/k, X/k ab. var. $g = \dim$

X^t dual,

$\mathcal{P}$ on $X \times X^t$

Fourier trans

$$\mathcal{F} : (CH(X)_{\mathbb{Q}}, *) \xrightarrow{\sim} (CH(X^t)_{\mathbb{Q}}, \cdot)$$

$$\begin{tikzcd} & X \times X^t \arrow[dl, "pr_X"'] \arrow[dr, "pr_{X^t}"] & \\ X & & X \end{tikzcd}$$

$\mathcal{F}(x) = pr_{X^t,*}(pr_X^*(x) \cdot ch(P$

$n \in \mathbb{Z}$, $[n] : X \longrightarrow X$ $\rightsquigarrow$

$$[n]^*, [n]_* : CH(X) \longrightarrow CH(X)$$

proj. form: $[n]_* [n]^* = n^{2g} \cdot \mathrm{id}$

L line bun on X

L is symm. if $[-1]^*L \cong L$

antisymm $[-1]^*L \cong L^{-1}$

If L symm then $[n]^*L \cong L^{n^2}$ ← quadr in n

L antisymm $[n]^*L \cong L^{n}$ ← lin in n

For any L :

$$L^2 = \underbrace{\left(L \otimes [-1]^*L\right)}_{\text{symm}} \otimes \underbrace{\left(L \otimes [-1]^*L^{-1}\right)}_{\text{anti-symm}}$$

$$\begin{array}{ccccc} CH^1(X)_{\mathbb{Q}} & = & CH^1(X)_{\mathbb{Q}}^{sym} & \oplus & CH^1(X)_{\mathbb{Q}}^{asym} \\ \circlearrowleft & & \parallel & & \parallel \\ [-1]^* & & CH^1_{(0)}(X) & & CH^1_{(1)}(X) \end{array}$$

<u>Def</u> $i, j, \; s \in \mathbb{Z}$:

$$CH^i(X)_{\mathbb{Q}} \supset CH^i_{(s)}(X) :=$$

Weight

$$\left\{ \alpha \in CH^i(X)_{\mathbb{Q}} \;\middle|\; \begin{array}{l} \forall n : \\ [n]^* \alpha = n^{2i-s} \cdot \alpha \end{array} \right\}$$

$$CH_j(X)_{\mathbb{Q}} \supset CH_{j,(s)}(X) := CH^{g-j}_{(s)}(X)$$

$$= \left\{ \alpha \in CH_j(X)_{\mathbb{Q}} \;\middle|\; \begin{array}{l} \forall n : \\ [n]_* \alpha = n^{2j+s} \cdot \alpha \end{array} \right\}$$

THEOREM (Beauville)

(i) $$CH^i_{(s)}(X) = \left\{ \alpha \in CH^i(X)_{\mathbb{Q}} \;\middle|\; \mathcal{F}(\alpha) \in CH^{g-i+s}(X^t)_{\mathbb{Q}} \right\}$$

and:

$$\mathcal{F} : CH^i_{(s)}(X) \xrightarrow{\sim} CH^{g-i+s}_{(s)}(X^t)$$

Similarly:

$$CH_{j,(s)}(X) = \left\{ \alpha \in CH_j(X)_{\mathbb{Q}} \;\middle|\; \mathcal{F}(\alpha) \in CH_{g-j-s}(X^t)_{\mathbb{Q}} \right\}$$

(ii) $$CH^i_{(s)}(X) \cdot CH^j_{(t)}(X) \subseteq CH^{i+j}_{(s+t)}(X)$$

$$CH_{i,(s)}(X) * CH_{j,(t)}(X) \subseteq CH_{i+j,(s+t)}(X)$$

$\rightsquigarrow$ bi-graded rings

(iii)

