newton polygons $\sigma \prec \ldots \prec \rho$ has length $\mathrm{sdim}(\rho) - \mathrm{sdim}(\sigma) = 4$.



$\triangle(\rho)$



$\triangle(\sigma)$

Now consider an abelian scheme $\mathcal{X} \to S$ over a base scheme S in characteristic p. Grothendieck proved, cf. [6, §IV.7], that if X_0 is a specialisation of $\mathcal{X}_\eta$, then $\mathcal{N}(X_0) \prec \mathcal{N}(\mathcal{X}_\eta)$, i.e. the Newton polygon goes up under specialisation. He conjectured the converse, which was proved by Oort (announced in [63], proved in [64] and [65, Corollary 3.2]): if $\alpha = \mathcal{N}(X_0)$ is the (necessarily symmetric) Newton polygon of a principally polarised abelian variety X_0, and $\alpha \prec \beta$ for some other symmetric Newton polygon β, then there exists an irreducible scheme S and a principally polarised abelian scheme $\mathcal{X} \to S$ such that its special fibre is X_0 and its generic fibre $\mathcal{X}_\eta$ has Newton polygon $\mathcal{N}(\mathcal{X}_\eta) = \beta$.

Remark 2.33. The principally polarised condition on X_0 is important: for any $g \geq 3$ there exist counterexamples to Grothendieck's conjecture with non-principally polarised varieties, cf. [64, Remark 6.4] and [38, Remark 6.10].

As alluded to above, Grothendieck's conjecture was proved by studying deformations of p-divisible groups: one needs both deformations within a Newton polygon stratum to obtain a scheme $\mathcal{X}$ with $a(\mathcal{X}_\eta) = 1$, and deformations of (p-divisible groups of) such varieties of a-number 1 to other Newton polygon strata. To deform within a Newton polygon stratum, a purity result due to de Jong and Oort [34, Theorem 4.1] is used, which says that if the Newton polygon jumps in a family of p-divisible groups (over an irreducible noetherian scheme) then it already jumps in codimension 1.

More importantly for us, these techniques imply the following results for Newton strata W_ξ:

Theorem 2.34. (cf. [63, Theorem 2.6], [64, Theorem 3.4], [65, Theorem 4.1]) *Let ξ be a symmetric Newton polygon and let $W \subseteq W_\xi$ be an irreducible component of the Newton stratum W_ξ.*

1. *Generically on W, the Newton polygon is ξ.*
2. *Generically on W, the a-number is 1, unless $\xi = \rho$ (for which the a-number is 0).*
3. *The dimension of W is* $\mathrm{sdim}(\xi)$.

It was already noted in [63, Theorem 2.6.(c)] that W_ξ is connected whenever $g > 1$ (since every irreducible component $W \subseteq W_\xi$ contains an irreducible component of the supersingular locus W_σ) and conjectured in [65, §5.1] that W_ξ is geometrically irreducible for any $\xi \neq \sigma$. The latter was proven ten years later by Chai and Oort using monodromy arguments:

Theorem 2.35. (cf. [5, Theorem 3.1]) *For any g-dimensional symmetric Newton polygon ξ such that $\xi \neq \sigma$, the Newton stratum $W_\xi \subseteq \mathcal{A}_g$ (and hence also W_ξ^0) is geometrically irreducible.*

2.4. **The a-number and Ekedahl-Oort stratifications.**

We now introduce two other stratifications on $\mathcal{A}_g$. They are respectively determined by the a-number of the abelian variety, cf. Definition 1.31, and by combinatorial data attached to the p-torsion scheme of an abelian variety, introduced by Ekedahl and Oort.

It is worth noting that both stratifications are determined by *isomorphism* invariants, while the p-rank and Newton stratifications introduced in Subsection 2.3 were defined by *isogeny* invariants (of the p-torsion and p-divisble group).

The a-number stratification is easier to define, but harder to analyse than the Ekedahl-Oort stratification. Moreover, the latter refines the former: that is, each a-number stratum is a disjoint union of Ekedahl-Oort strata. Therefore, we will say relatively little about a-number strata, focussing on setting up the theory needed for the Ekedahl-Oort stratification.

2.4.1. *The a-number stratification.*
Recall the definition of the a-number $a(X) := \dim_k \operatorname{Hom}(\alpha_p, X)$ of an abelian variety X over k (Definition 1.31).

The a-number is an isomorphism invariant, and we may use it to define a stratification with strata consisting of varieties with the same a-number; cf. [11, 80].

Definition 2.36. For any $0 \leq n \leq g$, consider the locally closed subsets

$$T_n = \mathcal{A}_g(a \geq n) := \{x = (X, \lambda) \in \mathcal{A}_g : a(X) \geq n\};$$
$$\mathcal{A}_g(n) := \{x = (X, \lambda) \in \mathcal{A}_g : a(X) = n\}.$$

We see that $T_n \subseteq T_{n+1}$ for any $n \leq g-1$, and hence the T_n form a good stratification of $\mathcal{A}_g$.

The locus $T_g = \mathcal{A}_g(g)$ consists of all superspecial abelian varieties by [62], and hence has dimension zero. It is reducible, since it consists of a number of superspecial points. For any $n \leq g-1$ however, T_n is irreducible, by [80, Theorem 2.11].

In [11, Theorem 12.5], Ekedahl and van der Geer compute the cycle classes of the T_n in the Chow ring $\mathrm{CH}^*_{\mathbb{Q}}(\widetilde{\mathcal{A}}_g)$. In the same paper, they also compute the cycle classes of the p-rank strata, and of the Ekedahl-Oort strata which we will soon define.

In Subsection 3.5.1 we will give more precise results on the a-number stratification on the supersingular locus $\mathcal{S}_g$, as defined in [45, § 9.9-9.11], which are due to Harashita. On $\mathcal{A}_g$, we generally obtain more interesting results than for a-number strata by considering their refinement by Ekedahl-Oort strata, which we introduce next.

2.4.2. *The Ekedahl-Oort stratification.*
As mentioned above, the definition of the Ekedahl-Oort stratification is more involved, since we will first need to define and characterise several types of filtrations on group schemes in characteristic p. We then apply this to the p-torsion group scheme $X[p]$ of a (principally polarised) abelian variety to obtain the stratification $\mathcal{A}_g = \sqcup_\varphi \mathcal{S}_\varphi$; some of its properties are listed in Theorem 2.51.

The main reference for the Ekedahl-Oort stratification is [66]. The description of the strata in terms of Weyl group elements can be found in [11].

Notation 2.37. Recall the relative Frobenius and Verschiebung morphisms from Definition 1.13. Here we will consider them for group schemes G over $S = \operatorname{Spec}(k)$ and should therefore denote them by $F_{G/k}$ and $V_{G/k}$, respectively. For ease of notation however, we will write F and V throughout this section.

Definition 2.38. A finite flat commutative group scheme G over k – or more generally over any base scheme in characteristic p – is a BT_1 ("Barsotti-Tate truncated level one group scheme") if it satisfies:

$$\operatorname{im}(V : G^{(p)} \to G) = \ker(F : G \to G^{(p)}), \quad \operatorname{im}(F : G \to G^{(p)}) = \ker(V : G^{(p)} \to G).$$

Since $V \circ F = F \circ V = [p]$, this implies that $[p]_G = 0$, i.e. G is annihilated by p.
A BT_1 is *symmetric* if it admits an isomorphism to its Cartier dual: $\iota : G \xrightarrow{\simeq} G^D$.

For an abelian variety X over k, or over any field K of characteristic p, we see that the p-torsion subscheme $X[p]$ is a BT_1. If X admits a polarisation of degree coprime to p, e.g. a principal polarisation, then $X[p]$ is symmetric.

On any BT_1, we can act by Frobenius and Verschiebung, their powers and their inverses. On a symmetric BT_1, we can moreover act on any finite subscheme $H \subseteq G$ via

$$-(H) := \ker(G \xrightarrow{\iota} G^D \to H^D).$$

We now use these actions to introduce filtrations on BT_1 group schemes over k.

Definition 2.39. Let G be a BT_1 over a field K of characteristic p.

(1) The *canonical filtration* of G

$$0 = G_0 \subseteq \ldots \subseteq G_s = V(G) \subseteq \ldots \subseteq G_t = G$$

is obtained inductively as the finite set

$$\{w(G) : w \text{ is a finite word in } V \text{ and } F^{-1}\};$$

if G is symmetric, its canonical filtration is equivalently obtained as the finite set

$$\{w'(G) : w' \text{ is a finite word in } V \text{ and } -\}.$$

One can think of first applying V^i to G for all $i > 0$, then applying F^{-j} to these images for all $j > 0$, et cetera; if the rank of G is p^r, this process stabilises after $2(r-1)$ steps, in the sense that we stop producing new group schemes.

(2) For G that is also symmetric, a *good filtration* of G is a filtration

$$0 = G_0 \subseteq \ldots \subseteq G_s = V(G) \subseteq \ldots \subseteq G_{2s} = G$$

into subgroup schemes G_i such that $G_i \neq G_{i+1}$ for all $0 \leq i \leq 2s-1$, and $-(G_j) = G_{2s-j}$ for all $0 \leq j \leq 2s$. Moreover, every G_i for $i \leq s$ is the image of Verschiebung acting on $G_j^{(p)}$ for some j and every such image occurs this way.
Every canonical filtration is a good filtration, by [66, Proposition 5.4], of minimal length.

(3) A *final filtration* of G of rank p^r is a good filtration of maximal (even) length r where each G_i has respective rank p^i.

Example 2.40. Let $g = 3$. Consider a supersingular abelian threefold X over k with a-number 2. Then it follows from [19, Theorem 5.1.(2)], building on results in [18] on supersingular Dieudonné modules, that the canonical filtration of $G = X[p]$ is of the form

$$0 = G_0 \subseteq G_1 \subseteq G_2 \subseteq G_3 \subseteq G_4 \subseteq G_5 \subseteq G_6 = G,$$

where as finite words in V and F^{-1}, we have

$$G_0 = 0, \quad G_1 = V^2(G), \quad G_2 = VF^{-1}V(G),$$

$$G_3 = V(G), \quad G_4 = F^{-1}V^2(G), \quad G_5 = F^{-1}V(G), \quad G_6 = G = X[p].$$

This is shown by choosing explicit bases and representing V and $-$ as matrices (note that in [19] words in $F, \perp$ are considered, which is equivalent).

To these filtrations, we now attach a *type*, which we will see in Theorem 2.43 determines G up to isomorphism over $k = \overline{\mathbb{F}}_p$.

Definition 2.41. (1) The *canonical type* attached to the canonical filtration of G is the triple of functions

$$\tau = \{v : \{0, \ldots, t\} \to \{0, \ldots, s\}, f : \{0, \ldots, t\} \to \{s, \ldots, t\}, \rho : \{0, \ldots, t\} \to \mathbb{Z}_{\geq 0}\}$$

such that:

- Via $V(G_i) = G_{v(i)}$ we keep track of the action of Verschiebung;
- Via $F^{-1}(G_i) = G_{f(i)}$ we keep track of the action of F^{-1};
- Via $\mathrm{rank}(G_i) = p^{\rho(i)}$ we encode the ranks.

The functions v and f are non-decreasing and surjective and by [66, Lemma 2.4] satisfy

$$v(i+1) > v(i) \Leftrightarrow f(i+1) = f(i);$$
$$v(i+1) = v(i) \Leftrightarrow f(i+1) > f(i);$$
$$f(i) + v(i) = t + i.$$

The function ρ is strictly increasing and satisfies $\rho(0) = 0$. More generally, any triple $\tau = (v, f, \rho)$ satisfying these conditions is called a canonical type.

(2) If G is symmetric, so $t = 2s$ for some s, then its canonical type further satisfies

$$f(j) = 2s - v(2s - j) = s + j - v(j);$$
$$\rho(j+1) - \rho(j) = \rho(2s - j) - \rho(2s - j - 1).$$

Any triple $\tau = (v, f, \rho)$ satisfying all conditions above forms a *symmetric canonical type.*

(3) On any good filtration, by [66, Proposition 5.5] the analogously defined functions v, f on $\{0, \dots, 2s\}$ further satisfy

$$v(j) = v(j+1) \Leftrightarrow v(2r - j) = v(2r - j - 1) + 1;$$
$$v(j) < v(j+1) \Rightarrow v(j+1) = v(j) + 1.$$

Example 2.42. In the setting of Example 2.40, where $s = g = 3$, the canonical type is given by:

$$v : \{0, 1, 2, 3, 4, 5, 6\} \to \{0, 1, 2, 3\}$$
$$v(0) = v(1) = v(2) = 0, \;\; v(3) = v(4) = 1, \;\; v(5) = 2, \;\; v(6) = 3;$$
$$f : \{0, 1, 2, 3, 4, 5, 6\} \to \{3, 4, 5, 6\}$$
$$f(0) = 3, f(1) = 4, f(2) = f(3) = 5, f(4) = f(5) = f(6) = 6;$$
$$\rho : \{0, 1, 2, 3, 4, 5, 6\} \to \mathbb{Z}_{\geq 0}$$
$$\rho(i) = i \text{ for all } 0 \leq i \leq 6.$$

Theorem 2.43. (cf. [72, Proposition 3.5], [49, Theorem 4.7], [66, Theorem 9.4]) *If the* BT_1 *group schemes G and G' have the same canonical type, then they are k-isomorphic: $G \simeq_k G'$.*

Remark 2.44. We see that every canonical filtration gives rise to a canonical type. Conversely, it is claimed in [66, Remark 2.8] that every canonical type arises from a canonical filtration of some BT_1; in [72, Remark 3.7] it is pointed out that every canonical type occurs through *some* filtration of a BT_1, but that might be a strict refinement of the canonical filtration.

Remark 2.45. As is explained in [80, § 2], for an abelian variety X we may equivalently define the canonical filtration of the p-torsion group scheme $X[p]$ and its canonical type by considering its de Rham cohomology $H^1_{\mathrm{dR}}(X)$; on this space we also have actions of Frobenius and Verschiebung, that are moreover adjoints under the symplectic form.

Definition 2.46. (1) A *final sequence* is a function $\psi : \{0, 1, \dots, 2s\} \to \mathbb{Z}_{\geq 0}$ satisfying

$$\psi(0) = 0;$$
$$\psi(2s) = s;$$
$$\psi(i) \leq \psi(i+1) \leq \psi(i) + 1 \text{ for all } 0 \leq i < 2s;$$
$$\psi(i) + 1 = \psi(i+1) \Leftrightarrow \psi(2s-1) = \psi(2s - i - 1).$$

(2) An *elementary sequence* is a function $\varphi : \{0, 1, \dots, s\} \to \mathbb{Z}_{\geq 0}$ satisfying

$$\varphi(0) = 0;$$
$$\varphi(i) \leq \varphi(i+1) \leq \varphi(i) + 1 \text{ for all } 0 \leq i < s.$$

Because of the conditions $\psi(0) = 0 = \varphi(0)$, we may view both types of sequences as functions on $\{1, \dots, 2s\}$ and $\{1, \dots, s\}$, respectively.

We define a partial ordering $\prec$ on the set of 2^s elementary sequences by

$$\varphi' \prec \varphi \Leftrightarrow \varphi(i)' \leq \varphi(i) \text{ for all } 0 \leq i \leq s. \tag{6}$$

The smallest stratum with this ordering is therefore $\varphi = (0, 0, \dots, 0)$; this corresponds to the superspecial locus.

We can turn a final sequence ψ into an elementary sequence by truncating it, i.e. by restricting ψ to $\{0, 1, \dots, s\}$. Conversely, we can "stretch" an elementary sequence to a final sequence by defining $\varphi(2s-i) = \varphi(i) + s - i$ for all $0 \le i \le s$. Thus, the data of an elementary sequence is equivalent to that of a final sequence, and we will use them interchangeably in the sequel.

Further, we can inductively define an elementary sequence φ corresponding to a symmetric canonical type $\tau = (v, f, \rho)$ as follows: having defined $\{\varphi(0), \varphi(1), \ldots, \varphi(\rho(i))\}$ with $\rho(i) < \rho(i+1) \le s$, we determine the next $\rho(i+1) - \rho(i)$ entries, yielding $\{\varphi(0), \varphi(1), \ldots, \varphi(\rho(i+1))\}$, by taking

$$\begin{cases} \varphi(\rho(i+1)) = \ldots = \varphi(\rho(i)+1) = \varphi(\rho(i)) & \text{if } v(i) = v(i+1); \\ \varphi(\rho(i+1)) > \ldots > \varphi(\rho(i)+1) > \varphi(\rho(i)) & \text{if } v(i) < v(i+1). \end{cases}$$

Alternatively, we may refine the canonical filtration giving rise to τ into a final filtration of length $2s$ and set $\varphi(i) = \dim(FG_i)$ for all $0 \le i \le s$. This final filtration may not be unique, but its type will be and hence also the final sequence. Conversely, Oort gives a "canonical construction" to obtain a canonical type from a final sequence, cf. [66, p. 18]. We will not need it in this course.

Example 2.47. Taking the canonical type of Example 2.42, we see that $\rho(i+1) - \rho(i) = 1$ for all i, and so we inductively define the following elementary sequence one step at a time:

$$\varphi = (\varphi(0), \varphi(1), \varphi(2), \varphi(3)) = (0, 0, 0, 1).$$

That is, the value of φ jumps exactly when that of v does.

Definition 2.48. For any $g \ge 1$, the *Weyl group* W_g of the symplectic group Sp_{2g} is the permutation group

$$\begin{aligned} W_g &= \{w \in S_{2g} : w(i) + w(2g+1-i) = 2g+1 \text{ for all } 1 \le i \le g\} \\ &= \langle \sigma_i = (i, i+1)(2g-i, 2g+1-i) \text{ for all } 1 \le i < g, \text{ and } \sigma_g = (g, g+1)\rangle \end{aligned}$$

generated by the reflections $\sigma_1, \ldots, \sigma_g$.

