GEOMETRY AND ARITHMETIC OF MODULI SPACES OF ABELIAN VARIETIES IN POSITIVE CHARACTERISTIC

VALENTIJN KAREMAKER

Abelian varieties are important objects in arithmetic geometry. When studying their rational points, we can make use of the fact that they are group varieties. That is, the rational points over a fixed field form a group, which provides us with useful extra structure. In this course, we will consider abelian varieties over fields of positieve characteristic p, and study geometric and arithmetic properties of their moduli spaces.

Abelian varieties over finite fields have been intensively studied, both for their inherent theoretical interest and for their applications in cryptography and coding theory. Let $\mathbb{F}_q$ be a finite field of characteristic p, so $q = p^r$ for some r. Every abelian variety A over $\mathbb{F}_q$ admits a *(relative) Frobenius endomorphism*, often denoted π. It also acts on the ℓ^n-torsion of A, denoted $A[\ell^n]$, for any $\ell \neq p$ and $n \geq 1$, and hence on its *Tate module* $T_\ell(A) = \varprojlim_n A[\ell^n]$ and on $V_\ell(A) = T_\ell(A) \otimes_{\mathbb{Z}_\ell} \mathbb{Q}_\ell$; as such it has a characteristic polynomial, which turns out to be independent of $\ell (\neq p)$ and is a so-called *Weil polynomial*.

The characteristic polynomial of π captures a lot of arithmetic information about A. For example, by a theorem of Tate [29, Theorem 1], it determines the *isogeny class* of A; two abelian varieties A_1, A_2 are isogenous if there exists a surjective morphism between them with finite kernel. Isogeny is an equivalence relation. It is a weaker notion than isomorphism; indeed, a lot of current research is concerned with determining which and how many isomorphism classes a given isogeny class contains.

For any $n \geq 1$, we may also consider the p^n-torsion $A[p^n]$. The direct limit $A[p^\infty] = \varinjlim_n A_{p^n}$ of the corresponding finite group schemes is the *p-divisible group* of A. While for any $\ell \neq p$ the Tate module satisfies $T_\ell(A) \simeq \mathbb{Z}_\ell^{2g}$ where $g = \dim(A)$, the p-divisible group is much more mysterious. There is however a useful anti-equivalence between p-divisible groups and so-called *Dieudonné modules* which we will also make use of in this course.

The p-torsion in characteristic p is also used for the following important classification: An elliptic curve E that has trivial p-torsion is called *supersingular*; otherwise it is called *ordinary*. A higher-dimensional abelian variety is supersingular if it is geometrically (i.e. over $\overline{\mathbb{F}}_p$) isogenous to a product of supersingular elliptic curves; it is *superspecial* if it is geometrically isomorphic to such a product. The *p-rank* of $A/\mathbb{F}_q$, denoted $f(A)$, is the integer such that $|A[p]| = p^f$. Thus $0 \leq f(A) \leq g = \dim(A)$ and supersingular abelian varieties have p-rank zero, but the converse of the latter only holds for $g \leq 2$. When $f(A) = g$, the variety is said to be ordinary.

Let $k = \overline{\mathbb{F}}_p$. We will consider the moduli space $\mathcal{A}_g$ of principally polarised g-dimensional abelian varieties over k. The locus of supersingular abelian varieties is denoted $\mathcal{S}_g$. It is of interest to study stratifications of these spaces, and especially to determine the number and (co)dimension of the strata.

We will introduce several stratifications, each determined by an invariant of the varieties, in the sense that each stratum consists of all varieties with the same value for that invariant:

- The *p-rank stratification*, where the p-rank is as defined above.
- The *Newton (polygon) stratification*: the geometric isogeny class of the p-divisible group $A[p^\infty]$ uniquely determines a Newton polygon. Importantly, all supersingular abelian varieties have isogenous p-divisible groups, so $\mathcal{S}_g$ is a Newton stratum. Since the p-rank is also the number of zero slopes in the Newton polygon, the Newton stratification refines the p-rank stratification.
- The *a-number stratification*: the a-number $a(A) := \dim(\mathrm{Hom}(\alpha_p, A))$ is an isomorphism invariant of $A[p]$; here α_p is a local finite additive group scheme. If A has dimension g and p-rank f, then $0 \leq a(A) \leq g - f$. The a-number is g if and only if A is superspecial [24].
- The *Ekedahl-Oort stratification*: we can combinatorially describe $A[p]$ in terms of symplectic group elements, or equivalently using so-called elementary sequences; this again yields an isomorphism invariant [27]. The corresponding stratification refines both the p-rank and a-number stratifications.