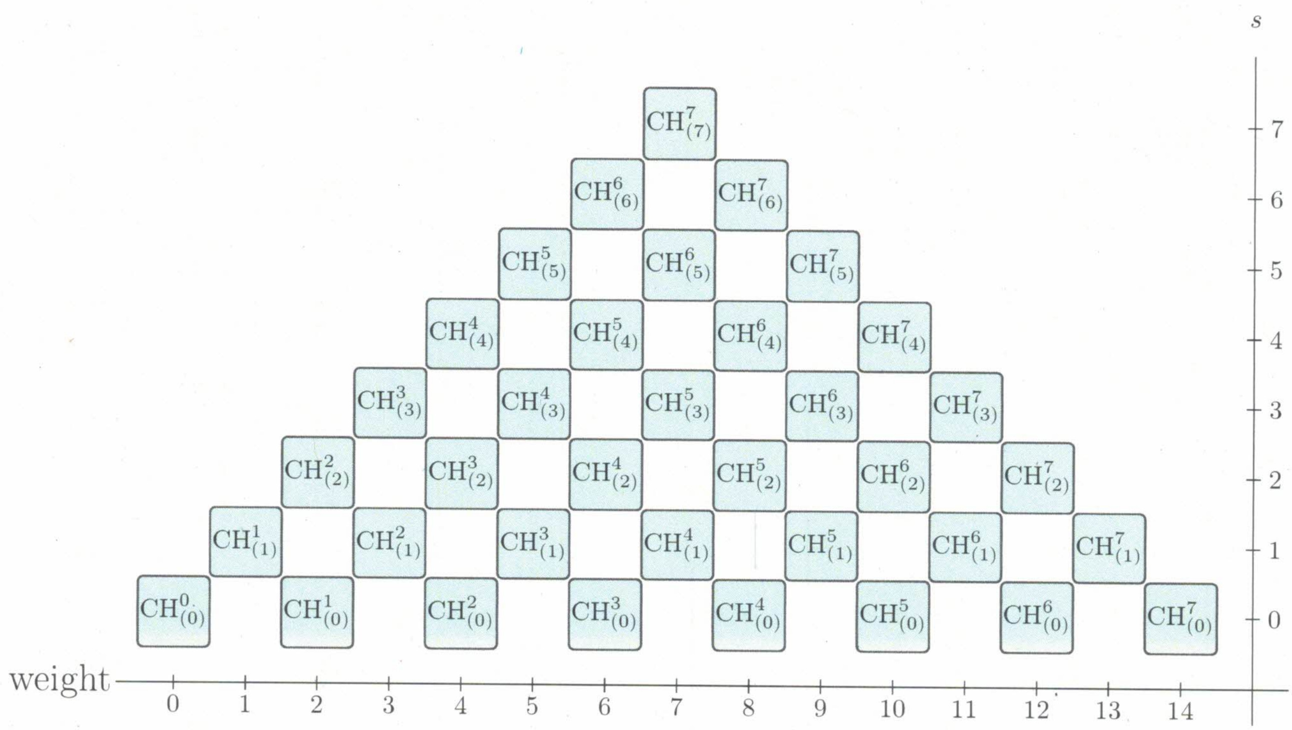
$$CH^i(X)_{\mathbb{Q}} = \bigoplus_{s=i-g}^{i} CH^i_{(s)}(X)$$