The Bruhat-Chevalley order on W_g, denoted $\prec_{BC}$, for any two elements $w : (1, \ldots, 2g) \mapsto (w(1), \ldots, w(2g))$ and $w' : (1, \ldots, 2g) \mapsto (w'(1), \ldots, w'(2g))$ is defined by

$$\begin{aligned} w \prec_{BC} w' \Leftrightarrow & \text{ for all } 1 \le d \le g, \text{ the } d\text{th-largest element of } (w(1), \ldots, w(d)) \\ & \le \text{ the } d\text{th-largest element of } (w'(1), \ldots, w'(d)). \end{aligned} \tag{7}$$

To a symmetric canonical type $\tau = (v, f, \rho)$ we can associate a Weyl group element of W_s as follows: write all $1 \le i \le s$ for which $v(i) = v(i-1)$ in increasing order as $S = \{i_1, i_2, \ldots\}$. Also write the complement of S in $\{1, \ldots, s\}$ in increasing order, as $S^c = \{j_1, j_2, \ldots\}$. Now define the permutation $w : (1, 2, \ldots, 2s) \mapsto (w(1), \ldots, w(2s))$ in S_{2s} via

$$w(\ell) = \begin{cases} k & \text{if } \ell = i_k \text{ for some } k; \\ s + k & \text{if } \ell = j_k \text{ for some } k; \\ 2s + 1 - w(i) & \text{if } \ell = 2s + 1 - i \text{ for some } 0 \le i \le s; \end{cases}$$

Then $w \in W_s$ by construction and by the symmetry properties of v. In particular, the sequence $(w(1), \ldots, w(2s))$ is uniquely determined by the subsequence $(w(1), \ldots w(s))$.

Example 2.49. Following up with Examples 2.40, 2.42, and 2.47, we see that $v(i) = v(i-1)$ holds for $i \in S = \{1, 2\}$. Its complement in $\{1, 2, 3\}$ is therefore $S^c = \{3\}$. This yields the permutation

$$w : (1, 2, 3, 4, 5, 6) \mapsto (1, 2, 4, 3, 5, 6),$$

which equals the transposition $(3, 4) = \sigma_3$.

We now apply the theory above to $G = X[p]$, the symmetric p-torsion scheme of rank p^{2g} of a principally polarised abelian variety X over k (where the symmetry is induced from the principal polarisation). So **from now on, we work with $r = 2s = 2g$.**

We have already seen how the canonical filtration on $X[p]$ is determined up to k-isomorphism by its (symmetric) canonical type $\tau = (v, f, \rho)$, and that we can equivalently express this information in terms of a final sequence ψ or an elementary sequence φ. Finally, the Weyl group construction allows us to attach a Weyl group element w to φ.

Definition 2.50. For each elementary sequence φ, we let

$$\mathcal{S}_\varphi := \{(X,\lambda) \in \mathcal{A}_g(k) : \text{ the elementary sequence corresponding to } X[p] \text{ is } \varphi\}.$$

Then $\mathcal{S}_\varphi$ is called the *Ekedahl-Oort stratum* in $\mathcal{A}_g$ corresponding to φ.

The result below collects the most important statements about the Ekedahl-Oort strata in $\mathcal{A}_g$, proven in several (cited) references.

Theorem 2.51. *Let $g \geq 1$ and consider $\mathcal{A}_g$ in characteristic p.*

(1) *Every Ekedahl-Oort stratum $\mathcal{S}_\varphi$ is non-empty and quasi-affine. All irreducible components of $\mathcal{S}_\varphi$ have dimension $\sum_{i=1}^{g} \varphi(i)$* (cf. [66, Theorem 1.2]).
(2) *If $\varphi \neq (0,\dots,0)$, i.e. outside of the superspecial locus, the Zariski closure $\overline{\mathcal{S}_\varphi}$ of $\mathcal{S}_\varphi$ is connected* (cf. [66, Theorem 1.3]).
(3) *In fact, if $\mathcal{S}_\varphi \not\subseteq \mathcal{S}_g$, where $\mathcal{S}_g$ is the supersingular locus, then $\mathcal{S}_\varphi$ is irreducible* (cf. [11, Theorem 11.5]). *Otherwise it is reducible for sufficiently large g and p* (cf. [18, Corollary 3.5.5]).
(4) *Any stratum is locally closed, and its Zariski closure is a union of the stratum itself and lower-dimensional strata* (cf. [66, Theorem 1.3 and Proposition 3.2]).
(5) *The a-number of a stratum $\mathcal{S}_\varphi$ is $g - \varphi(g)$* (cf. [66, p.56]).
(6) *The p-rank of a stratum $\mathcal{S}_\varphi$ is $\max\{i : \varphi(i) = i\}$* (cf. [66, p.56]).

Proof. We sketch the proof of the fact that $\dim(\mathcal{S}_\varphi) = \sum_{i=1}^{g} \varphi(i)$. Fix an abelian variety (X_0, λ_0) in $\mathcal{S}_\varphi$. By choosing an explicit ("standard") basis for the Dieudonné module of $X_0[p]$ and constructing deformations of (X_0, λ_0) that still lie inside $\mathcal{S}_\varphi$ explicitly in terms of this basis, it is shown that $\dim(\mathcal{S}_\varphi) \geq \sum_{i=1}^{g} \varphi(i)$, cf. [66, Proposition 10.1].

On the other hand, [66, Proposition 11.1] shows that if $\varphi' \prec \varphi$ with $\sum_{i=1}^{g}(\varphi(i) - \varphi'(i)) = 1$, then $\mathcal{S}_{\varphi'} \subseteq \overline{\mathcal{S}_\varphi}$ (note the typo in the statement in [66]). This follows again by using explicit computations with bases for Dieudonné modules to obtain a deformation of (Y_0, μ_0) in $\mathcal{S}_\varphi$ whose generic fibre corresponds to φ'. By forming chains of elementary sequences that differ at one place, and repeatedly applying the proposition, this shows that $\dim(\mathcal{S}_\varphi) \leq \sum_{i=1}^{g} \varphi(i)$, so we have equality. ☐

Corollary 2.52. *The Ekedahl-Oort strata form a good stratification of $\mathcal{A}_g$, in which the boundary of any stratum is the union of all lower-dimensional strata meeting it. Moreover, we see from Theorem 2.51.(5) that it refines both the a-number and the p-rank stratifications.*

Remark 2.53. Theorem 2.51 mentions the Zariski closure of the Ekedahl-Oort strata. These closures turn out to be rather complicated to describe in detail.

In particular, it follows from [66, Proposition 11.1] that if $\varphi' \prec \varphi$ (as in (6)), then $\mathcal{S}_{\varphi'} \subseteq \overline{\mathcal{S}_\varphi}$ is contained in the Zariski closure of $\mathcal{S}_\varphi$, but in [66, Example 14.3], we see that the converse does not hold: $\mathcal{S}_{\varphi'} \subseteq \overline{\mathcal{S}_\varphi} \not\Rightarrow \varphi' \prec \varphi$.

On the other hand, in [11] Ekedahl and van der Geer construct a flag space over $\mathcal{A}_g$ that admits a stratification by elements of the Weyl group W_g, where the inclusion relation between strata is given precisely by the order $\prec_{BC}$ (see (7)). While projecting these strata from the flag space to $\mathcal{A}_g$ yields the Ekedahl-Oort strata on $\mathcal{A}_g$, in [11, Example 9.5] they give examples that show that $\mathcal{S}_{\varphi'} \subseteq \overline{\mathcal{S}_\varphi} \not\Rightarrow w' \prec_{BC} w$, where w, w' are the Weyl group elements associated to φ, φ', respectively.

Finally, it was shown by Wedhorn (cf. [84, Theorem 5.4] and [69, Theorem 6.2]) that the closure relation for Ekedahl-Oort strata can be fully understood through so-called shuffles: i.e. $\mathcal{S}_{\varphi'} \subseteq \overline{\mathcal{S}_\varphi} \Leftrightarrow$ there exists $u \in W_I$ such that $uw'(w_{0,I}uw_{0,I}) \prec_{BC} w$, where as above w, w' are the respective Weyl group elements associated to φ, φ', and where $W_I = \{\ w \in W_g : w(\{1,2,\dots,g\}) = \{1,2,\dots,g\}\ \}$ and $w_{0,I} \in W_I$ is defined so that $w_{0,I}(i) = g+1-i$ for all $1 \leq i \leq g$.

Example 2.54. In Examples 2.40, 2.42 and 2.47 we have seen one example of a stratum in $g = 3$, namely $\mathcal{S}_{(0,0,1)}$ (omitting $\varphi(0) = 0$ from the notation), which determines the Weyl group element $w = \sigma_3$. It has a-number $2 = 3 - \varphi(3)$ and p-rank 0 by Theorem 2.51.(5). All strata with a-number $a = g - 1$ are classified in [66, Theorem 8.3].

The other elementary sequences for $g = 3$ and the corresponding Weyl group elements are as follows, cf. [80, p. 15]:

φ	w	a-number	p-rank
$(0,0,0)$	id	3	0
$(0,0,1)$	σ_3	2	0
$(0,1,1)$	$\sigma_2\sigma_3$	2	0
$(0,1,2)$	$\sigma_3\sigma_2\sigma_3$	1	0
$(1,1,1)$	$\sigma_1\sigma_2\sigma_3$	2	1
$(1,1,2)$	$\sigma_3\sigma_1\sigma_2\sigma_3$	1	1
$(1,2,2)$	$\sigma_2\sigma_3\sigma_1\sigma_2\sigma_3$	1	2
$(1,2,3)$	$\sigma_3\sigma_2\sigma_3\sigma_1\sigma_2\sigma_3$	0	3

Remark 2.55. We close this subsection with a historical remark. In 1975, Kraft classified BT_1 group schemes over an algebraically closed field k, cf. [42]. This classification was reobtained by Oort and is heavily used in [66] and subsequent papers about the Ekedahl-Oort stratification. Moonen generalises the stratification in [49] to Shimura varieties of PEL-type, also using Weyl groups. Later, in [50] Moonen and Wedhorn generalise even further, replacing canonical filtrations by other combinatorial constructions, called F-zips, which can be defined for any smooth proper morphism of schemes $X \to S$ in characteristic p.

3. The geometry of $\mathcal{S}_g$

3.1. Introduction and $\mathcal{S}_1$.

So far, we have considered the moduli space of g-dimensional principally polarised abelian varieties, and we have studied stratifications on $\mathcal{A}_g$ in characteristic p. We have seen in Subsection 1.3.3 that supersingularity is a phenomenon that only occurs in characteristic p. We define the *supersingular locus*

$$\mathcal{S}_g = \{x = (X, \lambda) \in \mathcal{A}_g : X \text{ is supersingular}\}.$$

This is a Zariski closed algebraic subset of $\mathcal{A}_g$ which can be given an induced reduced scheme structure. Moreover, it can be viewed as the (coarse) moduli space of supersingular abelian varieties, cf. [45, § 13.12-13.14]. Finally, we see from Remark 2.22, and from the fact that any two g-dimensional supersingular abelian varieties are k-isogenous (cf. Proposition 1.23), that every g-dimensional supersingular abelian variety over k has the same Newton polygon, namely the line segment with unique slope $1/2$, and therefore that $\mathcal{S}_g = W_\sigma$ is a Newton stratum in $\mathcal{A}_g$.

Example 3.1. When $g = 1$, the supersingular locus $\mathcal{S}_1$ consists of all supersingular elliptic curves. (Recall that elliptic curves are canonically pricipally polarised.) It is a zero-dimensional space, i.e. a finite set, whose cardinality is known by the work of Deuring [7], Eichler [9] and Igusa [33] to be

$$|\mathcal{S}_1| = \left\lfloor \frac{p-1}{12} \right\rfloor + \begin{cases} 0 & \text{if } p \equiv 1 \pmod{12}; \\ 1 & \text{if } p \equiv 2, 3, 5, 7 \pmod{12}; \\ 2 & \text{if } p \equiv 11 \pmod{12}. \end{cases} \tag{8}$$

We will revisit the idea of counting supersingular elliptic curves and higher-dimensional abelian varieties, also up to automorphisms, in Section 4. With $|\cdot|$ we will always mean honest cardinality.

In this section, we still study the geometry of $\mathcal{S}_g$. First, in Subsection 3.2, we will look closely at the case $g = 2$, before treating general g in Subsection 3.3. Next, in Subsection 3.4 we will put a foliation structure on $\mathcal{S}_g$ and in Subsection 3.5 we will see how the a-number and Ekedahl-Oort stratifications introduced in Subsection 2.4 restrict to $\mathcal{S}_g$.

3.2. Supersingular abelian surfaces: $\mathcal{S}_2$.

As a warm-up, in this subsection we treat the case $g = 2$. That is, we give an explicit construction of principally polarised supersingular abelian surfaces over an algebraically closed field k of characteristic $p > 0$, due to Moret-Bailly [52]. This description will have direct consequences for the geometry of $\mathcal{S}_2$, as shown by Katsura and Oort [37].

Recall from Definition 1.22 that a *superspecial* abelian variety X_0 of dimension g over k is isomorphic to a product of g supersingular elliptic curves. Equivalently, by [62, Theorem 2], it satisfies $a(X_0) = \dim_k \operatorname{Hom}(\alpha_p, X_0) = g$. Furthermore, recall from Proposition 1.23 that all g-dimensional superspecial abelian varieties are k-isomorphic; we use the latter fact as follows.

Notation 3.2. Fix a supersingular elliptic curve E_0 over k that is defined over $\mathbb{F}_{p^2}$, with Frobenius endomorphism $\pi_{E_0} = -p$.

Using Notation 3.2, any superspecial abelian surface over k satisfies

$$X_0 \simeq E_0 \times E_0.$$

A non-superspecial supersingular abelian surface X will have $a(X) = 1$. By [62, Corollary 7],

$$X \simeq (E_0 \times E_0)/\iota(\alpha_p) \tag{9}$$

for some immersion $\iota : \alpha_p \hookrightarrow \alpha_p \times \alpha_p \hookrightarrow E_0 \times E_0$. Since $\operatorname{End}_k(\alpha_p) \simeq k$, we can write $\iota = (a, b)$ for some $a, b \in k$; since the embedding only depends on the ratio a/b, we will view (a, b) as a point on $\mathbb{P}^1_k$.

Note that the above describes unpolarised varieties; we will now consider polarised varieties. In general, a superspecial abelian variety can be equipped with many different polarisations. The construction in Example 3.3 shows how polarisations descend from a superspecial surface $E_1 \times E_2$ to a supersingular surface obtained as its quotient.

Example 3.3. (Moret-Bailly, [52, II, Appendice]) Let E_1, E_2 be supersingular elliptic curves with respective points at infinity O_1, O_2. The superspecial abelian surface $X_0 = E_1 \times E_2$ admits a polarisation induced by the ample line bundle $\mathcal{L}_0 = \mathcal{O}_{X_0}(E_1 \times O_2 + O_1 \times E_2)^{\otimes p}$. The kernel $K(\mathcal{L}_0)$ of the polarisation is $X_0[p]$ and hence of order p^4, and it comes equipped with an alternating form $e^{\mathcal{L}_0} : K(\mathcal{L}_0) \times K(\mathcal{L}_0) \to \mathbb{G}_m$.

Via an explicit calculation on the Dieudonné module of $K(\mathcal{L}_0)$ one can find a subgroup H satisfying $H \simeq \alpha_p$ and $H^\perp/H \simeq \alpha_p \times \alpha_p$, where $H^\perp$ is orthogonal to H with respect to $e^{\mathcal{L}_0}$. Consider then the quotient surface $A = X_0/H$. By [52, Théorème 4.1 and Proposition 4.2], the line bundle $\mathcal{L}_0$ descends to a line bundle $\mathcal{L}$ on A which induces a polarisation with kernel $K(\mathcal{L}) \simeq \alpha_p \times \alpha_p$. In particular, it follows that $a(A) = 2$, so A is also superspecial. Moreover, we may assume that $\mathcal{L}$ is symmetric, i.e. $[-1]_A^*(\mathcal{L}) \simeq \mathcal{L}$.

We will now see how the polarised superspecial surface A constructed above is used to produce families of polarised supersingular abelian surfaces over $\mathbb{P}^1_k$. This may be viewed as a polarised analogue of Equation (9). The following holds in characteristic $p > 2$; for similar results when $p = 2$, see [51].

For ease of notation, let $S = \mathbb{P}^1_k$ with homogeneous coordinate (X, Y). Also let

$$K = \alpha_p \times \alpha_p = \mathrm{Spec} k[\alpha]/(\alpha^p) \times \mathrm{Spec} k[\beta]/(\beta^p);$$
$$K_S = K \times S = \mathrm{Spec}\mathcal{O}_S[\alpha, \beta]/(\alpha^p, \beta^p).$$

Consider the subgroup scheme N of K_S defined by $Y\alpha - X\beta = 0$ (denoted by H in [52]); since N has rank p, it is locally isomorphic to $\alpha_p \times S$. Next, form the quotient $\mathcal{X} = A_S/N$ of $A_S = A \times S$. These objects fit into the following diagram, where the top row is exact and the triangle and square commute.

$$\begin{tikzcd}
1 \arrow[r] & N \arrow[rr] & & A_S \arrow[rr, "\pi"] \arrow[dl, "\mathrm{pr}_1"] \arrow[dr, "\mathrm{pr}_2"] & & \mathcal{X} \arrow[r] \arrow[dl, "q"] & 1 \\
& & A \arrow[dr] & & S \arrow[dl] & & \\
& & & \mathrm{Spec}(k) & & &
\end{tikzcd}$$

FIGURE 1. The Moret-Bailly construction.

There is a unique line bundle $\mathcal{M}$ on $\mathcal{X}$ such that $\pi^*\mathcal{M} \simeq \mathcal{L}_S$ (or equivalently, $\pi^*\mathcal{M} \simeq \mathrm{pr}_1^*(\mathcal{L})$), which by construction induces a principal polarisation on $\mathcal{X}$. The cokernel of $q^*q_*(\mathcal{M}) \to \mathcal{M}$ is an effective relative ("theta") divisor $D \to S$.

The fibration, also denoted by $q : D \to S$, is nontrivial and defines a surface that is shown to be non-singular and of general type. For $s \in S(k)$, the fibre D_s is either a smooth genus 2 curve on the surface $\mathcal{X}_s$ or two elliptic curves meeting transversally; by [52, Proposition 2.5.(i)], the number of singular fibres is $5p-5$. In both cases the fibre induces a principal polarisation on $\mathcal{X}_s$. Finally, note that the commuting triangle in the diagram shows that each $\mathcal{X}_s$ is supersingular.

In conclusion, $q : (\mathcal{X}, D) \to S$ is a ("Moret-Bailly") family of principally polarised supersingular abelian surfaces over k. Such a family exists for any ample line bundle $\mathcal{L}$ (or polarisation) on A with kernel isomorphic to $\alpha_p \times \alpha_p$.

Remark 3.4. In [68], Pieper shows that the whole family is determined by two of its singular fibres. He moreover describes the family explicitly by finding the defining equations for the hyperelliptic curves $\mathcal{C}_s$ such that the irreducible fibres are $\mathcal{X}_s \simeq \mathrm{Jac}(\mathcal{C}_s)$ as principally polarised abelian varieties.

