ALGEBRAIC CYCLES
ON ABELIAN VARIETIES

Notes for the Arizona Winter School 2024

by

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Preface

These notes for the Arizona Winter School 2024 have three parts. Sections 1–6 contain material with which I hope most students are already somewhat familiar, or which could in any case be studied prior to the AWS. This doesn't mean that this is 'easy' material; but in order to get to more interesting topics, I have to assume some familiarity with the notions and results discussed in these sections. Of course, I'd be more than happy to explain and discuss this material during the working sessions.

Sections 7–11 of these notes directly correspond to the topics that I intend to discuss in my lectures. I have chosen to focus on results about Chow groups of abelian varieties, and especially those results that can be formulated (but not necessarily proven) without using the language of Chow motives. The first topic is Fourier duality and the so-called Beauville decomposition; the main results about this are due to Mukai [34] and Beauville [4], [5]. Next we shall discuss the action of the Lie algebra $\mathfrak{sl}_2$ on $\mathrm{CH}(X)_{\mathbb{Q}}$, which appears in the literature in several forms; the main results go back to work of Mukai and Polishchuk, and in the form presented here to work of Künnemann. In Section 10 we discuss some results about 0-cycles; these results work with integral coefficients, but as we shall see, a large part of $\mathrm{CH}_0(X)$ is already a $\mathbb{Q}$-vector space, at least over an algebraically closed base field. The last topic of my lectures, which corresponds to Section 11 of these notes, concerns an attempt to say something about how big (or small) Chow groups are. This is a topic about which many questions remain open (for instance, the precise nature of the torsion in Chow groups seems not yet fully understood), but at least there are some clear statements that tell us that, over a sufficiently big base field, Chow groups tend to be very large.

The third part of these notes, Sections 12 and 13, are about Chow motives; this is aimed mostly at students who want to work on one of the projects. I will set up my lectures in such a manner that you can follow them without ever looking at these sections. However, for several more advanced results about Chow groups, some knowledge of Chow motives seems indispensable, and as this is a very rich theory which remains full of mysteries, I can only encourage you to explore this.

A thorough treatment of these topics would require a sizeable volume, and much of what is in these notes is no more than a summary of the most important facts, though for some of the main results full proofs are given. These notes contain no new results, and nothing of what I present is due to me. The short sections entitled 'Further reading' give some pointers to the literature. While in many cases I have included references to the original papers, I apologise in advance if I have not given enough credit to someone's contributions.

I thank the organizers of the AWS 2024 for the invitation to be one of the lecturers and for their help and support. I look forward to what I'm sure is going to be a great event!

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Contents

PART 1

Basic notions

1. Abelian varieties

Throughout this section, k denotes a field.

1.1 Definition. (1) An *abelian variety* over k is a connected k-group scheme X such that the structural morphism $X \to \mathrm{Spec}(k)$ is smooth and proper.

(2) If X and Y are abelian varieties over k then by a *homomorphism of abelian varieties* $f\colon X \to Y$ we mean a homomorphism of k-group schemes.

1.2. Basic facts. We list a couple of standard properties of abelian varieties. Let X/k be an abelian variety.

- The group structure on X is commutative. In particular, if T is any k-scheme, the group of T-valued points $X(T)$ is abelian. We shall use additive notation; this means we have an origin $e = e_X \in X(k)$, and if P, $Q \in X(T)$ then we write $P + Q$ for their sum and $-P$ for the inverse of P. If $n \geq 0$ is a integer then nP means $P + \cdots + P$ (n terms) and $(-n)P = -(nP)$. We write $[n]\colon X \to X$ for the homomorphism given by $P \mapsto nP$. For $P \in X(k)$ we call the morphism $t_P\colon X \to X$ given by $Q \mapsto P + Q$ the translation by P.
- An abelian variety X over k is projective, i.e., there exist an embedding of X into a projective space over k. (This does not extend to abelian schemes: if S is a scheme, there is the notion of an abelian scheme over S, but such abelian schemes are not, in general, projective S-schemes.)
- Let X and Y be abelian varieties over k. If $f\colon X \to Y$ is a morphism of k-schemes such that $f(e_X) = e_Y$ then f is a homomorphism. Every morphism of k-schemes $f\colon X \to Y$ can be written as a composition $t \circ h$, where $t\colon Y \to Y$ is a translation and $h\colon X \to Y$ is a homomorphism.
- The endomorphisms $\mathrm{End}(X)$ of X (i.e., the homomorphisms from X to itself) form a ring, with addition given by the rule $(f+g)\bigl(P\bigr) = f(P) + g(P)$ and with composition of endomorphisms as multiplication. This ring $\mathrm{End}(X)$ is free of finite rank as a $\mathbb{Z}$-module. We write $\mathrm{End}^0(X) = \mathrm{End}(X) \otimes \mathbb{Q}$, which is a finite dimensional $\mathbb{Q}$-algebra.

1.3. Isogenies. Let X and Y be abelian varieties over k. Let $f\colon X \to Y$ be a homomorphism. Then the following conditions are equivalent:

(a) $\dim(X) = \dim(Y)$ and f is surjective;
(b) $\dim(X) = \dim(Y)$ and $\mathrm{Ker}(f)$ is a finite group scheme;
(c) f is finite flat and surjective.