In the second half of this course, we will focus our attention on $\mathcal{S}_g$ and discuss for instance what happens when we restrict the stratifications mentioned above to $\mathcal{S}_g$. We will introduce a very explicit geometric description of $\mathcal{S}_g$ due to Li and Oort [17], in terms of so-called *flag type quotients*. These in particular yield a useful description of the *minimal isogeny* to our supersingular abelian variety from some superspecial abelian variety.

Another geometric structure is provided by central leaves. The *central leaf* through a moduli point $x_0 \in \mathcal{A}_g$ corresponding to a principally polarised abelian variety A_0 is

$$\mathcal{C}(x_0) := \{A = x \in \mathcal{A}_g : A[p^\infty] \simeq A_0[p^\infty]\}. \tag{1}$$

Every irreducible component of a Newton stratum, such as $\mathcal{S}_g$, is foliated by central leaves [28]. We will consider this structure particularly for $\mathcal{S}_g$, because central leaves have finite cardinality only for supersingular abelian varieties [1].

Finally, we turn towards arithmetic questions, still focussing on $\mathcal{S}_g$. Two important arithmetic invariants of algebraic varieties in general are their endomorphism ring (and algebra) and automorphism group. For example, over $\mathbb{F}_q$ the endomorphism ring of an ordinary elliptic curve is a commutative order in an imaginary quadratic number field, while that of a supersingular elliptic curve can be an order in a non-commutative quaternion algebra. For abelian varieties over finite fields, the characteristic polynomial of Frobenius again gives us a lot of information about the endomorphism ring and algebra. And a theorem by Tate [29, Main Theorem] shows that the localisation of the endomorphism ring of a variety A at a prime $\ell \neq p$ is isomorphic to the endomorphism ring of the Tate module $T_\ell(A)$.

Working over $k = \overline{\mathbb{F}}_p$, we use automorphism groups to define the notion of *mass*. For a moduli point $x_0 \in \mathcal{S}_g$, its mass is defined as

$$\mathrm{Mass}(x_0) := \sum_{A = x \in \mathcal{C}(x_0)} \frac{1}{|\mathrm{Aut}(A)|}. \tag{2}$$

We will discuss formulae for masses of supersingular and superspecial abelian varieties, explaining how minimal isogenies are crucial tools to obtain the former from the latter.

Overview of the lectures

The following is a tentative plan for the lectures to cover the material described above, including some of the main references.

- **Lecture 1**: this will be an introduction to abelian varieties over finite fields of characteristic p. In particular, we will discuss their classification up to isogeny due to Honda and Tate ([30, Théorème 2], [8, §2, Theorem 1]) and important properties determined by the characteristic polynomial of the Frobenius endomorphism (cf. [32]).

- **Lecture 2:** we will introduce the moduli space $\mathcal{A}_g$ of g-dimensional principally polarised abelian varieties over $k = \overline{\mathbb{F}}_p$ (cf. [21]). Then we will study its geometric structure by means of stratifications by invariants, in particular the p-rank stratification (cf. [15,22]), the Newton stratification (cf. [11,19,26]), the a-number stratification (cf. [4,31]) and the Ekedahl-Oort stratification (cf. [27]).

- **Lecture 3**: from now on, we specialise to the supersingular locus $\mathcal{S}_g$. In this lecture, we will study its geometry, explicitly in low dimensions (cf. e.g. [13,14,20]), and generally using flag type quotients (cf. [16,17,23]) and the foliation by central leaves (cf. [10,28]). We will also consider the restrictions of the stratifications mentioned above from $\mathcal{A}_g$ to $\mathcal{S}_g$ (cf. [2,5,6]).

- **Lecture 4:** finally, we will study the arithmetic of $\mathcal{S}_g$, focussing on the endomorphism rings and algebras and automorphism groups of the abelian varieties (cf. [25,29]). We will introduce the notion of mass and discuss mass formulae for superspecial (cf. [3,7,33]) and supersingular (cf. [9,12,34]) varieties, showing how the latter may be deduced from the former through minimal isogenies and explicit computations with Dieudonné modules.

Projects

The following are suggestions for projects; the lecture notes will contain more detailed descriptions. While building on known results for special cases, these are open problems in general, so any progress made on them will be an interesting contribution to the topic.