The above has far-reaching implications for the geometry of $\mathcal{S}_2$. Previously, it was known that every irreducible component of $\mathcal{S}_2$ is a rational curve, cf. [61, proof of Corollary 4.7]. Katsura-Oort [37] build on Moret-Bailly's results and prove that moreover any irreducible component of $\mathcal{S}_2$ is the image of a Moret-Bailly family. From this, it follows that the number of irreducible components of $\mathcal{S}_2$ is equal to the number of isomorphism classes of Moret-Bailly families $(\mathcal{X}, D) \to S$, cf. [37, Theorem 2.7]. This number is determined in [37, Theorem 5.7], invoking [32, Theorem 2.15], to be the class number $h_2(1, p)$; we will introduce these in Subsection 4.2.2 and define them formally in Definition 4.12. Knowing the exact values of some of these class numbers, this implies the following result:

Theorem 3.5. (cf. [37, Theorem 5.8]) *$\mathcal{S}_2$ is irreducible if and only if $p \leq 11$.*

Remark 3.6. In the same article, the authors also describe the automorphisms of a Moret-Bailly family preserving the relative polarisation D, which turn out to be determined by their actions on the $5p-5$ singular fibres of the family, cf. [37, Theorem 4.1]. In addition, the normalisation of each irreducible component of $\mathcal{S}_2$ is shown to be isomorphic to $\mathbb{P}^1_k/G$ for some group $G \subseteq \mathrm{Aut}(\mathbb{P}^1_k)$ which is itself the quotient of the group of automorphisms acting on the singular fibres of the corresponding family by the -1-map; the final chapters of the article are devoted to studying the groups G that occur (depending on p) and their ramification groups.

3.3. Polarised flag type quotients: $\mathcal{S}_g$ for general g.

In this subsection, we will give a geometric description of g-dimensional supersingular abelian varieties for general $g \geq 1$ in terms of polarised flag type quotients (PFTQs), a construction due to Li-Oort [45]. We will see how this reduces to Moret-Bailly families when $g = 2$ and give an equally explicit description of the case $g = 3$. Furthermore, by studying the moduli space $\mathcal{P}_g$ of g-dimensional PFTQs we will determine the dimension and the number of components of $\mathcal{S}_g$ in Theorems 3.15 and 3.16, respectively.

The general idea behind (polarised) flag type quotients is that any supersingular abelian variety X can be connected to a superspecial variety through a purely inseparable isogeny. The kernel of this isogeny is formed out of successive extensions of α_p group schemes; we can use this information to break up the isogeny into a chain of isogenies with prescribed kernel ranks. If X is principally polarised, we may also equip the superspecial variety and all quotient varieties appearing in this chain with suitable – generally not principal! – polarisations that are compatible with the isogenies.

Before giving the formal definition of flag type quotients, we recall some notions and introduce some notation. As in Notation 3.2, let $E_0/\mathbb{F}_{p^2}$ be the supersingular elliptic curve with Frobenius endomorphism $\pi_{E_0} = -p$. And as in Definition 1.13, let S be a scheme of characteristic p, let $X \to S$ be an abelian scheme, and let

$$F_{X/S} : X \to X^{(p)} \quad \text{resp.} \quad V_{X/S} : X^{(p)} \to X$$

be the relative Frobenius resp. Verschiebung morphism on X, where we write $X^{(p)} := X \times_{S, F_S} S$. If there is no risk of confusion, we will drop the subscripts on the relative Frobenius and Verschiebung morphisms. The kernel $\ker(f)$ of a morphism $f : X \to Y$ of abelian varieties is also denoted $X[f]$.

Definition 3.7. (cf. [45, § 2.4]) An *α-group* G of α-rank r is a finite flat commutative group scheme over an $\mathbb{F}_p$-scheme S on which the relative Frobenius and Verschiebung satisfy $F_{G/S} = 0$ and $V_{G/S} = 0$; it is locally isomorphic to $\alpha_p^r \times S$.

Definition 3.8. (cf. [45, § 3.2, 3.6]) Let the notation be as above and let $g \geq 1$.

(1) A g-dimensional *flag type quotient (FTQ)* is a chain of abelian schemes, each over an $\mathbb{F}_{p^2}$-scheme S,
$$(Y_\bullet, \rho_\bullet) : Y_{g-1} \xrightarrow{\rho_{g-1}} Y_{g-2} \cdots \xrightarrow{\rho_2} Y_1 \xrightarrow{\rho_1} Y_0,$$
such that:
  (i) $Y_{g-1} = E_0^g \times_{\mathrm{Spec}(\mathbb{F}_{p^2})} S$, with E_0 chosen as in Notation 3.2;
  (ii) $\ker(\rho_i)$ is an α-group of α-rank i for all $1 \leq i \leq g-1$.
In particular, each Y_i is supersingular.

(2) Let μ be a polarisation on E_0^g such that $\ker(\mu) = E_0^g[F]$ if g is even and $\ker(\mu) = 0$ if g is odd, i.e. such that $\ker(p^{\lfloor (g-1)/2 \rfloor}\mu) = E_0^g[F^{g-1}]$. For any such μ, a g-dimensional *polarised flag type quotient (PFTQ)* with respect to μ is a chain of polarised abelian schemes over an $\mathbb{F}_{p^2}$-scheme S
$$(Y_\bullet, \lambda_\bullet, \rho_\bullet) : (Y_{g-1}, \lambda_{g-1}) \xrightarrow{\rho_{g-1}} (Y_{g-2}, \lambda_{g-2}) \cdots \xrightarrow{\rho_2} (Y_1, \lambda_1) \xrightarrow{\rho_1} (Y_0, \lambda_0),$$
such that:
  (i') $(Y_{g-1}, \lambda_{g-1}) = (E_0^g, p^{\lfloor (g-1)/2 \rfloor}\mu) \times_{\mathrm{Spec}(\mathbb{F}_{p^2})} S$;
  (ii) $\ker(\rho_i)$ is an α-group of α-rank i for all $1 \leq i \leq g-1$;
  (iii) $\ker(\lambda_i) \subseteq Y_i[V^j \circ F^{i-j}]$ for all $0 \leq i \leq g-1$ and $0 \leq j \leq \lfloor \frac{i}{2} \rfloor$, where $F = F_{Y_i/S}$ and $V = V_{Y_i/S}$.
In particular, λ_0 is a principal polarisation on Y_0.

(3) An isomorphism of g-dimensional PFTQs is a chain of isomorphisms $(\beta_i)_{0 \leq i \leq g-1}$ of polarised abelian varieties, compatible with the isogenies ρ_i, such that $\beta_{g-1} = \mathrm{id}_{Y_{g-1}}$. Isomorphism is denoted by $\simeq$.

(4) A g-dimensional (polarised) flag type quotient $(Y_\bullet, \rho_\bullet)$ is said to be *rigid* if
$$\ker(Y_{g-1} \to Y_i) = \ker(Y_{g-1} \to Y_0) \cap Y_{g-1}[F^{g-1-i}], \quad \text{for } 1 \leq i \leq g-1.$$
We will say more about the rigidity condition in Remark 3.14.

Remark 3.9. Note that to introduce polarisations on flag type quotients in the definition above, we worked with an $\mathbb{F}_{p^2}$-scheme S instead of an $\mathbb{F}_p$-scheme. This is because every polarisation μ on E_0^g, with E_0 as in Notation 3.2, is defined over $\mathbb{F}_{p^2}$; in particular, to be able to choose μ such that $\ker(\mu) = E_0^g[F]$ when g is even, we must work over $\mathbb{F}_{p^2}$. When dealing with moduli spaces, we will often choose $S = k = \overline{\mathbb{F}}_p$.

Example 3.10. We return to the case $g = 2$. That is, we consider E_0^2 and a polarisation μ such that $\ker(\mu) = E_0^2[F] = \alpha_p \times \alpha_p$. Then a polarised flag type quotient looks like
$$(E_0^2, \mu) \to (Y_0, \lambda_0) = (E_0^2/\alpha_p, \lambda_0) \tag{10}$$
where λ_0 is a principal polarisation. When Y_0 is not superspecial, there exists a unique μ on E_0^2 and a unique isogeny to Y_0 compatible with the polarisations. Note that rigidity (4) is automatically satisfied, since $\alpha_p \simeq \ker(E_0^2 \to Y_0)$ and $E_0^2[F] = \alpha_p \times \alpha_p$.

We see that the PFTQ in this case is determined by an embedding $\alpha_p \hookrightarrow E_0^2$; recall from Subsection 3.2 that such an embedding is determined by a point on $\mathbb{P}^1$. This point is also called a Moret-Bailly parameter. Indeed, comparing Equations (10) and (9) and recalling how Moret-Bailly families provide polarised analogues of (9), we conclude that a Moret-Bailly family and a 2-dimensional PFTQ carry the same information.

Definition 3.11. Let $\mathcal{P}_{g,\mu}$ (resp. $\mathcal{P}'_{g,\mu}$) denote the moduli space over $\mathbb{F}_{p^2}$ of g-dimensional (resp. rigid) polarised flag type quotients with respect to the polarisation μ. That is, $\mathcal{P}_{g,\mu}$ (resp. $\mathcal{P}'_{g,\mu}$) is the projective (resp. quasi-projective) scheme over $\mathbb{F}_{p^2}$ representing the functor
$$\begin{aligned} \mathbb{F}_{p^2}\text{-schemes} &\longrightarrow \text{Set} \\ S' &\mapsto \{ \text{ (resp. rigid) } g\text{-dim. PFTQs over } S' \text{ w.r.t. } \mu\}/\simeq . \end{aligned}$$

Indeed, $\mathcal{P}'_{g,\mu}$ is an open subscheme of $\mathcal{P}_{g,\mu}$. It is geometrically irreducible (in fact, non-singular and geometrically integral) of dimension $\lfloor \frac{g^2}{4} \rfloor$.

Example 3.12. For $g = 2$ it follows from Example 3.10 that $\mathcal{P}_{2,\mu} \simeq \mathbb{P}^1_{\mathbb{F}_{p^2}}$.

Example 3.13. (cf. [31, § 3.3.2]) Suppose now that $g = 3$. Then $\mathcal{P}_{3,\mu}$ is a two-dimensional geometrically irreducible scheme over $\mathbb{F}_{p^2}$ by [45, § 9.4]. Its structure is independent of the choice of μ by [45, § 3.10]. The map

$$\pi : ((Y_2, \lambda_2) \to (Y_1, \lambda_1) \to (Y_0, \lambda_0)) \mapsto ((Y_2, \lambda_2) \to (Y_1, \lambda_1))$$

induces a morphism $\pi : \mathcal{P}_{3,\mu} \to \mathbb{P}^2$ whose image is isomorphic to the Fermat curve

$$C : X_1^{p+1} + X_2^{p+1} + X_3^{p+1} = 0.$$

As a fibre space over C, $\mathcal{P}_{3,\mu}$ is isomorphic to $\mathbb{P}_C(\mathcal{O}(-1) \oplus \mathcal{O}(1))$; see [45, § 9.3-9.4] and [36, Proposition 3.5].

According to [45, § 9.4] (cf. [36, Definition 3.14]), there is a section $s : C \twoheadrightarrow T \subseteq \mathcal{P}_{3,\mu}$ of π, and $\mathcal{P}'_{3,\mu} = \mathcal{P}_{3,\mu} - T$.

We can derive several key facts about the geometry $\mathcal{S}_g$ from that of $\mathcal{P}'_{g,\mu}$, cf. [45, § 4]. The connection between these moduli spaces is the following: projection to the last member of a PFTQ gives an $\overline{\mathbb{F}}_p$-morphism

$$\begin{aligned} \mathrm{pr}_0 : \mathcal{P}'_{g,\mu} &\to \mathcal{S}_g, \\ (Y_\bullet, \lambda_\bullet, \rho_\bullet) &\mapsto (Y_0, \lambda_0). \end{aligned}$$

Moreover, for every supersingular principally polarised (Y_0, λ_0) there exists at least one, and at most finitely many PFTQs, each with respect to a suitable polarisation μ, whose last member is geometrically isomorphic to (Y_0, λ_0). That is, the $\overline{\mathbb{F}}_p$-morphism

$$\mathrm{pr}_0 : \coprod_{\mu} \mathcal{P}'_{g,\mu} \to \mathcal{S}_g, \tag{11}$$

where the disjoint union runs over all suitable polarisations μ of E_0^g, is surjective and generically finite. The generic fibre over any irreducible component of $\mathcal{S}_g$ has a-number 1 and is contained in the image of $\mathcal{P}'_\mu$ for a unique μ.

Remark 3.14. The projection pr_0 exists also for $\mathcal{P}_{g,\mu}$, but in this case it could blow down a component of $\mathcal{P}_{g,\mu}$ to a proper closed subset of $\mathcal{S}_g$. Only after restriction to $\mathcal{P}'_{g,\mu}$ we are guaranteed to obtain a surjective and generically finite morphism. This explains why we had to introduce the notion of rigidity. This condition is generally harmless, in the sense that for a general supersingular principally polarised abelian variety, a PFTQ of which it is the last member is unique and automatically rigid.

It follows that the dimension and the number of irreducible components of $\mathcal{S}_g$ are determined by those of $\mathcal{P}'_{g,\mu}$. For the dimension, we see that the closure of each $\mathrm{pr}_0(\mathcal{P}'_{g,\mu})$ yields an irreducible component of $\mathcal{S}_g$, which therefore has dimension $\lfloor \frac{g^2}{4} \rfloor$. Thus:

Theorem 3.15. (cf. [45, Theorem 4.9.(i)]) *For any $g \geq 1$, we have* $\dim(\mathcal{S}_g) = \lfloor \frac{g^2}{4} \rfloor$.

For the number of irreducible components, one shows that a generic supersingular abelian variety has a-number 1, and that in this case there is a unique polarisation μ and a PFTQ with respect to μ of which it is the last member [58, Theorem 2.2]. Hence, the number of irreducible components of $\mathcal{S}_g$ equals the number of polarisations μ on E_0^g satisfying $\ker(p^{\lfloor (g-1)/2 \rfloor}\mu) = E_0^g[F^{g-1}]$. We can deduce (from Proposition 4.21 for instance) that this number is again a class number (as in Definition 4.12). That is:

Theorem 3.16. (cf. [45, Theorem 4.9.(ii)]) *The number of irreducible components of* $\mathcal{S}_g$ *is*

$$\begin{cases} h_g(p,1) & \text{if } g \text{ is odd;} \\ h_g(1,p) & \text{if } g \text{ is even.} \end{cases}$$

One may ask when the number of components is 1, i.e. when $\mathcal{S}_g$ is geometrically irreducible. The following result gives a complete answer.

Theorem 3.17. *The supersingular locus* $\mathcal{S}_g$ *is geometrically irreducible if and only if one of the following three cases holds:*

(i) $g = 1$ *and* $p \in \{2, 3, 5, 7, 13\}$*;*
(ii) $g = 2$ *and* $p \in \{2, 3, 5, 7, 11\}$*;*
(iii) $(g, p) = (3, 2)$ *or* $(g, p) = (4, 2)$.

Proof. This is [31, Theorem 5.20.(i)], which itself follows from the class number one result [31, Theorem 2.10]. The first case is classical, and can be found e.g. in the list in [81, p. 155], though *loc. cit.* also provides an alternative proof. □

Example 3.18. Suppose again that $g = 3$ and use the notation of Example 3.13. We saw that $\dim(\mathcal{P}'_{3,\mu}) = 2$ in Example 3.13 and Theorem 3.15 confirms that $\dim(\mathcal{S}_g) = 2$; the projection map pr_0 contracts the section T to a point. The number of components of $\mathcal{S}_3$ is $h_3(p, 1)$ by Theorem 3.16 - this was proven separately in [38, Theorem 6.7]. This number is 1 for $p = 2$ and > 1 for all $p \geq 3$.

We may define the a-number of a point of $\mathcal{P}_{3,\mu}$ by putting $a(y) := a(\mathrm{pr}_0(y))$ for $y \in \mathcal{P}_{3,\mu}(k)$. Using this, we can refine our structural results on $\mathcal{P}_{3,\mu}$ as follows. Writing a point $y \in \mathcal{P}_{3,\mu}(k)$ as (t, u), where $t = \pi(y)$ and $u \in \pi^{-1}(t) =: \mathbb{P}^1_t(k)$, by [45, § 9.3-9.4] we see:

(i) If $y \in T$ then $a(y) = 3$.
(ii) If $t \in C(\mathbb{F}_{p^2})$, then $a(y) \geq 2$. Moreover, $a(y) = 3$ if and only if $u \in \mathbb{P}^1_t(\mathbb{F}_{p^2})$.
(iii) We have $a(y) = 1$ if and only if $y \notin T$ and $t \notin C(\mathbb{F}_{p^2})$.

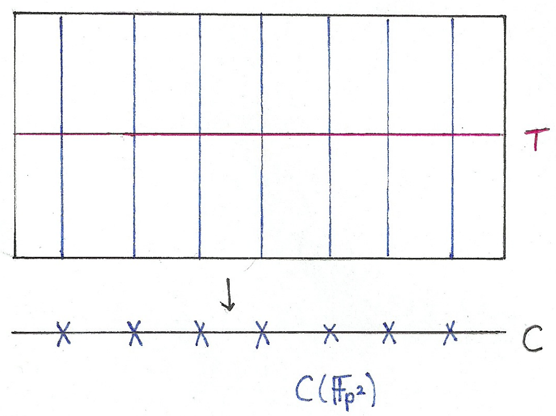


FIGURE 2. A schematic picture of $\mathcal{P}_{3,\mu}$ as a $\mathbb{P}^1$-bundle over the Fermat curve C.

Remark 3.19. Flag type quotients first appeared in 1978 in [58]. More precisely, [58, Theorem 2.2.(1)] describes any supersingular abelian variety as the quotient of a superspecial variety by a "flag type subgroup scheme" $K = K_0 \supseteq K_1 \supseteq \ldots \supseteq K_{g-1} = 0$ whose quotients K_{i-1}/K_i are α-groups of α-rank i for all $1 \leq i \leq g-1$. Further, [58, Theorem 3.3] classifies polarised flag type quotients for varieties with a-number 1 (above which the flag type quotient is unique and of maximal length) by quasi-polarised flag varieties of supersingular Dieudonné modules.

A little over a decade later, a slightly different definition of (polarised) flag type quotients was given in [38, Definitions 4.1-4.2], with any a-number. They are used to construct families of principally polarised supersingular abelian threefolds, and eventually to prove that the number of irreducible components of $\mathcal{S}_3$ is $h_3(p,1)$, cf. [38, Theorem 6.7].