$$CH_j(X)_{\mathbb{Q}} = \bigoplus_{s=-j}^{g-j} CH_{j,(s)}(X)$$

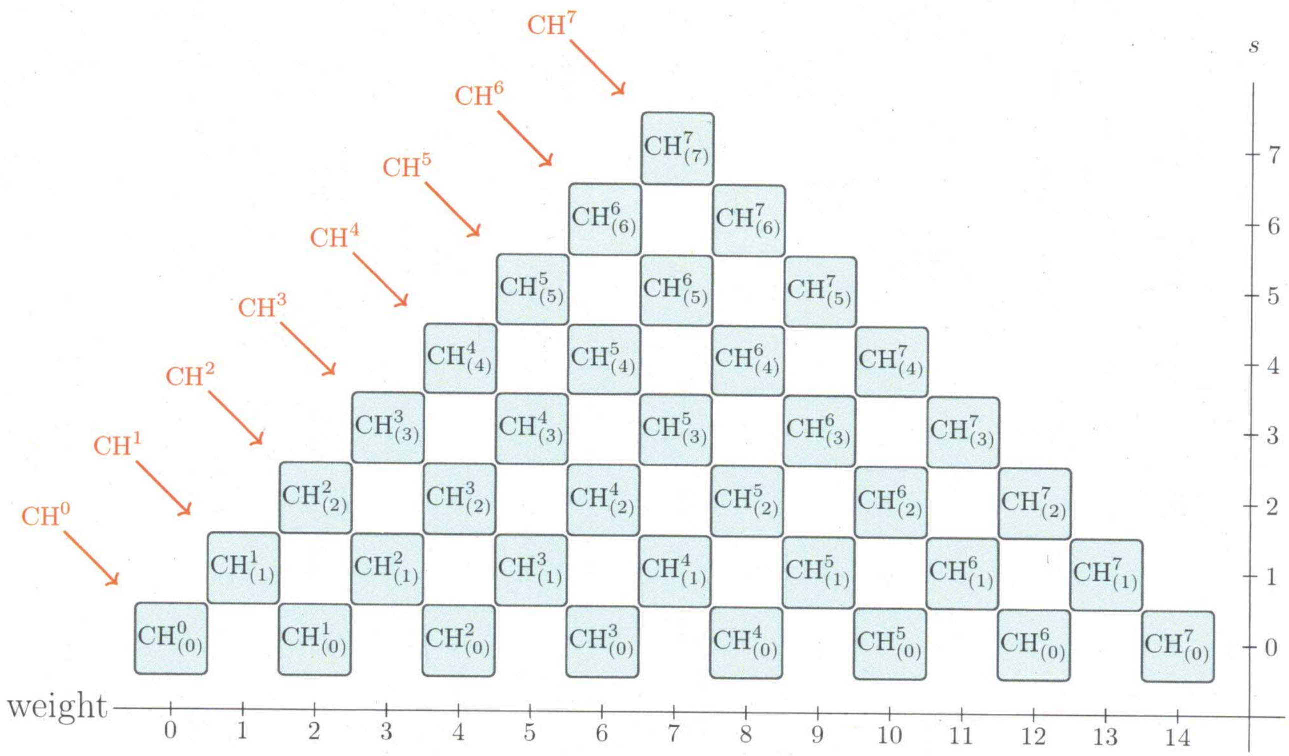
Represent the summand

$$\mathrm{CH}^i_{(s)}(X)$$