(1) **Weil polynomials of abelian varieties over finite fields.**
In this project, you will investigate which polynomials occur as characteristic polynomials of Frobenius endomorphisms of abelian varieties over finite fields. For any abelian variety $A/\mathbb{F}_q$ of dimension g, these characteristic polynomials $f_A(x) \in \mathbb{Z}[x]$ have the general shape

$$f_A(x) = x^{2g} + a_1 x^{2g-1} + \ldots + a_{g-1}x^{g+1} + a_g x^g + a_{g-1} q x^{g-1} + \ldots + a_1 q^{g-1} x + q^g.$$

To see when this is a characteristic polynomial of Frobenius, you need to determine first when it is a Weil polynomial, and then whether its Newton polygon is admissible. This has been worked out in dimensions $g \leq 5$ but is open for higher g. Using the LMFDB [18], you can then try to formulate and test heuristics for the distribution of various arithmetic properties of the isogeny classes.

(2) **Intersections of stratifications.**
In this project, you will investigate intersections of the various stratifications introduced in the course, in particular between Newton strata and Ekedahl-Oort strata, since this behaviour is generally not well understood.

(3) **Automorphism groups, mass and $|\mathcal{C}(x)|$.**
For any $x \in \mathcal{S}_g$, the following three things are (by definition) intimately related:
- The mass $\mathrm{Mass}(x)$, cf. (2);
- (The cardinality of) the central leaf $\mathcal{C}(x)$, cf. (1);
- The automorphism groups $\mathrm{Aut}(A)$ of the principally polarised abelian varieties A corresponding to the points in $\mathcal{C}(x)$.

In this project, you will work on obtaining information on one of these three objects from the others, also using results from the literature.

(4) **Mass functions on $\mathcal{A}_g$.**
In this project, you will work on extending the notion of mass from $\mathcal{S}_g$ to $\mathcal{A}_g$. For a moduli point $x_0 \in \mathcal{S}_g$ the mass was defined in (2) above. The sum is over the points in the central leaf, cf. (1), which is finite if and only if the underlying abelian variety is supersingular. For a possibly non-supersingular point $x_0 \in \mathcal{A}_g$ corresponding to a principally polarised abelian variety A_0, we can remedy this by considering only principally polarised varieties A (quasi-)isogenous to A_0, denoted $A \sim A_0$. That is, we consider instead

$$\mathcal{C}'(x_0) = \{A = x \in \mathcal{A}_g : A \sim A_0, A[p^\infty] \simeq A_0[p^\infty]]\}.$$

Then it follows from the results in [28] that $\mathcal{C}'(x)$ is again finite. (Note that any two principally polarised supersingular abelian varieties over k are isogenous.) Can we find mass formulae for $\mathcal{A}_g$, starting with low g, making use of its stratifications and foliation structure or of explicit geometric families?

References

1. Ching-Li Chai, *Every ordinary symplectic isogeny class in positive characteristic is dense in the moduli*, Invent. Math. **121** (1995), no. 3, pp. 439–479.
2. Ching-Li Chai and Frans Oort, *Monodromy and irreducibility of leaves*, Ann. of Math. (2) **173** (2011), no. 3, pp. 1359–1396.
3. Torsten Ekedahl, *On supersingular curves and abelian varieties*, Math. Scand. **60** (1987), no. 2, pp. 151–178.
4. Torsten Ekedahl and Gerard van der Geer, *Cycle classes of the E-O stratification on the moduli of abelian varieties*, Algebra, arithmetic, and geometry: in honor of Yu. I. Manin. Vol. I, Progr. Math., vol. 269, Birkhäuser Boston, Boston, MA, 2009, pp. 567–636.
5. Shushi Harashita, *The a-number stratification on the moduli space of supersingular abelian varieties*, J. Pure Appl. Algebra **193** (2004), no. 1-3, pp. 163–191.
6. ———, *Ekedahl-Oort strata contained in the supersingular locus and Deligne-Lusztig varieties*, J. Algebraic Geom. **19** (2010), no. 3, pp. 419–438.
7. Ki-ichiro Hashimoto and Tomoyoshi Ibukiyama, *On class numbers of positive definite binary quaternion Hermitian forms*, J. Fac. Sci. Univ. Tokyo Sect. IA Math. **27** (1980), no. 3, pp. 549–601.
8. Taira Honda, *Isogeny classes of abelian varieties over finite fields*, J. Math. Soc. Japan **20** (1968), pp. 83–95.
9. Tomoyoshi Ibukiyama, *Principal polarizations of supersingular abelian surfaces*, J. Math. Soc. Japan **72** (2020), no. 4, pp. 1161–1180.
10. Tomoyoshi Ibukiyama, Valentijn Karemaker, and Chia-Fu Yu, *The Gauss problem for central leaves*, arXiv e-prints 2205.13180 (2022).