At around the same time, [44] also considers flag type quotients (here called "flag type level structures"); these are equipped with an *index*, which is an increasing sequence of g integers between 0 and g prescribing the α-ranks of the kernels of the isogenies ρ_i as α-groups. This extra structure yields a fine moduli space, and we will see it is used in the proof of Theorem 3.29. This article is where the notion of rigidity is first mentioned (as corresponding to the smallest possible index).

3.4. **Foliation of $\mathcal{S}_g$ by central leaves and isogeny leaves.**
In this subsection, we will put a geometric, so-called foliation structure on $\mathcal{S}_g$ using the notions of central leaves and isogeny leaves. These are introduced in [67] and defined more generally as closed subsets of Newton strata W^0_ξ, so considering them for $\mathcal{S}_g = W^0_\sigma$ amounts to considering a special case of the general theory. We will study some geometric properties of the leaves and the "almost-product" structure they form.

We first give the definition of a central leaf, which you should view as a geometric isomorphism class of p-divisible groups.

Definition 3.20. Let $g \geq 1$. For a point $x = (X_0, \lambda_0) \in \mathcal{A}_g(k)$, define the *central leaf* passing through x to be

$$\mathcal{C}(x) := \{(X,\lambda) \in \mathcal{A}_g(k) : (X,\lambda)[p^\infty] \simeq (X_0,\lambda_0)[p^\infty]\}.$$

Suppose that (X_0, λ_0) has Newton polygon ξ. We collect some first facts about the dimensions of central leaves.

Proposition 3.21. (cf. [67, Theorem 3.3, Theorem 3.13])

(1) *With notation as in Definition 3.20, the central leaf passing through x is a closed subset*

$$\mathcal{C}(x) \subseteq W^0_\xi.$$

It is also a locally closed smooth subscheme of $\mathcal{A}_g$ which is pure of dimension c_ξ depending only on the Newton polygon ξ; that is, all irreducible components of $\mathcal{C}(x)$ have the same dimension.

(2) *An isogeny between principally polarized abelian varieties $x = (X_0, \lambda_0) \to y = (Y_0, \mu_0)$ induces a finite-to-finite isogeny correspondence between the central leaves through x and y, i.e. a k-scheme T and finite surjections $T \twoheadrightarrow \mathcal{C}(x)$, $T \twoheadrightarrow \mathcal{C}(y)$, so that* $\dim(\mathcal{C}(x)) = \dim(\mathcal{C}(y))$.

In other words, since the Newton polygon ξ is an isogeny invariant, we see that all central leaves in the same Newton polygon stratum have the same dimension c_ξ.

Thus, the dimension of a central leaf passing through $x = (X_0, \lambda_0)$ depends only on the Newton polygon ξ of (X_0, λ_0) – and conversely, every Newton polygon stratum W^0_ξ is a disjoint union of central leaves. We have the following dichotomy, cf. [3, Proposition 1], see also [31, Proposition 5.1]:

Proposition 3.22. *With notation as above, we have* $\dim(\mathcal{C}(x)) = 0$ *if and only if* (X_0, λ_0) *is supersingular, i.e. if and only if $\xi = \sigma$. In other words, the central leaf passing through a non-supersingular principally polarised abelian variety is positive-dimensional.*

When considering the zero-dimensional central leaves through supersingular points, one may ask when they have the smallest possible cardinality 1; then the supersingular abelian variety is uniquely determined by its p-divisible group. We answered this in the following result, where p denotes the characteristic of $k = \overline{\mathbb{F}}_p$.

Theorem 3.23. (cf. [31, Theorem 5.20.(ii)]) *Let $\mathcal{C}(x)$ be the central leaf in $\mathcal{A}_g$ passing through a point $x = (X_0, \lambda_0) \in \mathcal{S}_g(k)$. Then $\mathcal{C}(x)$ consists of one element if and only if one of the following three cases holds:*

(i) $g = 1$ *and* $p \in \{2, 3, 5, 7, 13\}$*;*
(ii) $g = 2$ *and* $p = 2, 3$*;*
(iii) $g = 3$*,* $p = 2$ *and* $a(x) \geq 2$*.*

Theorem 3.23 immediately implies that a central leaf passing through a supersingular point $x \in \mathcal{S}_g(k)$ is irreducible if and only if one of the conditions (i)–(iii) is satisfied. By contrast, Chai-Oort prove the following:

Proposition 3.24. (cf. [5, Theorem 4.1]) *The central leaf* $\mathcal{C}(x)$ *passing through any non-supersingular point* $x \in \mathcal{A}_g(k)$ *is irreducible.*

Recall the moduli space $\mathcal{A}_{g,1,n}$ of principally polarised abelian varieties with level n structure, defined before Theorem 2.4. Below, we will consider its characteristic p fibre $\mathcal{A}_{g,1,n} \otimes \mathbb{F}_p$, which we will again denote by $\mathcal{A}_{g,1,n}$ for ease of notation, as for $\mathcal{A}_g$, cf. Notation 2.10. Furthermore, assume that $n \geq 3$ is coprime to p. The reason that $\mathcal{A}_{g,1,n}$ appears is that it is a *fine* moduli space, while $\mathcal{A}_g$ is a coarse one; so in particular $\mathcal{A}_{g,1,n}$ carries a universal family, say $(\mathcal{X}, \lambda_{\mathcal{X}})$.

Definition 3.25. (cf. [67, §4.1-4.3]) Let $g \geq 1$. An *isogeny leaf* of $\mathcal{A}_g$ is a maximal closed integral subscheme I of $\mathcal{A}_g$ such that there exist: a principally polarised abelian variety (M, μ) over k, a scheme T of finite type over k, a surjective morphism $T \to I_n$, where $I_n := I \times_{\mathcal{A}_g} \mathcal{A}_{g,1,n}$ is the base change of I to $\mathcal{A}_{g,1,n}$, and an isogeny $\varphi : (M, \mu) \to (\mathcal{X}, \lambda_{\mathcal{X}}) \otimes_{\mathcal{A}_{g,1,n}} T$, such that every geometric fibre of φ is formed out of successive extensions of α_p group schemes.

For each $x \in \mathcal{A}_g(k)$, there is a closed reduced subscheme $I(x)$ of $\mathcal{A}_g$ whose irreducible components are the isogeny leaves containing x. In other other words, there are only finitely many isogeny leaves containing x and $I(x)$ is their union, with the induced reduced scheme structure.

The scheme $\mathcal{I}(x)$ is a proper k-scheme [67, Proposition 4.11] and for x and y in the same central leaf, the formal completions of $\mathcal{I}(x)$ and $\mathcal{I}(y)$ are isomorphic [67, Proposition 4.12].

You should think of an isogeny leaf through $x = (X_0, \lambda_0)$ as consisting of all abelian varieties $(Y_0, \mu_0) \in \mathcal{A}_g(k)$ that are isogenous to (X_0, λ_0) via an iterated α_p-isogeny (i.e. whose kernel is a repeated α_p-extension). In particular, such isogenies have p-power degree and can change the p-divisible group. By contrast, prime-to-p isogenies leave the p-divisible group unchanged. So while the former move you along an isogeny leaf, the latter move you within a central leaf.

Applying degree-ℓ isogenies can be viewed as an action on $\mathcal{A}_g(k)$, the so-called Hecke-ℓ-action, which restricts to an action on individual central leaves by the previous observation. (Similarly, we can define Hecke-α-actions on isogeny leaves using iterated α_p-isogenies.) In fact, Ekedahl-Oort strata are also preserved under Hecke-ℓ-actions.

The orbits in $\mathcal{A}_g$ of this action are called Hecke-(ℓ-)orbits. The Hecke Orbit Conjecture (cf. [67, Conjectures 6.1-6.2]) asserts that the Hecke-ℓ-orbit in $\mathcal{A}_g$ through a moduli point x is Zariski dense in its central leaf $\mathcal{C}(x)$. It was proven by Chai in [3, Theorem 2] for ordinary abelian varieties – showing in fact that the orbit is dense in $\mathcal{A}_g$ – and in [4] for any principally polarised abelian variety.

The following result explains the geometric interplay between central and isogeny leaves.

Proposition 3.26. (cf. [67, Theorem 5.3, Corollary 5.7]) *Let* $V \subseteq W^0_\xi \subseteq \mathcal{A}_g$ *be any irreducible component of a Newton stratum. Then there exists a finite surjective* k*-morphism*

$$\Phi : D \times J \to V,$$

where D, J *are integral* k*-schemes of finite type, such that*

(1) *For any* $d \in D(k)$*, the image* $\Phi(\{d\} \times J)$ *is an isogeny leaf in* V *and any isogeny leaf in* V *can be found this way;*
(2) *For any* $j \in J(k)$*, the image* $\Phi(D \times \{j\})$ *is a central leaf in* V *and any central leaf in* V *can be found this way.*

Hence, every central leaf in V intersects every isogeny leaf in V non-trivially, creating an "almost-product structure".

We derive the following result on the dimensions of the isogeny leaves.

Proposition 3.27. *All isogeny leaves in W_ξ^0 have the same dimension i_ξ, which only depends on the Newton polygon ξ.*

Proof. For a fixed Newton polygon ξ, the dimension of each irreducible component W of W_ξ^0 is the same, write $d_\xi = \mathrm{sdim}(\xi)$, cf. Definition 2.30. In Proposition 3.21.(2) we also saw that each central leaf in $W_\xi^0(k)$ has the same dimension c_ξ. The almost-product structure then implies that the dimension of any isogeny leaf in V must be $i_\xi = d_\xi - c_\xi$ and hence only depends on ξ. □

Remark 3.28. In the notation of the previous proposition, it follows from Proposition 3.22, together with Theorem 3.15 and the paragraph preceding it, that in the supersingular case

$$i_\sigma = d_\sigma = \left\lfloor \frac{g^2}{4} \right\rfloor.$$

3.5. **Stratifications restricted to $\mathcal{S}_g$.**
In Section 2 we introduced the p-rank, Newton polygon, a-number, and Ekedahl-Oort stratifications on $\mathcal{A}_g$. Recall that the supersingular locus $\mathcal{S}_g$ is itself a Newton stratum, which is contained in the p-rank zero stratum. In this subsection, we will restrict the a-number and Ekedahl-Oort stratifications to $\mathcal{S}_g$ and study their properties.

3.5.1. *The a-number stratification on $\mathcal{S}_g$.*
The a-number strata on $\mathcal{S}_g$ were first defined in [45, § 9.9–9.11] and are comprehensively dealt with by Harashita in [16]. We define

$$\begin{aligned} \mathcal{S}_g(a \geq n) &:= \{x = (X, \lambda) \in \mathcal{S}_g : a(X) \geq n\}; \\ \mathcal{S}_g(n) &:= \{x = (X, \lambda) \in \mathcal{S}_g : a(X) = n\}. \end{aligned}$$

The former is a closed subscheme of $\mathcal{S}_g$, the latter is locally closed.

The projection morphism of Equation (11) induces a surjective and generically finite k-morphism

$$\mathrm{pr}_0 : \coprod_\mu \mathcal{P}'_{g,\mu}(a) \to \mathcal{S}_g(a), \tag{12}$$

where the disjoint union again runs over all suitable polarisations μ of E_0^g and where $\mathcal{P}'_{g,\mu}(a)$ is the moduli space of rigid PFTQs whose last member (Y_0, λ_0) has a-number a.

Thus, the results in [16] are obtained by studying $\mathcal{P}'_{g,\mu}(a)$. As in [45] for the results in Subsection 3.3, (moduli spaces of) PFTQs of abelian varieties in turn are studied by considering the corresponding (moduli spaces of) chains – also called PFTQs – of Dieudonné modules.

Theorem 3.29. (cf. [16, Theorem 3.15, Theorem 4.17])

(1) *The Zariski closure $\mathcal{S}_g^c(a)$ of $\mathcal{S}_g(a)$ is connected unless $a = g$ and satisfies*

$$S_g^c(a) = \cup_{a' \geq a} \mathcal{S}_g(a').$$

(2) *Every irreducible component of $\mathcal{S}_g(a)$ has the same dimension*

$$\left\lfloor \frac{g^2 - a^2 + 1}{4} \right\rfloor.$$

(3) *The number of irreducible components of $\mathcal{S}_g(a)$ is*

$$\begin{cases} \binom{(g-2)/2}{(g-a-1)/2} h_g(1,p) & \text{if } g \text{ is even, } a \text{ is odd;} \\ \binom{(g-1)/2}{(g-a)/2} h_g(p,1) & \text{if } g \text{ is odd, } a \text{ is odd;} \\ \binom{g/2-1}{(g-a)/2} h_g(p,1) + \binom{g/2-1}{(g-a)/2-1} h_g(1,p) & \text{if } g \text{ is even, } a \text{ is even;} \\ \binom{(g-1)/2-1}{(g-a-1)/2} h_g(1,p) + \binom{(g-1)/2-1}{(g-a-1)/2-1} h_g(p,1) & \text{if } g \text{ is odd, } a \text{ is even.} \end{cases}$$

Sketch of the proof. By introducing new ("good") bases Θ for the Dieudonné module of the first and last members of a PFTQ (respectively (Y_{g-1}, λ_{g-1}) and (Y_0, λ_0)), we get an open covering $\coprod_\Theta U^\Theta$ of the moduli space $\mathcal{N}_g$ of rigid PFTQs of Dieudonné modules, by for each Θ grouping together in U^Θ those PFTQs whose last member is written in basis Θ. The moduli space $\mathcal{N}_g$ is isomorphic to $\mathcal{P}'_{g,\mu}$ up to inseparable isomorphism. Let $U^\Theta(a)$ denote the subscheme of $\mathcal{N}_g$ of PFTQs of Dieudonné modules with a-number a.

For any choice Θ, the action of Frobenius and Verschiebung on the Dieudonné module of (Y_0, λ_0) can be nicely expressed in terms of the chosen basis, and the a-number of Y_0 can be read off from the rank of the matrix of the coefficients. All such matrices with the same rank therefore form a *period domain* $\nabla_{g,a}$, such that there is an étale surjective map $U^\Theta(a) \to \nabla_{g,a}$.

The irreducible components of the $\nabla_{g,a}$ are determined by completely explicit computations, that immediately also determine the connected Zariski closure $\nabla^c_{g,a} = \cup_{a' \geq a} \nabla_{g,a'}$, dimension $\left\lfloor \frac{g^2-a^2+1}{4} \right\rfloor$, and number of irreducible components of each $\nabla_{g,a}$. This shows parts (1) and (2) of the theorem, using the connectedness result [66, Theorem 1.1].

For part (3), the number of irreducible components of $\mathcal{S}_g(a)$ is shown to be $\sum_{x \in I_{g,a}} |\Lambda_x|$, where $I_{g,a}$ denotes the set of irreducible components of the moduli space $D_g(a)$ of supersingular Dieudonné modules with a-number a, and where $|\Lambda_x|$ denotes the number of suitable polarisations on E_0^g with kernel prescribed by x. In other words, for each component of $D_g(a)$ there are $|\Lambda_x|$ components of $\mathcal{S}_g(a)$. Finally, $|I_{g,a}|$ is explicitly and combinatorially determined using results from [44] and shown to be equal to the number of components of $\nabla_{g,a}$, while $|\Lambda_x|$ is proved to be a class number $h_g(p, 1)$ or $h_g(1, p)$ (cf. Definition 4.12); multiplying yields (3). $\square$

3.5.2. *The Ekedahl-Oort stratification on $\mathcal{S}_g$.*
In general, the intersections of Ekedahl-Oort strata and Newton strata in $\mathcal{A}_g$ is not well understood. Restricting to $\mathcal{S}_g$ however, we can say a few things.

First of all, there is a combinatorial criterion for when an Ekedahl-Oort stratum is supersingular, i.e. is fully contained in $\mathcal{S}_g$:

Proposition 3.30. (cf. [5, Theorem 4.8, Step 2]), [66, Theorem 8.3.(II)] *Let $\mathcal{S}_\varphi$ be the Ekedahl-Oort stratum in $\mathcal{A}_g$ associated with an elementary sequence φ. Then $\mathcal{S}_\varphi \subseteq \mathcal{S}_g$ if and only if $\varphi(r) = 0$ for $r = \lfloor \frac{g+1}{2} \rfloor$.*

Sketch of the proof. Suppose first that $\varphi(r) = 0$ and let $N_1 \subseteq N_2 \subseteq \ldots \subseteq N_{2g} = X[p]$ be a corresponding final filtration. The condition $\varphi(r) = 0$ means that F and V are both zero on N_{g+r}/N_r, which in turn means that X/N_r is superspecial. Since $X/N_r \sim X$, we conclude that X is supersingular, hence $\mathcal{S}_\varphi \subseteq \mathcal{S}_g$.

The other implication is shown by constructing a counterexample, namely by exhibiting a Newton polygon and corresponding "minimal" p-divisible group such that the associated elementary sequence φ' satisfies $0 = \varphi'(1) = \varphi'(2) = \ldots = \varphi'(r-1)$ but $\varphi'(r) = 1$, and $\mathcal{S}_{\varphi'} \not\subseteq \mathcal{S}_g$. $\square$

Recall from the discussion following Definition 2.48 that Ekedahl-Oort strata are also classified by elements of the Weyl group W_g of Sp_{2g}; in fact, the set of elementary sequences of length g is in bijection with the subset

$$^I W_g = \{w \in W_g : w^{-1}(1) < \ldots < w^{-1}(g)\} \subseteq W_g.$$

Next, for any $0 \leq c \leq g$, define

$$^I W_g^{[c]} = \{w \in W_g : w(i) = i \text{ for all } i \leq g - c\}$$

and

$$^I W_g^{(c)} = {^I W_g^{[c]}} - {^I W_g^{[c-1]}} \text{ for } 0 < c \leq g \ , \quad {^I W_g^{(0)}} = \mathrm{id}.$$

With this notation, we can equivalently reformulate Proposition 3.30 as follows:

Proposition 3.31. (cf. [18, Lemma 2.5.4, Remark 2.5.7, Proposition 3.1.5]) *Let w be the Weyl group element associated with an elementary sequence φ. Then $\mathcal{S}_\varphi \subseteq \mathcal{S}_g$ if and only if $w \in {}^I W_g^{(c)}$ for $c \leq \lfloor \frac{g}{2} \rfloor$.*

Remark 3.32. Also in [18], Harashita gives descriptions of certain unions of supersingular Ekedahl-Oort strata in terms of Deligne-Lusztig varieties. This description is then used to confirm that supersingular Ekedahl-Oort strata are reducible (whereas the non-supersingular strata are irreducible, by [11, Theorem 11.5]). It was refined by Hoeve [26], who described single supersingular Ekedahl-Oort strata in terms of so-called fine Deligne-Lusztig varieties.