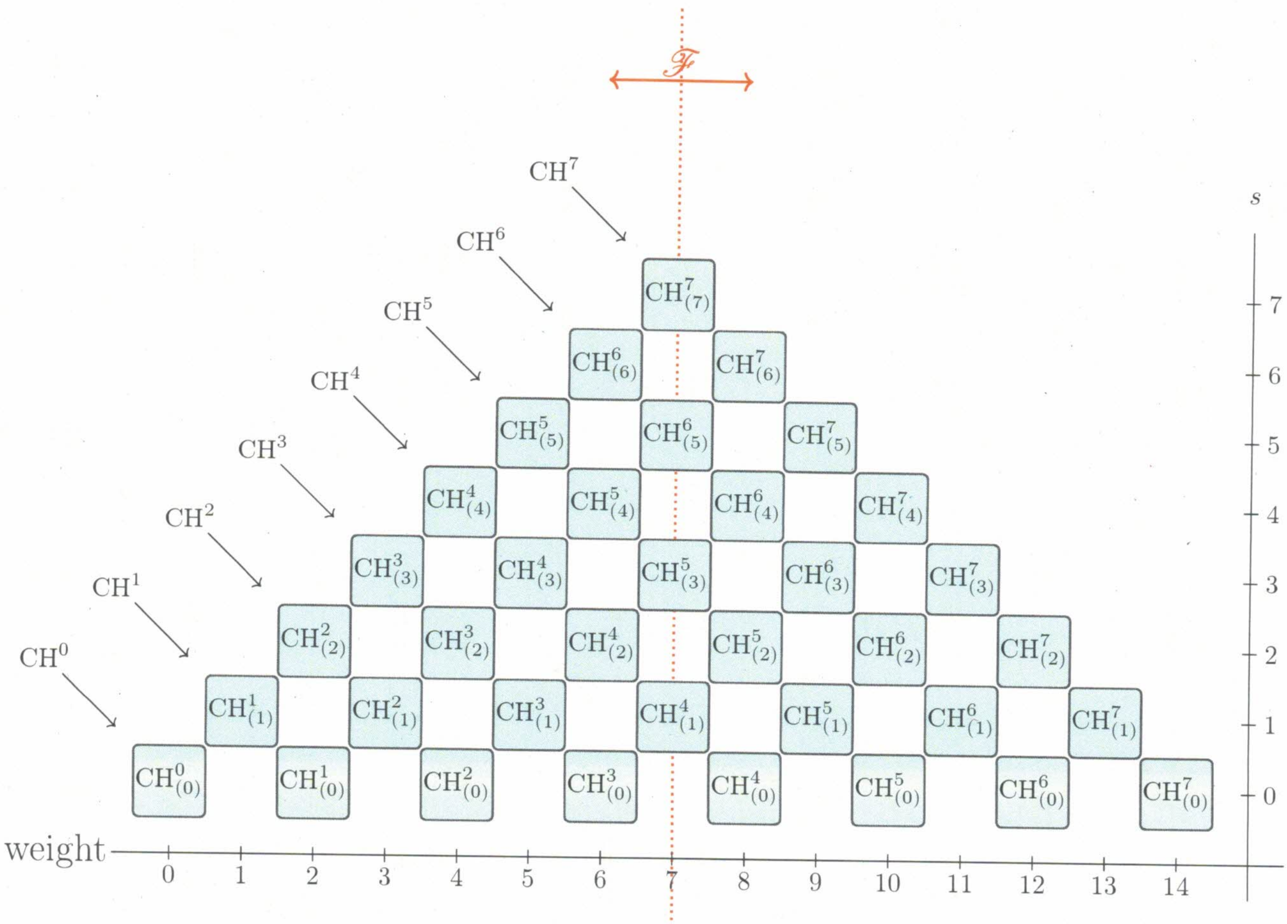
by a box in position $(2i - s, s)$, and call $2i - s$ the weight. Example with $g = 7$:



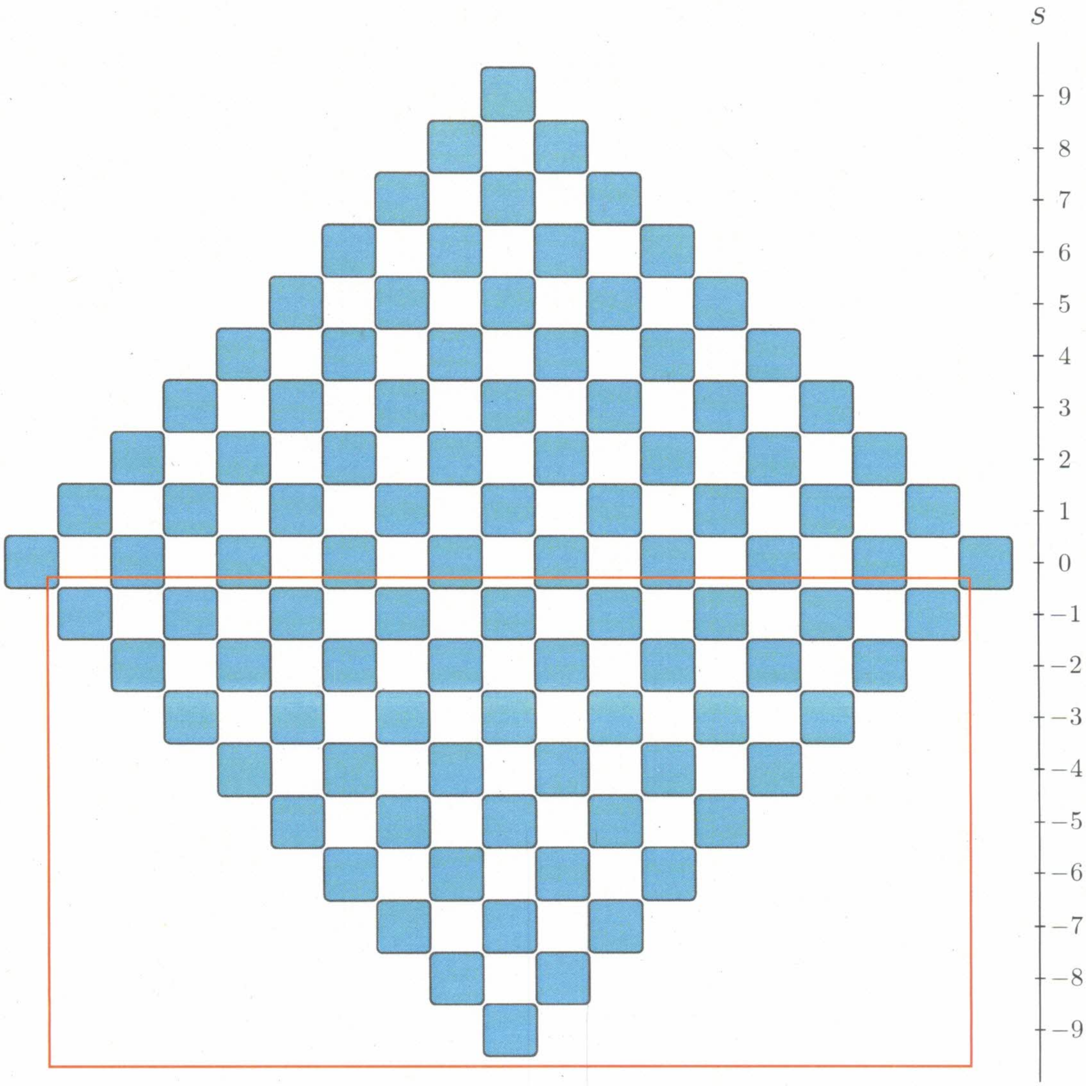
The usual grading by codimension of cycles is then represented by diagonal lines:



Fourier duality is now simply a reflection in the central vertical axis:

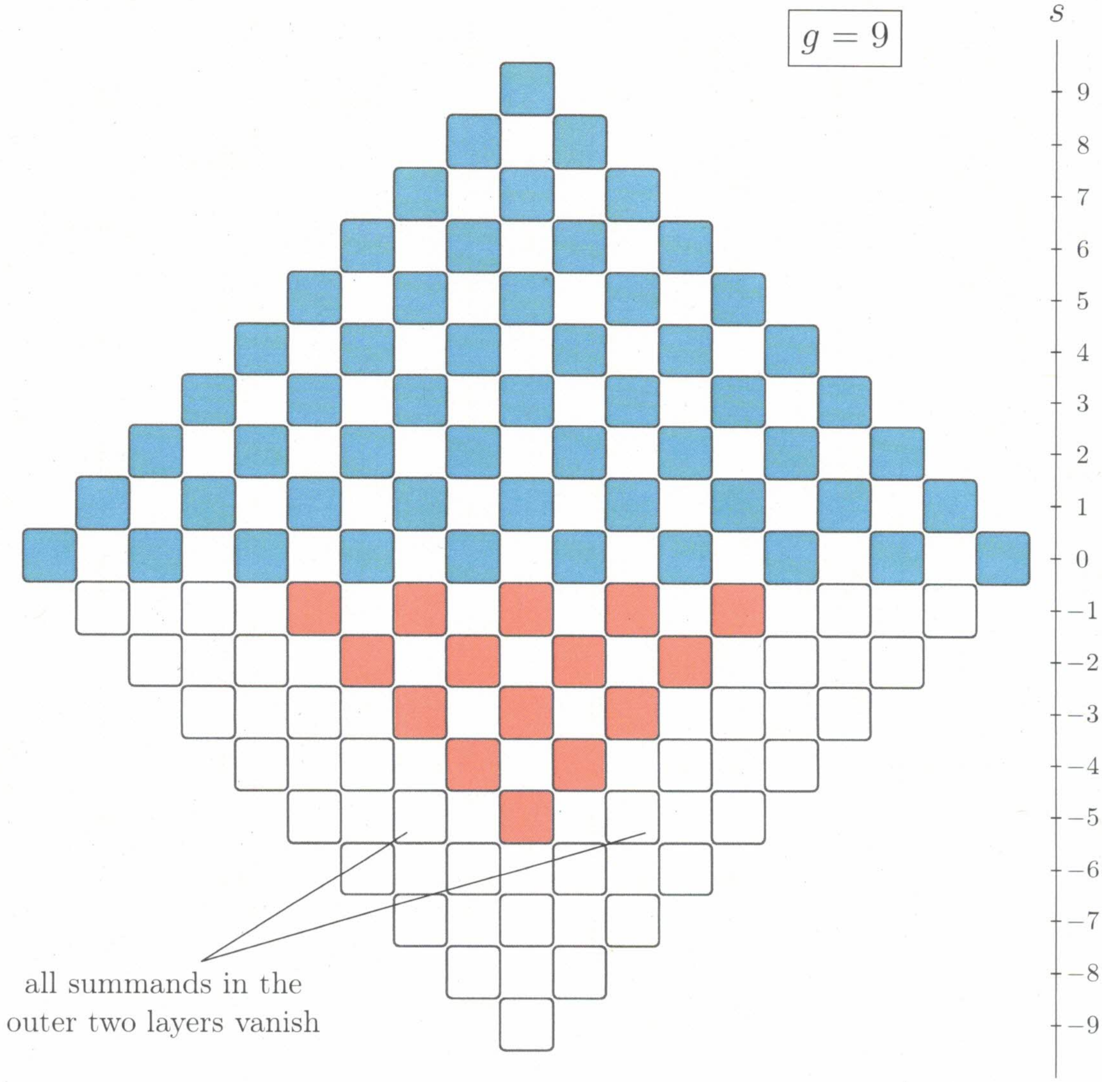


Conjecturally, all summands with $s < 0$ are zero; this is part of the *Bloch–Beilinson Conjectures.* As long as we do not know this, the picture would be as follows (example with $g = 9$):



conjecturally, all summands
in this region vanish

In general, we only know the vanishing of the summands with $s < 0$ for the outer two layers:



Let H be any Weil cohom. th.
for sm. proj. / k

Exa :

- $k = \mathbb{C}$: singular cohom. of $X(\mathbb{C})^{an}$
- any k, prime $\ell \neq \mathrm{char}(k)$:
  ℓ-adic cohom.
- ~~dR cohom.~~

Cycle class maps

$$cl : CH^i(X) \longrightarrow H^{2i}(X)$$

X/k ab. var. in any theory :

- $H^m(X) = \Lambda^m H^1(X)$
- $[n]^* =$ mult. by n^m on $H^m(X)$

By weights:

$cl = 0$ on all $CH^i_{(s)}$ with $s \neq 0$

Conj. (?):

$cl \hookrightarrow$ on $CH^i_{(0)}$

If $\alpha \in CH^i(X)_{\mathbb{Q}}$ w/ $cl(\alpha) = 0$

then try Abel–Jacobi map

$\downarrow$

target space "built out of" H^{2i-1}

If again:

$$AJ(\alpha) = 0$$

then (ℓ-adic coh): go on using "higher AJ maps".