11. Aise J. de Jong and Frans Oort, *Purity of the stratification by Newton polygons*, J. Amer. Math. Soc. **13** (2000), no. 1, pp. 209–241.
12. Valentijn Karemaker, Fuetaro Yobuko, and Chia-Fu Yu, *Mass formula and Oort's conjecture for supersingular abelian threefolds*, Adv. Math. **386** (2021), Paper No. 107812, 52.
13. Toshiyuki Katsura and Frans Oort, *Families of supersingular abelian surfaces*, Compositio Math. **62** (1987), no. 2, pp. 107–167.
14. ______, *Supersingular abelian varieties of dimension two or three and class numbers*, Algebraic geometry, Sendai, 1985, Adv. Stud. Pure Math., vol. 10, North-Holland, Amsterdam, 1987, pp. 253–281.
15. Neal Koblitz, *p-adic variation of the zeta-function over families of varieties defined over finite fields*, Compositio Math. **31** (1975), no. 2, pp. 119–218.
16. Ke-Zheng Li, *Classification of supersingular abelian varieties*, Math. Ann. **283** (1989), no. 2, pp. 333–351.
17. Ke-Zheng Li and Frans Oort, *Moduli of supersingular abelian varieties*, Lecture Notes in Mathematics, vol. 1680, Springer-Verlag, Berlin, 1998.
18. The LMFDB Collaboration, *The L-functions and modular forms database*, `https://www.lmfdb.org`, 2023.
19. Yuri I. Manin, *Theory of commutative formal groups over fields of finite characteristic.*, Uspehi Mat. Nauk **114** (1963), no. 6, pp. 3–90.
20. Laurent Moret-Bailly, *Familles de courbes et de variétés abéliennes sur* $\mathbb{P}^1$, no. 86, 1981, Seminar on Pencils of Curves of Genus at Least Two, pp. 109–140.
21. David Mumford, *The structure of the moduli spaces of curves and Abelian varieties*, Actes du Congrès International des Mathématiciens (Nice, 1970), Tome 1, 1971, pp. 457–465.
22. Peter Norman and Frans Oort, *Moduli of abelian varieties*, Ann. of Math. (2) **112** (1980), no. 3, pp. 413–439.
23. Tadao Oda and Frans Oort, *Supersingular abelian varieties.*, Proceedings of the International Symposium on Algebraic Geometry (Kyoto Univ., Kyoto, 1977), 1978, pp. 595–621.
24. Frans Oort, *Which abelian surfaces are products of elliptic curves?*, Math. Ann. **214** (1975), pp. 35–47.
25. ______, *Endomorphism algebras of abelian varieties*, Algebraic geometry and commutative algebra, Vol. II, Kinokuniya, Tokyo, 1988, pp. 469–502.
26. ______, *Newton polygons and formal groups: conjectures by Manin and Grothendieck*, Ann. of Math. (2) **152** (2000), no. 1, pp. 183–206.
27. ______, *A stratification of a moduli space of abelian varieties*, Moduli of abelian varieties (Texel Island, 1999), Progr. Math., vol. 195, Birkhäuser, Basel, 2001, pp. 345–416.
28. ______, *Foliations in moduli spaces of abelian varieties*, J. Amer. Math. Soc. **17** (2004), no. 2, pp. 267–296.
29. John Tate, *Endomorphisms of abelian varieties over finite fields*, Invent. Math. **2** (1966), pp. 134–144.
30. ______, *Classes d'isogénie des variétés abéliennes sur un corps fini (d'après T. Honda)*, Séminaire Bourbaki. Vol. 1968/69: Exposés 347–363, Lecture Notes in Math., vol. 175, Exp. 352, Springer, Berlin, 1971, pp. 95–110.
31. Gerard van der Geer, *Cycles on the moduli space of abelian varieties*, Moduli of curves and abelian varieties, Aspects Math., vol. E33, Friedr. Vieweg, Braunschweig, 1999, pp. 65–89.
32. William C. Waterhouse, *Abelian varieties over finite fields*, Ann. Sci. École Norm. Sup. (4) **2** (1969), pp. 521–560.
33. Chia-Fu Yu, *The supersingular loci and mass formulas on Siegel modular varieties*, Doc. Math. **11** (2006), pp. 449–468.
34. Chia-Fu Yu and Jeng-Daw Yu, *Mass formula for supersingular abelian surfaces*, J. Algebra **322** (2009), no. 10, pp. 3733–3743.

Mathematical Institute, Utrecht University, Utrecht, The Netherlands
Email address: `V.Z.Karemaker@uu.nl`