We have seen two equivalent ways of determining which Ekedahl-Oort strata are fully contained in $\mathcal{S}_g$; recall also that $\mathcal{S}_g$ is a Newton stratum.

In [17], Harashita extends the above to other Newton strata, by giving a necessary and sufficient condition for an Ekedahl-Oort stratum $\mathcal{S}_\varphi$ to be fully contained in the Newton locus Z_λ consisting of moduli points in $\mathcal{A}_g$ for which the first slope (when the slopes are written in increasing order) of their associated Newton polygon is greater than or equal to a rational number λ. In Harashita's notation, we have $\mathcal{S}_g = Z_{\frac{1}{2}}$. The condition is derived from the main result [17, Theorem 4.1], which combinatorially determines the first Newton slope λ_φ associated with any generic moduli point in $\mathcal{S}_\varphi$, and is as follows:

Proposition 3.33. (cf. [17, Corollary 4.2]) *With notation as above, we have $\mathcal{S}_\varphi \subseteq Z_\lambda$ if and only if $\lambda_\varphi \geq \lambda$.*

In addition to supersingular Ekedahl-Oort strata, there might also be strata that intersect $\mathcal{S}_g$ non-trivially, without being fully contained in it. Below, we give a few low-dimensional examples.

Example 3.34. Let $g = 2$. The Ekedahl-Oort strata of p-rank zero are those corresponding to the elementary sequences $(0, 0)$ and $(0, 1)$ by Theorem 2.51.(6). Since $\lfloor \frac{g+1}{2} \rfloor = \lfloor \frac{3}{2} \rfloor = 1$ and both these sequences φ satisfy $\varphi(1) = 0$, we see that both Ekedahl-Oort strata of p-rank zero are supersingular, as expected: for $g = 2$, the notions of p-rank zero and supersingularity coincide.

Example 3.35. Let $g = 3$. The Ekedahl-Oort strata of p-rank zero are precisely the $\mathcal{S}_\varphi$ for $\varphi \in \{(0, 0, 0), (0, 0, 1), (0, 1, 1), (0, 1, 2)\}$, by Theorem 2.51.(6). These strata have respective a-numbers 3, 2, 2 and 1, also by Theorem 2.51.(5). In particular, we conclude that $\mathcal{S}_{(0,1,2)} \cap \mathcal{S}_3$ is the a-number 1 locus of $\mathcal{S}_3$, so it is Zariski dense in $\mathcal{S}_3$ by [45, Theorem 4.9(iii)].

Next, we have $\lfloor \frac{g+1}{2} \rfloor = \lfloor \frac{4}{2} \rfloor = 2$ so Proposition 3.30 implies that $\mathcal{S}_\varphi$ is supersingular for $\varphi = (0, 0, 0), (0, 0, 1)$.

It remains to consider the stratum $\mathcal{S}_{(0,1,1)}$. However, for $\varphi = (0, 1, 1)$ we compute that $\lambda_\varphi = \frac{1}{3}$, cf. [17, Definition 3.1], and using Proposition 3.33 we see that this stratum is fully contained in another Newton polygon stratum, corresponding to the slope sequence $\left(\frac{1}{3}, \frac{1}{3}, \frac{1}{3}, \frac{2}{3}, \frac{2}{3}, \frac{2}{3}\right)$, and therefore does not intersect $\mathcal{S}_3$.

We conclude that

$$\mathcal{S}_3 = \mathcal{S}_{(0,0,0)} \sqcup \mathcal{S}_{(0,0,1)} \sqcup \left(\mathcal{S}_{(0,1,2)} \cap \mathcal{S}_3\right)$$

describes the Ekedahl-Oort stratification of $\mathcal{S}_3$. In particular, we see that the other a-number strata are given by $\mathcal{S}_3(2) = \mathcal{S}_{(0,0,1)}$ and $\mathcal{S}_3(3) = \mathcal{S}_{(0,0,0)}$. See [19, Theorem 5.1] for the same result with a different proof, using Weyl group elements.

Example 3.36. (cf. [31, Proposition 5.13]) Let $g = 4$.

The Ekedahl-Oort strata of p-rank zero are precisely the $\mathcal{S}_\varphi$ for those φ appearing in Figure 3, according to Theorem 2.51.(6). Their a-numbers are as indicated by their colours, by Theorem 2.51.(5).

By Proposition 3.30, the strata fully contained in $\mathcal{S}_4$ are precisely the $\mathcal{S}_\varphi$ for $\varphi = (0, 0, 0, 0)$, $(0, 0, 0, 1)$, $(0, 0, 1, 1)$, and $(0, 0, 1, 2)$.

The other Newton strata of p-rank zero correspond to the slope sequences $\left(\frac{1}{3}, \frac{1}{3}, \frac{1}{3}, \frac{1}{2}, \frac{1}{2}, \frac{2}{3}, \frac{2}{3}, \frac{2}{3}\right)$ and $\left(\frac{1}{4}, \frac{1}{4}, \frac{1}{4}, \frac{1}{4}, \frac{3}{4}, \frac{3}{4}, \frac{3}{4}, \frac{3}{4}\right)$, and are denoted respectively by $W_{\frac{1}{3}}$ and $W_{\frac{1}{4}}$.

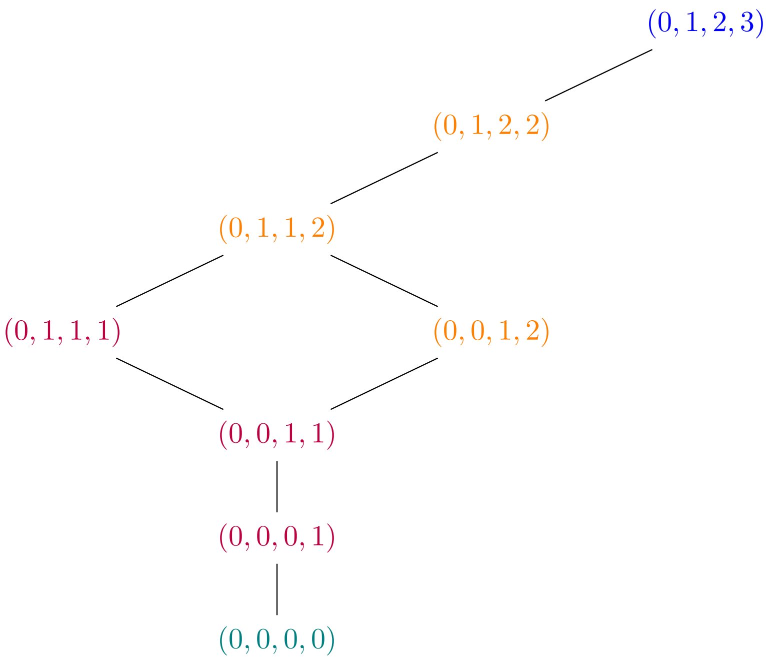


FIGURE 3. Ekedahl-Oort strata of p-rank zero in dimension $g = 4$. The blue stratum has a-number 1, the orange strata have a-number 2, the red strata have a-number 3 and the green stratum has a-number 4. Strata are connected by a line if the lower one is contained in the Zariski closure of the upper one.

We read off from Figure 3 that $\mathcal{S}_{(0,1,2,3)} \cap \mathcal{S}_4$ is the a-number 1 locus of $\mathcal{S}_4$, so it is Zariski dense by [45, Theorem 4.9(iii)].

By [17, Corollary 4.2 and Lemma 5.12] we see that $\mathcal{S}_{(0,1,2,2)} \subseteq W_{\frac{1}{4}}$ by minimality of the associated p-divisible group. Similarly, from [17, Corollary 4.2 and Proposition 7.1], we obtain that $\mathcal{S}_{(0,1,1,1)} \subseteq W_{\frac{1}{3}}$, again by minimality.

Finally, we read off from Figure 3 that

$$\mathcal{S}_{(0,1,1,2)} = \left(\mathcal{S}_{(0,1,1,2)} \cap W_{\frac{1}{3}}\right) \sqcup \left(\mathcal{S}_{(0,1,1,2)} \cap \mathcal{S}_4\right).$$

Now Theorem 3.29.(3) implies that $\mathcal{S}_4(2)$ has $h_4(1,p) + h_4(p,1)$ many irreducible components of two types, of which those of the type corresponding to $\mathcal{S}_{(0,0,1,2)}$ yield $h_4(1,p)$ many; see also [45, § 9.9]. Hence, the intersection $\mathcal{S}_{(0,1,1,2)} \cap \mathcal{S}_4$ must yield the other $h_4(p,1)$ components and thus be non-empty.

We conclude that

$$\begin{aligned} \mathcal{S}_4 = &\left(\mathcal{S}_{(0,1,2,3)} \cap \mathcal{S}_4\right) \sqcup \mathcal{S}_{(0,0,0,0)} \sqcup \mathcal{S}_{(0,0,0,1)} \\ &\sqcup \mathcal{S}_{(0,0,1,1)} \sqcup \mathcal{S}_{(0,0,1,2)} \sqcup \left(\mathcal{S}_{(0,1,1,2)} \cap \mathcal{S}_4\right), \end{aligned}$$

where each intersection is non-empty and $\mathcal{S}_{(0,1,2,3)} \cap \mathcal{S}_4$ is dense. In particular, we read off the a-number strata as

$$\begin{aligned} \mathcal{S}_4(4) &= \mathcal{S}_{(0,0,0,0)}; \\ \mathcal{S}_4(3) &= \mathcal{S}_{(0,0,0,1)} \sqcup \mathcal{S}_{(0,0,1,1)}; \\ \mathcal{S}_4(2) &= \mathcal{S}_{(0,0,1,2)} \sqcup \left(\mathcal{S}_{(0,1,1,2)} \cap \mathcal{S}_4\right). \end{aligned}$$

4. The arithmetic of $\mathcal{S}_g$

4.1. **Introduction.**

In the previous section, we saw different *geometric* aspects of $\mathcal{S}_g$ as the moduli space of supersingular abelian varieties. In this section, we will use these notions to prove *arithmetic* results about supersingular abelian varieties. In particular, we will be looking at the key question: *How many supersingular abelian varieties are there?*

This question is not very precisely stated. First of all, we will always fix a dimension g and a characteristic p (> 0) of the field $k = \overline{\mathbb{F}}_p$. Recall also that the varieties in $\mathcal{S}_g$ are principally polarised by definition.

It turns out to be useful to first ask how many *superspecial* abelian varieties there are. This is because there is a direct connection between superspecial abelian varieties and equivalence classes of lattices in quaternion Hermitian spaces; hence, the final number is a class number. (This connection is maybe not completely surprising, if you remember from Example 1.10 that the endomorphism algebra of a supersingular elliptic curve over k is a quaternion algebra!)

In Subsection 4.2 we will therefore first spend some time on quaternion algebras and quaternion Hermitian spaces and state what is known about their class numbers. It turns out that these are very hard to compute in general. A more accessible quantity is the *mass*, which you should view as a weighted count, namely, weighted by automorphisms: the mass of a finite set S, whose elements have a notion of automorphisms, is

$$\mathrm{Mass}(S) := \sum_{s \in S} \frac{1}{|\mathrm{Aut}(s)|}.$$

Masses of genera of lattices in quaternion Hermitian spaces have been determined in full generality; see Proposition 4.17.

In Subsection 4.3 we explain the connection between these lattices and superspecial abelian varieties; the latter may also be non-principally polarised. We let Λ_{g,p^c} denote the set of isomorphism classes of superspecial g-dimensional abelian varieties with degree-p^{2c} polarisations (so $0 \le c \le \lfloor g/2 \rfloor$). Using the connection with lattices, we determine mass of any Λ_{g,p^c} in Theorem 4.23.

Finally, in Subsection 4.4 we explain how to use superspecial masses to compute the mass of supersingular central leaves, through so-called minimal isogenies. Once you know the mass of a central leaf, knowing the cardinality of the central leaf is equivalent to understanding the automorphism groups of the varieties; and these groups are key arithmetic invariants in many applications.

4.2. **Class numbers for quaternion algebras.**

We now take a break from abelian varieties for a while to consider quaternion algebras and quaternion Hermitian spaces. We introduce the class number in this setting, as a count of equivalence classes, and the mass, which is a weighted count of the classes. Then we will briefly discuss what is known about these quantities, starting with the work of Eichler from 1938. A comprehensive reference for quaternion algebras is [82].

4.2.1. *Quaternion algebras.*

Let B be a quaternion algebra over $\mathbb{Q}$. Denote the natural involution on B by $x \mapsto \bar{x}$.

An *order* in B is a $\mathbb{Z}$-lattice (of maximal rank) that is also a subring. Let $\mathcal{O}$ be a maximal order of B, i.e. maximal with respect to containment. For example, the matrix ring $M_2(\mathbb{Z})$ is a maximal order in $M_2(\mathbb{Q})$; it is in fact the unique maximal order up to conjugacy, cf. [82, Corollary 10.5.5].

For any prime p we may consider the completion $B_p = B \otimes_{\mathbb{Q}} \mathbb{Q}_p$ of B at p. This is either split, i.e. isomorphic to the matrix algebra $M_2(\mathbb{Q}_p)$, or ramified, i.e. isomorphic the unique division algebra over $\mathbb{Q}_p$. We also consider the place ∞ at infinity, i.e. $B_\infty = B \otimes_{\mathbb{Q}} \mathbb{R}$: then B ramifies at ∞ if B_∞ is isomorphic to the Hamilton quaternions and split if it is isomorphic to $M_2(\mathbb{R})$.

A quaternion algebra B over $\mathbb{Q}$ ramifies only at finitely many places, and the number of ramified places is even by class field theory. Moreover, a quaternion algebra over $\mathbb{Q}$, or indeed over any global field, is determined up to isomorphism by its finite set of ramified places. The finite square-free product of finite ramified places is called the *discriminant* of B. Further, B is called *indefinite* if it is split at ∞, and *definite* if it is ramified at ∞.

Notation 4.1. For any prime number p, let $Q_{p,\infty}$ be the quaternion algebra over $\mathbb{Q}$ that is ramified exactly at p and ∞. It has discriminant p.

Example 4.2. Explicit representations of $Q_{p,\infty}$ for any p are given for example in [82, Example 14.2.13]. When $p = 2$ for instance, we can take

$$B = (-1, -1)_{\mathbb{Q}} = \langle 1, i, j, ij : i^2 = -1, j^2 = -1, ji = -ij\rangle.$$

To any lattice L in B we can associate its left order

$$O^L_B(L) = \{b \in B : bL \subseteq L\}$$

and its right order

$$O^R_B(L) = \{b \in B : Lb \subseteq L\}.$$

It is *invertible* if there exists another lattice L' such that

$$LL' = O^L_B(L) = O^R_B(L') \quad \text{and} \quad L'L = O^L_B(L') = O^R_B(L).$$

For any order A in B, a *(right) A-ideal* I of B is a lattice I in B such that $A \subseteq O^R_B(I)$. The *(right) class* of I is

$$[I]_R := \{J = aI : a \in B^\times\}$$

and the *right class set* of A is

$$\mathrm{Cl}_R(A) := \{[I]_R : I \text{ is an invertible right } A\text{-ideal}\}.$$

We could have equivalently defined left ideals, left classes and the left class set; the latter is in bijection with the right class set through the involution on B. Both are finite, cf. [82, Theorem 17.1.1], and their cardinality is called the *class number* of A, denoted $h(A)$. An ideal that is both a left and a right ideal is called a two-sided ideal.

Eichler computed the class number for the maximal orders of definite quaternion algebras over $\mathbb{Q}$. A literal translation yields:

Theorem 4.3. (cf. [9, Satz 2]) *Let B be a definite quaternion algebra over $\mathbb{Q}$ with maximal order $\mathcal{O}$ and discriminant d. The class number of $\mathcal{O}$ is 1 if $d = 2$ or 3, and for $d \geq 5$ it equals*

$$h(\mathcal{O}) = \frac{1}{12}\prod_{p|d}(p-1) + \frac{h_2}{2} + \frac{2h_3}{3}, \qquad \textit{where}$$

$$h_2 = \begin{cases} 2^{u-1} & \textit{if } d \textit{ is divisible by } u \textit{ odd primes, all congruent to } 3 \bmod 4; \\ 0 & \textit{otherwise}; \end{cases} \tag{13}$$

$$h_3 = \begin{cases} 2^{v-1} & \textit{if } d \textit{ is divisible by } v \textit{ primes unequal to } 3\textit{, all congruent to } 2 \bmod 3; \\ 0 & \textit{otherwise.} \end{cases}$$

Corollary 4.4. *When $B = Q_{p,\infty}$ for $p \geq 5$, with discriminant $d = p$, Theorem 4.3 gives, cf.* [7, p. 266]:

$$h(\mathcal{O}) = \begin{cases} \frac{p-1}{12} & \textit{if } p \equiv 1 \pmod{12}; \\ \frac{p-5}{12} + 1 & \textit{if } p \equiv 5 \pmod{12}; \\ \frac{p-7}{12} + 1 & \textit{if } p \equiv 7 \pmod{12}; \\ \frac{p-11}{12} + 2 & \textit{if } p \equiv 11 \pmod{12}. \end{cases} \tag{14}$$

Comparing Equation (14) with (8) shows that $h(\mathcal{O}) = |\mathcal{S}_1|$. As we will see in the next subsection, this is not a coincidence.

In the same article, Eichler also proves a formula for a "weighted" class number, which for definite quaternion algebras over $\mathbb{Q}$ simplifies to the following, cf. [82, Theorem 25.1.1]:

Theorem 4.5. (cf. [9, Satz 1]) *Let B be a definite quaternion algebra over $\mathbb{Q}$ with maximal order $\mathcal{O}$ and discriminant d. Then*

$$\sum_{[I]_L \in \mathrm{Cl}_L(\mathcal{O})} \frac{1}{|O_B^L(I)^\times/\{\pm 1\}|} = \frac{1}{12}\prod_{p|d}(p-1). \tag{15}$$

The significance of Theorem 4.5 is the following: The elements of the unit group $O_B^L(I)^\times$ are the automorphisms of the $\mathcal{O}$-ideal I, whereas the units of the lattice $\mathbb{Z}$ in $\mathbb{Q}$ are ± 1. The finite quotient $O_B^L(I)^\times/\{\pm 1\}$ is also called the *reduced automorphism group* of I. In other words, the left hand side of (15) counts the classes in $\mathrm{Cl}_L(\mathcal{O})$, but by dividing by the (reduced) automorphisms, we are counting them up to symmetry. Note that the right hand side of (15) is a lot cleaner than that of (13).

4.2.2. *Quaternion Hermitian spaces.*
The definite quaternion algebra B over $\mathbb{Q}$ with involution $x \mapsto \overline{x}$, discriminant d and maximal order $\mathcal{O}$ as above can be viewed as a one-dimensional quaternion Hermitian space. We now generalise to higher-dimensional quaternion Hermitian spaces, following [21, § 1], cf. [31, § 2.2].

Definition 4.6. A *positive-definite quaternion Hermitian space* over B of rank n is a pair (V, f) where V is a $\mathbb{Q}$-vector space and an n-dimensional left B-module, and $f : V \times V \to B$ is a $\mathbb{Q}$-bilinear form satisfying:

(i) $f(ax, y) = af(x, y)$ and $f(x, ay) = f(x, y)\bar{a}$;
(ii) $f(y, x) = \overline{f(x, y)}$;
(iii) $f(x, x) \geq 0$ and $f(x, x) = 0$ only when $x = 0$,

for all $a \in B$ and $x, y \in V$.

For each rank n there is a unique isomorphism class (V, f); we could take $V = B^{\oplus n}$ and the Hermitian form $f((x_i)_i, (y_i)_i) = \sum_i x_i \overline{y}_i$.

Notation 4.7. For each prime p, we define $\mathcal{O}_p := \mathcal{O} \otimes_{\mathbb{Z}} \mathbb{Z}_p$, $B_p := B \otimes_{\mathbb{Q}} \mathbb{Q}_p$ and $V_p := V \otimes_{\mathbb{Q}} \mathbb{Q}_p$. We further let $G = G(V, f)$ be the group of similitudes of (V, f):

$$G = \{\ \alpha \in \mathrm{GL}_B(V) : f(x\alpha, y\alpha) = n(\alpha) f(x, y) \quad \forall\, x, y \in V\ \}, \tag{16}$$

where $n(\alpha) \in \mathbb{Q}^\times$ is a scalar depending only on α, and similarly let $G_p = G(V_p, f_p)$ be the group of similitudes of (V_p, f_p). Taking $V = B^{\oplus n}$ and $f((x_i)_i, (y_i)_i) = \sum_i x_i \overline{y}_i$ as above, we see that

$$G = \{\alpha \in \mathrm{GL}_n(B) : \alpha \overline{\alpha}^t = n(\alpha)\mathbb{I}_n,\ n(\alpha) \in \mathbb{Q}^\times\}. \tag{17}$$

A lattice $L \subseteq V$ is called a left $\mathcal{O}$-lattice if $\mathcal{O}L \subseteq L$. An $\mathcal{O}$-submodule M of an $\mathcal{O}$-lattice L is called an $\mathcal{O}$-sublattice of L; then M is an $\mathcal{O}$-lattice in the B-module BM, possibly of smaller rank.

Definition 4.8. Two $\mathcal{O}$-lattices L_1 and L_2 are *equivalent*, denoted $L_1 \sim L_2$, if there exists an $\alpha \in G$ such that $L_2 = L_1\alpha$; equivalence of two O_p-lattices is defined analogously. Two $\mathcal{O}$-lattices L_1 and L_2 are in the same *genus* if $(L_1)_p \sim (L_2)_p$ for all primes p, i.e. if they are everywhere locally equivalent.

Definition 4.9. The *norm* $N(L)$ of an $\mathcal{O}$-lattice L is the two-sided $\mathcal{O}$-ideal generated by $f(x, y)$ for all $x, y \in L$. If L is maximal among the $\mathcal{O}$-lattices having the same norm $N(L)$, then it is called a *maximal $\mathcal{O}$-lattice*. Maximal $\mathcal{O}_p$-lattices in V_p are defined analogously. An $\mathcal{O}$-lattice L is maximal if and only if the $\mathcal{O}_p$-lattice $L_p := L \otimes_{\mathbb{Z}} \mathbb{Z}_p$ is maximal for all primes p.

If a prime p does not divide the discriminant d of B, then there is a unique equivalence class of maximal $\mathcal{O}_p$-lattices in V_p, represented by the standard unimodular lattice $(\mathcal{O}_p^n, f = \mathbb{I}_n)$.

If $p|d$ and $n > 1$, then there are two equivalence classes of maximal $\mathcal{O}_p$-lattices in V_p, represented respectively by the *principal lattice* $(\mathcal{O}_p^n, f = \mathbb{I}_n)$ and the *non-principal* lattice $((\Pi_p\mathcal{O}_p)^{\oplus(n-c)} \oplus \mathcal{O}_p^{\oplus c}, \mathbb{J}_n)$, where $c = \lfloor n/2 \rfloor$, where Π_p is a uniformising element in $\mathcal{O}_p$ with $\Pi_p\overline{\Pi}_p = p$, and where $\mathbb{J}_n = \text{anti-diag}(1, \ldots, 1)$ is the anti-diagonal identity matrix of size n. (This is equivalent to the lattice N_p in [45, (4.6.3)] and [32, p. 140].)

Since a genus is determined by choosing an equivalence class at every prime, we see that there are 2^t genera of maximal $\mathcal{O}$-lattices in V when $n \geq 2$, where t is the number of primes dividing the discriminant d of B.

Definition 4.10. For any positive integer n and any pair (d_1, d_2) of positive integers such that $d = d_1d_2$, let $\mathcal{L}_n(d_1, d_2)$ be the genus consisting of maximal $\mathcal{O}$-lattices in (V, f) of rank n such that for all primes $p|d_1$ (resp. $p|d_2$) the local $\mathcal{O}_p$-lattice (L_p, f) belongs to the principal class (resp. the non-principal class).

There are two extreme cases: the genus $\mathcal{L}_n(d, 1)$ is the *principal genus*, and $\mathcal{L}_n(1, d)$ is the *non-principal genus.*

Let $[\mathcal{L}_n(d_1, d_2)]$ be the set of (global) equivalence classes of lattices in $\mathcal{L}_n(d_1, d_2)$.

By considering all completions of our lattices, i.e. by viewing them adelically, the following lemma follows from the definitions.

Lemma 4.11. *Let $\mathbb{A}_f$ denote the finite adeles of $\mathbb{Q}$ and let $\widehat{\mathbb{Z}}$ be the profinite completion of $\mathbb{Z}$. Fix a lattice $L_0 \in \mathcal{L}_n(d_1, d_2)$. There is a natural map*

$$[\mathcal{L}_n(d_1, d_2)] \simeq U\backslash G(\mathbb{A}_f)/G(\mathbb{Q}), \tag{18}$$

where U is the stabiliser of $L_0 \otimes \widehat{\mathbb{Z}}$ in $G(\mathbb{A}_f)$, which is an isomorphism of pointed sets, sending L_0 to the trivial element.

Definition 4.12. The cardinality of $[\mathcal{L}_n(d_1, d_2)]$,

$$h_n(d_1, d_2) := |[\mathcal{L}_n(d_1, d_2)]|$$

is called the *class number* of the genus $\mathcal{L}_n(d_1, d_2)$.

Thus, we see that Theorem 4.3 computed the class number $h(\mathcal{O}) = h_1(p, 1)$. Analogous to Theorem 4.5, we also introduce a version of the class number that is weighted by automorphisms.

Definition 4.13. The *mass* $M_n(d_1, d_2)$ of $[\mathcal{L}_n(d_1, d_2)]$ is

$$M_n(d_1, d_2) = \text{Mass}([\mathcal{L}_n(d_1, d_2)]) := \sum_{L \in [\mathcal{L}_n(d_1, d_2)]} \frac{1}{|\text{Aut}(L)|}, \tag{19}$$

where $\text{Aut}(L) := \{\alpha \in G : L\alpha = L\}$.

Remark 4.14. We see that if $\alpha \in \text{Aut}(L)$ then $n(\alpha) = 1$ in (16), since $n(\alpha) > 0$ and also $n(\alpha) \in \mathbb{Z}^\times = \{\pm 1\}$. We could set $G^1 := \{\alpha \in G : n(\alpha) = 1\}$ and define the genus, $[\mathcal{L}_n(d_1, d_2)]$, the class number and the mass with respect to G^1 instead. It turns out that the latter three are not affected by this change, cf. [31, Lemma 2.5].

We finish this subsection by giving a brief account of known results for the class numbers and masses just defined.

After the one-dimensional results of Eichler ($n = 1$), the class numbers in the two-dimensional case ($n = 2$) were determined by Hashimoto-Ibukiyama in a series of papers from the 1980s, using an arithmetic trace formula. In [21] they compute the class number of the principal genus. In [22] and [23], they consider every other genus for $n = 2$; the former contains the statements, while the latter contains the proofs. For any genus, they first compute the mass and then the class number; generally, the mass is a more accessible quantity than the class number.

Proposition 4.15. (cf. [21, Proposition 9], attributed to Ihara) *For any $n \geq 2$, we have*

$$M_n(d,1) = \frac{\zeta(2)\cdot\zeta(4)\cdot\ldots\cdot\zeta(2n)\cdot 1!\cdot 3!\cdot\ldots\cdot(2n-1)!}{(2\pi)^{n(n+1)}}\prod_{p|d}\prod_{i=1}^{n}\left(p^i+(-1)^i\right), \tag{20}$$

where $\zeta(s)$ denotes the Riemann zeta function.

Proposition 4.16. (cf. [23, Proposition 2.3]) *For any d_1, d_2, we have*

$$M_2(d_1,d_2) = \frac{1}{2^7\cdot 3^2\cdot 5}\prod_{p|d_1}(p-1)(p^2+1)\prod_{p|d_2}(p^2-1). \tag{21}$$

In [32, § 2], Ibukiyama-Katsura-Oort determine explicit representations of lattices: The class number results of Eichler [9] imply that these are all of the form $L = \mathcal{O}^n x$ for some $x \in \mathrm{GL}_n(B)$, and Lemmas 2.3 and 2.6 of *loc. cit.* give explicit forms of x for $\mathcal{L}_n(d,1)$ for any $n \geq 2$ and for $\mathcal{L}_2(1,d)$, respectively.

In [20], Hashimoto computed the class number of the principal genus when $n = 3$ for prime discriminants $d = p$.

The class number of any genus, for any n, d_1, d_2 is currently still out of reach. However, we did find the mass in this generality, by comparing it to the mass $M_n(d,1)$ in (20) of the principal genus and computing arithmetic volumes of the automorphism groups.

Proposition 4.17. (cf. [31, Proposition 2.6]) *We have*

$$M_n(d_1,d_2) = v_n\cdot\prod_{p|d_1}L_n(p,1)\cdot\prod_{p|d_2}L_n(1,p), \tag{22}$$

where

$$v_n := \prod_{i=1}^{n}\frac{|\zeta(1-2i)|}{2}, \tag{23}$$

for each $n \geq 1$, where

$$L_n(p,1) := \prod_{i=1}^{n}(p^i+(-1)^i) \tag{24}$$

for each prime p and $n \geq 1$, and where

$$L_n(1,p) := \begin{cases}\prod_{i=1}^{c}(p^{4i-2}-1) & \text{if } n = 2c \text{ is even;}\\ \frac{(p-1)(p^{4c+2}-1)}{p^2-1}\cdot\prod_{i=1}^{c}(p^{4i-2}-1) & \text{if } n = 2c+1 \text{ is odd.}\end{cases} \tag{25}$$

4.3. **Mass formulae for superspecial abelian varieties.**
In the previous subsection, we saw how we may count certain equivalence classes of lattices, either directly to obtain the class number, or weighted by automorphisms to obtain the mass. Now, we would like to do something similar for abelian varieties over k. This will turn out to be a very closely related problem, as we have already seen several times in Section 3 (in Theorems 3.5, 3.16, 3.17 and 3.29).

In this setting, a genus corresponds to a set of isomorphism classes of abelian varieties in an isogeny class that are "everywhere locally isomorphic", i.e. that have isomorphic ℓ-adic Tate modules for all primes $\ell \neq p$ and isomorphic p-divisible groups (or equivalently, Dieudonné modules). Since k is algebraically closed, in fact any two abelian varieties of the same dimension are locally isomorphic at all $\ell \neq p$.

Now we focus again on *supersingular* abelian varieties over k, which are all inseparably isogenous. A genus of supersingular abelian varieties is nothing other than a central leaf, consisting of all abelian varieties with isomorphic p-divisible group, and $\mathcal{S}_g$ is a disjoint union of a finite number of genera.

The mass of the central leaf $\mathcal{C}(x)$ through a point $x = (X, \lambda) \in \mathcal{S}_g(k)$ is defined to be

$$\mathrm{Mass}(\mathcal{C}(x)) := \sum_{(X',\lambda')\in\mathcal{C}(x)} \frac{1}{|\mathrm{Aut}(X',\lambda')|}. \tag{26}$$

Computing these in low dimension will be the topic of Subsection 4.4.

For *superspecial* abelian varieties, we can say even more: the p-divisible group of a superspecial abelian variety of a given dimension is unique up to isomorphism. For the analogous statement for polarised abelian varieties, we proceed as follows. For each integer $0 \leq c \leq \lfloor g/2 \rfloor$, let Λ_{g,p^c} denote the set of isomorphism classes of g-dimensional polarised superspecial abelian varieties (X'_0, λ'_0) whose polarisation λ'_0 satisfies $\ker(\lambda'_0) \simeq \alpha_p^{2c}$. (Recall from Definition 1.4 that the degree of any polarisation is a square.) Then the polarised p-divisible group associated to any member in Λ_{g,p^c} is unique up to isomorphism, cf. [45, Proposition 6.1]. In particular, if $x = (X_0, \lambda_0)$ is superspecial and principally polarised, then $\mathcal{C}(x) = \Lambda_{g,1}$.

In this subsection, we will determine the mass

$$\mathrm{Mass}(\Lambda_{g,p^c}) = \sum_{(X'_0,\lambda'_0)\in\Lambda_{g,p^c}} \frac{1}{|\mathrm{Aut}(X'_0,\lambda'_0)|}$$

of Λ_{g,p^c} for any $g \geq 1$ and any $0 \leq c \leq \lfloor g/2 \rfloor$. First, we will explain the general idea of the connection between polarisations and quaternion Hermitian spaces, cf. [45, § 8.7] and [32, § 2.2].

4.3.1. *Deuring's correspondence.*
Let us first consider the case $g = 1$ again. Elliptic curves are superspecial if and only if they are supersingular, and they are canonically and uniquely principally polarised, so the set $\Lambda_{1,1}$ consists of all isomorphism classes of supersingular elliptic curves over k.

The endomorphism algebra of any supersingular elliptic curve E over k (with principal polarisation λ) is isomorphic to the definite quaternion algebra $B = Q_{p,\infty}$ ramified at p and ∞, cf. Example 1.10, and its endomorphism ring is a maximal order in $Q_{p,\infty}$. Under the isomorphism we identify the involution $x \mapsto \bar{x}$ on $Q_{p,\infty}$ with the involution

$$f \mapsto \overline{f} = \lambda^{-1} \circ f^\vee \circ \lambda$$

on $\mathrm{End}^0(E)$, where $f^\vee : E^\vee \to E^\vee$ is the dual of f (cf. Definition 1.2); this is called the *Rosati involution* relative to λ.

In 1941, Deuring (cf. [7, §10.2]) described a bijective correspondence between the ideal classes in $Q_{p,\infty}$ and isomorphism classes of supersingular elliptic curves over k, using mostly algebraic language. Using Eichler's results, he concludes the following.

Corollary 4.18. (cf. [7, § 10.3]) *The number $|\Lambda_{1,1}|$ of isomorphism classes of supersingular elliptic curves over k equals the class number $h(\mathcal{O})$, given in Corollary 4.4.*

Remark 4.19. Deuring remarked (cf. [7, p. 266]) that deriving the number of isomorphism classes of supersingular elliptic curves directly seemed to be "nicht leicht" (not easy). In 1958, Igusa proved in [33] that it was possible, by computing the number of supersingular j-invariants by algebraic methods.

Here, we will use more modern terminology to (roughly) describe Deuring's correspondence, see also e.g. [66, 7.12-7.13], [35], [43, Appendice], [31, §4].

Choose the supersingular elliptic curve E_0 defined over $\mathbb{F}_{p^2}$ as in Notation 3.2, and fix isomorphisms

$$\mathrm{End}_k^0(E_0) \simeq Q_{p,\infty}, \quad \mathrm{End}_k(E_0) \simeq \mathcal{O}.$$

For any supersingular elliptic curve E over k (including $E = E_0$), we consider the map

$$E \mapsto \mathrm{Hom}_k(E_0, E). \tag{27}$$

The right-hand side of (27) is a (left) $\mathrm{End}_k(E_0)$-ideal via pre-composition, and the (right) order of the ideal is identified with $\mathrm{End}_k(E)$. Since $\mathrm{End}_k(E_0) \simeq \mathcal{O}$ is maximal, the right order of a left $\mathcal{O}$-ideal is automatically also maximal. Moreover, taking the right orders of representatives of all left $\mathcal{O}$-ideals yields all isomorphism classes of maximal orders in $B_{p,\infty}$. Conversely, there is a map

$$I \mapsto \mathrm{Hom}(E_0, I \otimes_{\mathcal{O}} E_0) \tag{28}$$

from $\mathcal{O}$-ideals to supersingular elliptic curves. Both (27) and (28) define functors. Together they show one can go back and forth between supersingular elliptic curves and $\mathcal{O}$-ideals, in a way which implies that the number of isomorphism classes of supersingular elliptic curves equals the class number $h(\mathcal{O})$.

Remark 4.20. Waterhouse (cf. [83, Theorem 4.5]) establishes an analogous correspondence to Deuring's for finite fields, using that every ideal in a maximal order is a so-called kernel ideal. See also [41, §5.3] where the correspondence is turned into a categorical equivalence.

4.3.2. *From polarisations to quaternion Hermitian spaces.*
Deuring's correspondence has analogues in higher dimensions and for non-principal polarisations. Superspecial abelian varieties of dimension g are unique up to isomorphism, so without loss of generality they are isomorphic to E_0^g with E_0 as in Notation 3.2. Counting their isomorphism classes thus corresponds to counting the number of polarisations on E_0^g. In particular, for any $g > 1$ and $0 \leq c \leq \lfloor g/2 \rfloor$ there is a one-to-one correspondence

$$\Lambda_{g,p^c} \longleftrightarrow \{ \text{ polarisations } \mu \text{ on } E_0^g \text{ such that } \ker(\mu) \simeq \alpha_p^{2c} \}. \tag{29}$$

The polarisations on E_0^g are translated into quaternionic language by the following proposition. Note that one polarisation on E_0^g is $\lambda = \lambda_0^{\oplus g}$, where λ_0 is the canonical polarisation on E_0.

Proposition 4.21. *For $g \geq 2$, we have one-to-one correspondences*

$$\begin{aligned} \{ \textit{ polarisations } \mu \textit{ on } E_0^g \}/\simeq &\longleftrightarrow \{ f \in M_g(\mathcal{O}) : f = \overline{f}^\vee \textit{ is positive-definite} \}/\approx \\ &\longleftrightarrow \{ \textit{ left } \mathcal{O}\textit{- lattices in } B^{\oplus g} \}/\sim . \end{aligned} \tag{30}$$

Here, the first map is induced from mapping a polarisation μ on E_0^g to $\lambda^{-1} \circ \mu \in \mathrm{End}(E_0^g)$. This map restricts to equivalence classes: on the left hand side of (30)*, polarisations are equivalent if they differ up to an automorphism of E_0^g and on the right hand side, $f \approx f'$ are equivalent if there exists $k \in \mathrm{GL}_g(\mathcal{O})$ such that $\overline{k}^\vee \circ f \circ k = f'$. The second map is given by $f \mapsto \mathcal{O}f\overline{f}^\vee$; here, equivalence $\sim$ of $\mathcal{O}$-lattices is as in Definition 4.8.*

Equation (29) implies that to conclude anything about Λ_{g,p^c}, we need to show how the correspondences in Proposition 4.21 keep track of the kernels of the polarisations. This is done in [45, Theorem 8.7], which says that a polarisation uniquely determines a genus of $\mathcal{O}$-lattices, and conversely, that a genus uniquely determines the polarisation through its kernel (equipped with a quasi-polarisation, i.e. a map between the kernel and its Cartier dual).

In particular, we obtain that the genus corresponding to principal polarisations is the principal genus $\mathcal{L}_g(p,1)$ (cf. [32, Theorem 2.10]) and that the genus corresponding to polarisations with maximal kernel $\simeq \alpha_p^{2\lfloor g/2 \rfloor}$ is the non-principal genus $\mathcal{L}_g(1,p)$ (cf. [45, § 4.6–4.8], see also [32, Theorem 2.15] for the case $g = 2$). We will confirm this below in Remark 4.24.

More generally, for any genus we have a double coset description, analogous to that for quaternion Hermitian lattices in Lemma 4.11. To state it, recall the definition of the group of similitudes

$$G = \{\alpha \in \mathrm{GL}_n(Q_{p,\infty}) : \alpha\overline{\alpha}^t = n(\alpha)\mathbb{I}_n,\ n(\alpha) \in \mathbb{Q}^\times\}.$$

from (17), and that of $G^1 = \{\alpha \in G : n(\alpha) = 1\}$ from Remark 4.14. For any $x_0 = (X_0, \lambda_0)$ in Λ_{g,p^c}, we now define the group scheme G_{x_0} over $\mathbb{Z}$ so that its group of R-valued points for any commutative ring R is

$$G_{x_0}(R) = \{\alpha \in (\mathrm{End}(X_0) \otimes_{\mathbb{Z}} R)^\times : \alpha^t \lambda_0 \alpha = \lambda_0\}. \tag{31}$$

Then $G_{x_0} \otimes \mathbb{Q}$ does not depend on our choice of variety (X_0, λ_0), since any two are isogenous, so we may choose $(X_0, \lambda_0) = (E_0^g, \lambda_{E_0}^{\oplus g})$ where λ_{E_0} is the canonical polarisation on E_0, and deduce that moreover $G_{x_0} \otimes \mathbb{Q} \simeq G^1$. We slightly abusively view $U_{g,p^c} := G_{x_0}(\widehat{\mathbb{Z}})$ as an open compact subgroup of both $G_{x_0}(\mathbb{A}_f)$ and of the isomorphic group $G^1(\mathbb{A}_f)$.

Lemma 4.22. (cf. [87, Theorem 2.1]) *Fix any $x_0 = (X_0, \lambda_0)$ in Λ_{g,p^c} and define G_{x_0} as in* (31) *and U_{g,p^c} as above. Then there is a natural bijection of pointed sets, mapping (X_0, λ_0) to the trivial element:*

$$\Lambda_{g,p^c} \simeq G_{x_0}(\mathbb{Q})\backslash G_{x_0}(\mathbb{A}_f)/G_{x_0}(\mathbb{Z}) \simeq G^1(\mathbb{Q})\backslash G^1(\mathbb{A}_f)/U_{g,p^c}. \tag{32}$$

4.3.3. *Mass computations.*
The correspondences in (29) and Proposition 4.21 enable the computation of the mass, if not the class number, of Λ_{g,p^c} in general, by using the results for masses and class numbers of quaternion Hermitian spaces. Let us summarise the main results in the literature.

In dimension $g = 2$, similar to Igusa's result [33], Katsura-Oort counted the isomorphism classes of superspecial principally polarised abelian surfaces over k in [38] using geometric methods (exploiting that these surfaces are all Jacobians) to confirm the results of Hashimoto-Ibukiyama in [21].

For principally polarised superspecial abelian varieties of any dimension g, Ekedahl determined $\mathrm{Mass}(\Lambda_{g,1})$ as a direct result of the computation of $M_n(d,1)$ in Proposition 4.15, and separately computed a mass formula for the set of superspecial varieties with *indecomposable* principal polarisation in [10, Theorem 7.2].

For non-principally polarised superspecial abelian varieties, Yu gave a mass formula for the case $c = \lfloor g/2 \rfloor$, cf. [88, Theorem 6.6]. Finally, Harashita provided the formula for general $0 \le c \le \lfloor g/2 \rfloor$ by applying to G a mass formula for certain algebraic groups due to Prasad [70]; using the functional equation for $\zeta(s)$, we can write it as follows, cf. [31, Theorem 3.1].

Theorem 4.23. (cf. [18, Proposition 3.5.4]) *For any $g \ge 1$ and $0 \le c \le \lfloor g/2 \rfloor$, we have*

$$\mathrm{Mass}(\Lambda_{g,p^c}) = v_g \cdot L_{g,p^c},$$

where v_g is as defined in Equation (23)*, and where*

$$L_{g,p^c} = \prod_{i=1}^{g-2c} (p^i + (-1)^i) \cdot \prod_{i=1}^{c} (p^{4i-2} - 1) \cdot \frac{\prod_{i=1}^{g} (p^{2i} - 1)}{\prod_{i=1}^{2c} (p^{2i} - 1) \prod_{i=1}^{g-2c} (p^{2i} - 1)}. \tag{33}$$

Remark 4.24. Comparing Equation (33) with Equations (24) and (25), we see that $L_{g,p^0} = L_g(p,1)$ and that for $c = \lfloor g/2 \rfloor$,

$$L_{g,p^c} = \begin{cases} \prod_{i=1}^{c} (p^{4i-2} - 1) & \text{if } g = 2c \text{ is even;} \\ \frac{(p-1)(p^{4c+2}-1)}{p^2-1} \cdot \prod_{1=1}^{c} (p^{4i-2} - 1) & \text{if } g = 2c+1 \text{ is odd,} \end{cases} \tag{34}$$

so that $L_{g,p^c} = L_g(1,p)$. That is, the extremal values 0 and $\lfloor g/2 \rfloor$ of c correspond to the mass of the principal and non-principal mass, respectively. On the other hand, the values $0 < c < \lfloor g/2 \rfloor$ have no direct interpretation in terms of quaternion Hermitian spaces; in the next subsection we will see how they are still related through minimal isogenies.

Remark 4.25. With the notation as above, the functor $\mathrm{Hom}(E_0, -)$ induces an equivalence between the category of fractionally polarised superspecial abelian varieties over k and the category of positive-definite Hermitian right $\mathcal{O}$-lattices (cf. [31, Corollary 4.9], see also [66, 7.12–7.14] for an integral statement). So, also in this sense, "superspecial abelian varieties are directly determined by Hermitian lattices".

4.4. **Minimal isogenies and mass formulae for supersingular varieties.**
The previous subsection showed how to compute masses, and in some cases class numbers, of *superspecial* abelian varieties, by linking them to lattices in quaternion Hermitian spaces. In this subsection, we will discuss how to compute masses, and in some cases class numbers, for *supersingular* abelian varieties. That is, we aim to compute the mass $\mathrm{Mass}(\mathcal{C}(x))$ (cf. (26)) of the central leaf passing through any supersingular variety $x = (X, \lambda) \in \mathcal{S}_g(k)$, and ultimately the cardinality $|\mathcal{C}(x)|$.

These computations are sometimes enabled by the existence of *minimal isogenies.* That is, we exploit the fact that any supersingular abelian variety is ("minimally") isogenous to a unique, possibly non-principally polarised, superspecial abelian variety. The minimal isogeny then allows us to compare the mass of the supersingular variety x with that of a superspecial one, by comparing $\mathrm{Mass}(\mathcal{C}(x))$ with a suitable $\mathrm{Mass}(\Lambda_{g,p^c})$.

Until now, masses of supersingular abelian varieties have only been explicitly computed for surfaces [30,91] and threefolds [36]; in these cases, the comparison factors between supersingular and superspecial masses have been worked out explicitly using Dieudonné module computations. We will present these results in Subsections 4.4.2 and 4.4.3, after explaining the general theoretical idea in Subsection 4.4.1.

4.4.1. *Minimal isogenies.*
The following lemma defines minimal isogenies of supersingular abelian varieties through their universal (minimality) property.

Lemma 4.26. *Let X be a supersingular abelian variety over k. Then there exists a pair $(\widetilde{X}, \varphi)$, where $\widetilde{X}$ is a superspecial abelian variety and $\varphi : \widetilde{X} \to X$ is an isogeny such that for any pair $(\widetilde{X}', \varphi')$ as above there exists a unique isogeny $\rho : \widetilde{X}' \to \widetilde{X}$ such that $\varphi' = \varphi \circ \rho$.*

Proof. See [45, Lemma 1.8], though its proof contains a gap, as pointed out in [36, Remark 3.17]. See also [89, Corollary 4.3] for an independent proof. □

Definition 4.27. Let X be a supersingular abelian variety over k. We call the isogeny $\varphi : \widetilde{X} \to X$ of Lemma 4.26 *the minimal isogeny* of X.

Remark 4.28. There is the following dual notion, sometimes also called the minimal isogeny: for any X as above, there exists a pair (Z, γ), where Z is a superspecial abelian variety and $\gamma : X \to Z$ is an isogeny such that for any other pair (Z', γ') there exists a unique isogeny $\rho : Z \to Z'$ such that $\gamma' = \rho \circ \gamma$. We will not use this in this course.

When $x = (X, \lambda)$ is a (principally) polarised supersingular variety with minimal isogeny $\varphi : \widetilde{X} \to X$, we may consider the (not necessarily principally) polarised superspecial variety $\widetilde{x} = (\widetilde{X}, \widetilde{\lambda})$ where $\widetilde{\lambda} = \varphi^*\lambda$ is the pullback of the polarisation on X.

Recall from Lemma 4.22 that for any $0 \leq c \leq \lfloor g/2 \rfloor$ we have a double coset description

$$\Lambda_{g,p^c} \simeq G_{\widetilde{x}}(\mathbb{Q}) \backslash G_{\widetilde{x}}(\mathbb{A}_f)/G_{\widetilde{x}}(\widehat{\mathbb{Z}}) \simeq G^1(\mathbb{Q}) \backslash G^1(\mathbb{A}_f)/U_{g,p^c}, \tag{35}$$

where the group scheme $G_{\widetilde{x}}/\mathbb{Z}$ satisfies

$$G_{\widetilde{x}}(R) = \{\alpha \in (\mathrm{End}(\widetilde{X}) \otimes_{\mathbb{Z}} R)^\times : \alpha^t \widetilde{\lambda} \alpha = \widetilde{\lambda}\}$$

for any commutative ring R, and where we fix an isomorphism $G_{\widetilde{x}} \otimes \mathbb{Q} \simeq G^1$. Analogously defining the group scheme G_x for $x = (X, \lambda)$, fixing an isomorphism $G_x \otimes \mathbb{Q} \simeq G^1$, and considering the open compact subgroup $U_x = G_x(\widehat{\mathbb{Z}})$ also as an open compact subgroup of $G^1(\mathbb{A}_f)$, a similar double coset description also holds for the central leaf $\mathcal{C}(x)$ of the variety x, cf. [90, Theorems 2.2 and 4.6]:

$$\mathcal{C}(x) \simeq G^1(\mathbb{Q}) \backslash G^1(\mathbb{A}_f)/U_x. \tag{36}$$

Lemma 4.29. (cf. [31, Lemma 5.2]) *For every point $x \in \mathcal{S}_g(k)$, there exists a (non-canonical) surjective morphism*

$$\pi : \mathcal{C}(x) \twoheadrightarrow \Lambda_{g,p^c}$$

for some integer $0 \leq c \leq \lfloor g/2 \rfloor$. Moreover, we can choose a base point x_c in Λ_{g,p^c} so that $G_x(\mathbb{Z}_p)$ is contained in $G_{x_c}(\mathbb{Z}_p)$ and π is induced from the identity map

$$G^1(\mathbb{Q}) \backslash G^1(\mathbb{A}_f)/U_x \longrightarrow G^1(\mathbb{Q}) \backslash G^1(\mathbb{A}_f)/U_{x_c}, \tag{37}$$

where $U_{x_c} \simeq G_{x_c}(\widehat{\mathbb{Z}})$.

Remark 4.30. Since any two supersingular abelian varieties $x = (X, \lambda)$ and $x' = (X', \lambda')$ have isomorphic ℓ-adic Tate modules at all primes $\ell \neq p$, the corresponding groups $G_x(\prod_{\ell \neq p} \mathbb{Z}_\ell)$ and $G_{x'}(\prod_{\ell \neq p} \mathbb{Z}_\ell)$ are conjugate inside $G^1(\mathbb{A}_f^p)$, where $\mathbb{A}_f^p$ denotes the prime-to-p adeles. That is, "the corresponding groups $G_x(\widehat{\mathbb{Z}})$ and $G_{x'}(\widehat{\mathbb{Z}})$ only differ at p".

This observation also explains why in the statement of Lemma 4.29 we are comparing the groups $G_x(\mathbb{Z}_p)$ and $G_{x_c}(\mathbb{Z}_p)$ at p, while in Equation (37) we see the adelic groups $U_x \simeq G_x(\widehat{\mathbb{Z}})$ and $U_{x_c} \simeq G_{x_c}(\widehat{\mathbb{Z}})$.

Moreover, by Tate's theorem at p, cf. Theorem 1.28, at p we have that $G_x(\mathbb{Z}_p) \simeq \mathrm{Aut}((X, \lambda)[p^\infty])$ is isomorphic to the automorphism group of the p-divisible group.

The existence of the surjection $\pi : \mathcal{C}(x) \twoheadrightarrow \Lambda_{g,p^c}$ in Lemma 4.29 follows from abstract results about the algebraic group G^1; however, its relation with the minimal isogeny can be seen as follows. Let $\varphi : \widetilde{x} = (\widetilde{X}, \widetilde{\lambda}) \to x = (X, \lambda)$ the be minimal isogeny for x and pick $0 \le c \le \lfloor g/2 \rfloor$ such that that $\widetilde{x} \in \Lambda_{g,p^c}$. Then $U_x \subseteq U_{\widetilde{x}} := G_{\widetilde{x}}(\widehat{\mathbb{Z}})$. Further, viewing all groups inside $G^1(\mathbb{A}_f)$, we see from (35) and (36) that the natural map

$$G^1(\mathbb{Q}) \backslash G^1(\mathbb{A}_f)/U_x \longrightarrow G^1(\mathbb{Q}) \backslash G^1(\mathbb{A}_f)/U_{\widetilde{x}} \tag{38}$$

induces a surjection $\mathcal{C}(x) \twoheadrightarrow \Lambda_{g,p^c}$.

If the open compact subgroup $U_{\widetilde{x}}$ is maximal, then $U_{\widetilde{x}}$ is conjugate to U_{g,p^c} for some $0 \le c \le \lfloor g/2 \rfloor$ and the map $\pi : \Lambda_x \twoheadrightarrow \Lambda_{g,p^c}$ in Lemma 4.29 is realised by the minimal isogeny φ. Maximality holds for $g \le 4$, so in small dimensions, we may use Lemma 4.29 to compare supersingular masses to superspecial masses. In general, this comparison is achieved using minimal isogenies via the following proposition.

Proposition 4.31. (cf. [36, Proposition 2.12]) *The minimal isogeny $\varphi : \widetilde{x} = (\widetilde{X}, \widetilde{\lambda}) \to x = (X, \lambda)$ induces an injective map $\varphi^* : \mathrm{End}(X[p^\infty]) \hookrightarrow \mathrm{End}(\widetilde{X}[p^\infty])$, and if $U_{\widetilde{x}}$ is conjugate to U_{g,p^c} for some $0 \le c \le \lfloor g/2 \rfloor$, then we have*

$$\mathrm{Mass}(\mathcal{C}(x)) = [\mathrm{Aut}((\tilde{X}, \tilde{\lambda})[p^\infty]) : \mathrm{Aut}((X, \lambda)[p^\infty])] \cdot \mathrm{Mass}(\Lambda_{g,p^c}). \tag{39}$$

Proof. The injectivity of φ^* follows since every endomorphism of $X[p^\infty]$ lifts uniquely to an endomorphism of $\tilde{X}[p^\infty]$ by [89, Proposition 4.8]. The comparison factor can be seen to equal

$$\frac{[U_{\widetilde{x}} : U_{\widetilde{x}} \cap U_x]}{[U_x : U_{\widetilde{x}} \cap U_x]},$$

cf. Remark 4.30. $\square$

In conclusion, to compute the mass of (the central leaf of) a supersingular principally polarised abelian variety $x = (X, \lambda)$, we first need to find a suitable surjection $\mathcal{C}(x) \twoheadrightarrow \Lambda_{g,p^c}$ for some $0 \le c \le \lfloor g/2 \rfloor$, which exists by Lemma 4.29, and is in some cases induced from the minimal isogeny of (X, λ). If so, then we determine $\mathrm{Mass}(\Lambda_{g,p^c})$ using Theorem 4.23, and the comparison factor $[\mathrm{Aut}((\tilde{X}, \tilde{\lambda})[p^\infty]) : \mathrm{Aut}((X, \lambda)[p^\infty])]$ from Proposition 4.31, to compute $\mathrm{Mass}(\mathcal{C}(x))$.

4.4.2. *Supersingular abelian surfaces.*
Let $x = (X, \lambda)$ be a principally polarised supersingular abelian surface over k. If X is superspecial, then $\mathcal{C}(x) = \Lambda_{2,p^0}$ so we know its mass by Theorem 4.23 with $c = 0$ and its class number $|\Lambda_{2,p^0}|$ by Proposition 4.15 with $n = 2$, $d = p$ (or equivalently, by Proposition 4.16, with $d_1 = p$, $d_2 = 1$).

Assume then that X is not superspecial, so it has $a(X) = 1$. The latter implies that that there exists a unique PFTQ lying above (X, λ); cf. Example 3.10. That is, there is a unique (up to isomorphism) polarised superspecial abelian surface (Y_1, λ_1) such that $\ker(\lambda_1) \simeq \alpha_p^2$ and an isogeny $\phi : (Y_1, \lambda_1) \to (X, \lambda)$ of degree p that is compatible with polarisations. There is also a unique polarisation μ_1 on E_0^2 such that $\ker(\mu_1) \simeq \alpha_p^2$ and for which $(Y_1, \lambda_1) \simeq (E_0^2, \mu_1) \otimes_{\mathbb{F}_{p^2}} k$. Let t in $\mathbb{P}^1(k) = \mathbb{P}^1_{\mu_1}(k) := \{\phi_1 : (E^2, \mu_1) \otimes k \to (X, \lambda) \text{ an isogeny of degree } p\}$ be the Moret-Bailly parameter for (X, λ).

The condition $a(X) = 1$ moreover implies that $t \in \mathbb{P}^1(k) \setminus \mathbb{P}^1(\mathbb{F}_{p^2}) = k \setminus \mathbb{F}_{p^2}$. We distinguish two different cases: in the first case (I) we have $t \in k \setminus \mathbb{F}_{p^4}$, and in the second case (II) we have $t \in \mathbb{F}_{p^4} \setminus \mathbb{F}_{p^2}$. Roughly speaking, these cases correspond to the structure of $\mathrm{End}(X)$ in the sense that a larger field of definition of t yields a smaller endomorphism ring.

The following results respectively give the class number $|\mathcal{C}(x)|$ and mass $\mathrm{Mass}(\mathcal{C}(x))$ in each case.

Theorem 4.32. (cf. [30, Theorems 1.1 and 3.6]) *Let $x = (X, \lambda)$ be a principally polarised supersingular abelian surface over k with $a(X) = 1$ and Moret-Bailly parameter t, and let $h = |\mathcal{C}(x)|$ be the corresponding class number.*

(1) *In Case (I), i.e. when $t \in k \setminus \mathbb{F}_{p^4}$, we have*

$$h = \begin{cases} 1 & \text{if } p = 2; \\ \frac{p^2(p^4-1)(p^2-1)}{5760} & \text{if } p \geq 3. \end{cases}$$

(2) *In Case (II), i.e. when $t \in \mathbb{F}_{p^4} \setminus \mathbb{F}_{p^2}$, we have*

$$h = \begin{cases} 1 & \text{if } p = 2; \\ \frac{p^2(p^2-1)^2}{2880} & \text{if } p \equiv \pm 1 \bmod 5 \text{ or } p = 5; \\ 1 + \frac{(p-3)(p+3)(p^2-3p+8)(p^2+3p+8)}{2880} & \text{if } p \equiv \pm 2 \bmod 5. \end{cases}$$

(3) *For each case, we have $h = 1$ if and only if $p = 2, 3$.*

Theorem 4.33. (cf. [91, Theorem 1.1]), [30, Proposition 3.3]) *Let $x = (X, \lambda)$ and $t \in \mathbb{P}^1(k)$ be as in Theorem 4.32. Then*

$$\mathrm{Mass}(\mathcal{C}(x)) = \frac{L_p}{5760}, \tag{40}$$

where

$$L_p = \begin{cases} 2^{-e(p)}(p^4 - 1)(p^4 - p^2) & \text{if } t \in k \setminus \mathbb{F}_{p^4} & \text{(Case (I))}; \\ (p^2 - 1)(p^4 - p^2), & \text{if } t \in \mathbb{F}_{p^4} \setminus \mathbb{F}_{p^2} & \text{(Case (II))}, \end{cases}$$

with $e(p) = 0$ if $p = 2$ and $e(p) = 1$ if $p > 2$.

By combining Theorems 4.32 and 4.33, we can derive quite precise information about the automorphism groups of the supersingular surfaces, as the next result demonstrates.

Corollary 4.34. *Let $p = 2$, and let $x' = (X', \lambda')$ be a principally polarised supersingular abelian surface over k with $a(X') = 1$. Let $\phi_1 : (E_0^2 \otimes k, \mu_1) \to (X', \lambda')$ be the isogeny yielding a Moret-Bailly parameter $t \in k \setminus \mathbb{F}_{p^2}$, where μ_1 is a polarisation on E^2 such that $\ker(\mu_1) \simeq \alpha_p^2$. Then*

$$|\mathrm{Aut}(X', \lambda')| = \begin{cases} 32 & \text{if } t \in k \setminus \mathbb{F}_{p^4} & \text{(Case (I))}; \\ 160, & \text{if } t \in \mathbb{F}_{p^4} \setminus \mathbb{F}_{p^2} & \text{(Case (II))}. \end{cases} \tag{41}$$

Proof. By Theorem 4.32, we have $|\mathcal{C}(x')| = 1$ in both cases. Then Theorem 4.33 for $p = 2$ yields

$$\mathrm{Mass}(\mathcal{C}(x')) = \begin{cases} 1/32 & \text{if } t \in k \setminus \mathbb{F}_{p^4} & \text{(Case (I))}; \\ 1/160, & \text{if } t \in \mathbb{F}_{p^4} \setminus \mathbb{F}_{p^2} & \text{(Case (II))}. \end{cases}$$

$\square$

4.4.3. *Supersingular abelian threefolds.*
Recall the description of $\mathcal{P}_{3,\mu}$ from Example 3.13 via the truncation map π as a $\mathbb{P}^1$-bundle over the Fermat curve $C : X_1^{p+1} + X_2^{p+1} + X_3^{p+1} = 0$, independent of the choice of μ.

Recall also that we defined the a-number on points $y \in \mathcal{P}_{3,\mu}$ via $a(y) := a(\mathrm{pr}_0(y))$ and described the a-number loci in Example 3.18. As in that example, we will write a point $y \in \mathcal{P}_{3,\mu}(k)$ as a pair (t, u), where $t = \pi(y)$ is a point on C and where $u \in \pi^{-1}(t) =: \mathbb{P}^1_t(k)$ is a point on the projective line above it.

The mass calculation will depend on the a-number, since the a-number of a supersingular principally polarised abelian threefold (X, λ) tells us how to derive its minimal isogeny from the PFTQ lying over it, [36, Proposition 3.16]. If $a(X) = 3$, then X is superspecial already, so the minimal isogeny is the identiy. On the other extreme, if $a(X) = 1$, then the PFTQ $(Y_2, \lambda_2) \to (Y_1, \lambda_1) \to (Y_0, \lambda_0) = (X, \lambda)$ itself is the minimal isogeny. And if $a(X) = 2$, then the minimal isogeny is $(Y_1, \lambda_1) \to (X, \lambda)$. In particular, then the minimal isogeny is of degree p and $\ker(\lambda_1) \simeq \alpha_p^2$, so that this case can be compared to the surface case from Subsection 4.4.2.

We define the mass of a point $y = (t, u)$ in $\mathcal{P}_{3,\mu}(k)$ by setting $\mathrm{Mass}(y) = \mathrm{Mass}(\mathcal{C}(x))$ for $x = \mathrm{pr}_0(y)$. In [36] we determined the mass for any $y \in \mathcal{P}_{3,\mu}(k)$; the following theorems summarise the main results.

Theorem 4.35. (cf. [36, Theorem A]) *Let $y = (t, u) \in \mathcal{P}_{3,\mu}(k)$ be a point such that $t \in C(\mathbb{F}_{p^2})$; then $a(y) \geq 2$ by Example 3.18. Then we have*

$$\mathrm{Mass}(y) = \frac{L_p}{2^{10} \cdot 3^4 \cdot 5 \cdot 7},$$

where

$$L_p = \begin{cases} (p-1)(p^2+1)(p^3-1) & \text{if } u \in \mathbb{P}^1_t(\mathbb{F}_{p^2}); \\ (p-1)(p^3+1)(p^3-1)(p^4-p^2) & \text{if } u \in \mathbb{P}^1_t(\mathbb{F}_{p^4}) \setminus \mathbb{P}^1_t(\mathbb{F}_{p^2}); \\ 2^{-e(p)}(p-1)(p^3+1)(p^3-1)p^2(p^4-1) & \text{if } u \notin \mathbb{P}^1_t(\mathbb{F}_{p^4}), \end{cases}$$

where $e(p) = 0$ if $p = 2$ and $e(p) = 1$ if $p > 2$.

Theorem 4.35 gives the mass formula for points with a-number greater than or equal to 2. To describe the mass of points with a-number 1, we need to construct an auxiliary divisor $\mathcal{D} \subseteq \mathcal{P}'_{3,\mu}$, cf. [36, Definition 5.16], and a function $d : C(k) \setminus C(\mathbb{F}_{p^2}) \to \{3, 4, 5, 6\}$, cf. [36, Definition 5.12]. In [36, Proposition 5.13] it is shown how the value of this function is related to the field of definition of the parameter t; roughly speaking, the larger the field of definition, the higher the value of d. Further, the function d is surjective when $p \neq 2$, and it only takes value 3 when $p = 2$. On the other hand, the divisor $\mathcal{D}$ encodes information about both parameters t and u.

Using this terminology, we have the following result.

Theorem 4.36. (cf. [36, Theorem B]) *Let $y = (t, u) \in \mathcal{P}'_{3,\mu}(k)$ be a point such that $t \notin C(\mathbb{F}_{p^2})$; then $a(y) = 1$ by Example 3.18. Then we have*

$$\mathrm{Mass}(y) = \frac{p^3 L_p}{2^{10} \cdot 3^4 \cdot 5 \cdot 7},$$

where

$$L_p = \begin{cases} 2^{-e(p)} p^{2d(t)}(p^2-1)(p^4-1)(p^6-1) & \text{if } y \notin \mathcal{D}; \\ p^{2d(t)}(p-1)(p^4-1)(p^6-1) & \text{if } t \notin C(\mathbb{F}_{p^6}) \text{ and } y \in \mathcal{D}; \\ p^6(p^2-1)(p^3-1)(p^4-1) & \text{if } t \in C(\mathbb{F}_{p^6}) \text{ and } y \in \mathcal{D}, \end{cases}$$

where again $e(p) = 0$ if $p = 2$ and $e(p) = 1$ if $p > 2$.

As in the two-dimensional setting, in some cases we obtain precise information about the automorphism groups of the threefolds, this time by considering reductions of endomorphism rings (modulo a uniformiser of the maximal order of the quaternion division $\mathbb{Q}_p$-algebra). So rather than finding the automorphism groups from the combination of masses and class numbers, we now combine our knowledge of the mass and the automorphism groups in these cases to obtain the class number. The results for the generic case are given below.

Theorem 4.37. (cf. [36, Theorem 6.4]) *Let $x = (X, \lambda)$ be a supersingular principally polarised abelian threefold with $a(X) = 1$, whose associated PFTQ is described by parameters $(t, u) \notin \mathcal{D}$.*

(1) *If $p = 2$, then* $\mathrm{Aut}(X, \lambda) \simeq C_2^3$.
(2) *If $p \geq 5$, or $p = 3$ and $d(t) = 6$, then* $\mathrm{Aut}(X, \lambda) \simeq C_2$.

Corollary 4.38. (cf. [36, Corollary 6.5]) *Under the same notation as above and assumptions as in Theorem 4.37, we have:*

(1) *If* $p = 2$, *then* $|\mathcal{C}(x)| = 4$.
(2) *If* $p = 3$ *and* $d(t) = 6$, *then* $|\mathcal{C}(x)| = 3^{11} \cdot 13$.
(3) *If* $p \geq 5$, *then*

$$|\mathcal{C}(x)| = \frac{p^{3+2d(t)}(p^2-1)(p^4-1)(p^6-1)}{2^{10} \cdot 3^4 \cdot 5 \cdot 7}.$$

5. Projects

(1) **Weil polynomials of abelian varieties over finite fields.**
In this project, you will investigate which polynomials occur as characteristic polynomials of Frobenius endomorphisms of abelian varieties over finite fields. For any abelian variety $X/\mathbb{F}_q$ of dimension g, these characteristic polynomials $h(x) \in \mathbb{Z}[x]$ have the general shape

$$h(x) = x^{2g} + a_1 x^{2g-1} + \ldots + a_{g-1}x^{g+1} + a_g x^g + a_{g-1}qx^{g-1} + \ldots + a_1 q^{g-1}x + q^g.$$

(a) Suppose that $g = 6$ or 7. What are conditions on the coefficients $a_i \in \mathbb{Z}$ such that all roots of $h(x)$ are Weil numbers, i.e. such that the roots have absolute value $\sqrt{q}$ under all complex embeddings?
(This implies that $h(x)$ is a Weil q-polynomial, which is a necessary condition for $h(x)$ to be a characteristic polynomial of Frobenius.)
Results for dimensions 2 up to 5 can be found in [14, 15, 47, 73, 76], respectively; note that in [8] a few mistakes in [15, 76] were pointed out.

(b) Suppose now that $g = 7$. By considering the Newton polygon of $h(x)$, find the p-adic valuations of the coefficients $a_i \in \mathbb{Z}$ so that a Weil polynomial $h(x)$ appears as the characteristic polynomial of Frobenius for some abelian variety $X/\mathbb{F}_q$.
Results for dimensions 1 up to 6 can be found in the above-mentioned references and [83] ($g = 1$), [86] ($g = 3, 4$), [24] ($g = 5$), [77] ($g = 6$), cf. [25].

(c) Suppose that $g = 6$. By comparing the results of parts (a) and (b) for simple varieties of dimension $g = 6$ with those for simple varieties of dimensions $1, 2, 3$, formulate a heuristic for the proportion of simple isogeny classes among all isogeny classes over a fixed $\mathbb{F}_q$. You can use the LMFDB [46] to test this, at least in characteristic $p = 2$.
When are the (simple and non-simple) isogeny classes isogeny twists of each other in the sense of [8], i.e. when do they coincide after a finite extension of the base field $\mathbb{F}_q$?

(2) **Intersections of stratifications.**
In this project, you will investigate intersections of the various stratifications introduced in the course.

(a) We understand how Ekedahl-Oort strata refine p-rank and a-number strata. For $g = 5, 6$, work out the Ekedahl-Oort strata with fixed p-rank, keeping track of their a-numbers (or vice versa), and determine their closure relations. Tables for $g = 2, 3$ can be found in [11, Appendix], and for $g = 4$ this was carried out in [71, §4.4].

(b) Since the p-rank of a Newton stratum is the number of zero slopes, we also understand how the Newton stratification refines the p-rank stratification. For $g \leq 5$, work out the Newton strata with fixed p-rank and determine their closure relations.

(c) A significantly harder problem is to determine the intersections between Newton strata and Ekedahl-Oort strata. For $g = 3$ and 4 the intersections between the Ekedahl-Oort strata and $W_\sigma = \mathcal{S}_g$ are given in Examples 3.35 and 3.36, respectively. For $g = 5$, work out how to write the Newton stratum $\mathcal{S}_g$ as a disjoint union of Ekedahl-Oort strata $\mathcal{S}_\varphi \cap \mathcal{S}_g$ restricted to $\mathcal{S}_g$. You may do this separately for each a-number stratum of $\mathcal{S}_g$.

(d) What can you say about other Newton strata when $g = 5$, starting with those of p-rank zero?

(3) **Automorphism groups, mass and $|\mathcal{C}(x)|$.**
For any $x \in \mathcal{S}_g$, the following three things are intimately related by definition:

- (The cardinality of) the central leaf $\mathcal{C}(x)$ (cf. Definition 3.20 and Equation (36) for the definition of the central leaf);

- The mass $\mathrm{Mass}(\mathcal{C}(x))$, cf. (26);
- The automorphism groups $\mathrm{Aut}(X, \lambda)$ of the principally polarised abelian varieties (X, λ) corresponding to the points in $\mathcal{C}(x)$.

In this project, you will work on obtaining information on one of these three objects from the others. Some possible questions along these lines are:

(a) For $g = 2$ and any p: determine the cardinalities of the central leaves, using known results on mass formulae for, and automorphism groups of, supersingular abelian surfaces, cf. [29, 30, 32, 37, 52, 91].

(b) For $g = 3$ and $p > 2$: determine the automorphism groups of supersingular abelian threefolds, and hence the cardinalities of the central leaves, using mass formulae and some known cases from [31, 36]. Already determining automorphism groups of *superspecial* threefolds from their mass formulae and class number formulae (cf. [10, 21]) would be interesting.

(c) For $g = 4$ and $p > 2$: determine automorphism groups of superspecial abelian fourfolds, using known results on mass formulae and class number formulae (cf. [10,21]). When only a mass formula is known, you could try to give a finite list of possible automorphism groups and cardinalities of central leaves.

(4) **Mass functions on $\mathcal{A}_g$.**

In this project, you will work on extending the notion of mass from $\mathcal{S}_g$ to $\mathcal{A}_g$.

For a moduli point $x_0 \in \mathcal{S}_g$ the mass (of its central leaf) was defined in Equation (26) of the notes. The sum is taken over the points in the central leaf of x_0, cf. Definition 3.20, which is finite if and only if the underlying abelian variety is supersingular.

For a possibly non-supersingular point $x_0 \in \mathcal{A}_g$ corresponding to a principally polarised abelian variety (X_0, λ_0), we can remedy this by considering only principally polarised varieties (X, λ) (quasi-)isogenous to (X_0, λ_0), denoted $(X, \lambda) \sim (X_0, \lambda_0)$. That is, we consider instead

$$\mathcal{C}'(x_0) = \{(X, \lambda) = x \in \mathcal{A}_g : (X, \lambda) \sim (X_0, \lambda_0), X[p^\infty] \simeq X_0[p^\infty]]\}.$$

Then it follows from the results in [67] that $\mathcal{C}'(x)$ is again finite. (Recall that any two principally polarised supersingular abelian varieties over k are isogenous.)

Can we find mass formulae for $\mathcal{A}_g$, starting with low g, making use of its stratifications and foliation structure or of explicit geometric families?

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