If these conditions are satisfied, the rank of the group scheme $\mathrm{Ker}(f)$ equals the degree of the function field extension $f^*\colon k(Y) \hookrightarrow k(X)$.

1.4 Definition. A homomorphism $f\colon X \to Y$ is called an *isogeny* if the equivalent conditions in 1.3 are satisfied. If f is an isogeny, we define its *degree* by $\deg(f) = \bigl[k(X) : k(Y)\bigr] = \mathrm{rk}\bigl(\mathrm{Ker}(f)\bigr)$.

Two abelian varieties X and Y are said to be isogenous, notation $X \sim Y$, if there exists an isogeny $X \to Y$.

1.5. Basic facts (continued).

- If $f\colon X \to Y$ is an isogeny and k is algebraically closed, the map $f\colon X(k) \to Y(k)$ on k-valued points is surjective.
- Let X be an abelian variety of dimension g. If n is an integer, $n \neq 0$, the multiplication by n map $[n]_X\colon X \to X$, given by $P \mapsto nP$ is an isogeny of degree n^{2g}.
- Being isogenous is an equivalence relation: if there exists an isogeny $f\colon X \to Y$ of degree d, there exists an isogeny $g\colon Y \to X$ such that $g \circ f = [d]_X$ and $f \circ g = [d]_Y$. In this case, $\mathrm{End}^0(X)$ and $\mathrm{End}^0(Y)$ are isomorphic $\mathbb{Q}$-algebras.
- An abelian variety X is said to be *simple* if X has no abelian subvarieties other than 0 and X itself. This is equivalent to the condition that $\mathrm{End}^0(X)$ is a division algebra. Note that this notion is relative to the base field: if $k \subset L$ is a field extension and X is a simple abelian variety over k, the abelian variety X_L over L may not be simple. However, if k is separably closed, the property of being simple is preserved under arbitrary field extensions.
- Let X be an abelian variety. Then there exist simple abelian varieties $Y_1, \ldots, Y_t$, no two of which are isogenous, and positive integers $m_1, \ldots, m_t$, such that

  $$X \sim Y_1^{m_1} \times \cdots \times Y_t^{m_t}\,.$$

  Up to isogeny and a renumbering, the factors Y_j that occur, as well as their multiplicities, are uniquely determined.

2. Line bundles on abelian varieties

We start with two general results from algebraic geometry that give criteria for when a line bundle on a product of varieties is trivial. These criteria lie at the basis of several important results about abelian varieties.

As a general convention, if k is a base field then by a *variety* over k we shall mean a reduced and irreducible k-scheme that is separated and of finite type over k.

2.1 See-saw Principle. *Let X and Y be varieties over a field k such that X is complete and geometrically integral. Let $\mathrm{pr}_Y\colon X \times Y \to Y$ be the projection map. Let $\mathscr{L}$ be a line bundle such that $\mathscr{L}|_{X\times\{y\}} \cong \mathscr{O}_{X_{k(y)}}$ for every $y \in Y$, where $k(y)$ denotes the residue field of y. Then there exists a line bundle $\mathscr{M}$ on Y such that $\mathscr{L} \cong \mathrm{pr}_Y^*\mathscr{M}$. If additionally there exists a point $x \in X$ such that $\mathscr{L}|_{\{x\}\times Y} \cong \mathscr{O}_{Y_{k(x)}}$ then $\mathscr{L} \cong \mathscr{O}_{X\times Y}$.*

2.2 Theorem. *Let X, Y, Z be smooth complete varieties over k with base points $x \in X(k)$, $y \in Y(k)$ and $z \in Z(k)$. If $\mathscr{L}$ is a line bundle on $X \times Y \times Z$ whose restrictions to the three faces*

$$\{x\} \times Y \times Z\,, \qquad X \times \{y\} \times Z\,, \qquad X \times Y \times \{z\}$$

are all trivial then $\mathscr{L} \cong \mathscr{O}_{X\times Y\times Z}$.

Now assume X is an abelian variety with origin e, and take $Y = Z = X$. In the following result, write $\mathrm{pr}_i\colon X^3 \to X$ (for $i = 1, 2, 3$) for the projection map $(x_1, x_2, x_3) \mapsto x_i$, write m_{ij} for the map $(x_1, x_2, x_3) \mapsto x_i + x_j$, and let $m_{123}\colon X^3 \to X$ be the summation map $(x_1, x_2, x_3) \mapsto x_1 + x_2 + x_3$.

2.3 Theorem of the Cube. *Let X be an abelian variety over a field k. Let $\mathscr{L}$ be a line bundle on X. Then the line bundle*

$$\Theta(\mathscr{L}) = m_{123}^*\mathscr{L} \otimes m_{12}^*\mathscr{L}^{-1} \otimes m_{13}^*\mathscr{L}^{-1} \otimes m_{23}^*\mathscr{L}^{-1} \otimes \mathrm{pr}_1^*\mathscr{L} \otimes \mathrm{pr}_2^*\mathscr{L} \otimes \mathrm{pr}_3^*\mathscr{L}$$

on $X \times X \times X$ is trivial.

We could also write

$$\Theta(\mathscr{L}) = \bigotimes_{J \subset \{1,2,3\}} m_J^*(\mathscr{L})^{(-1)^{1+\#J}} ,$$

where $m_J\colon X^3 \to X$ is the map $(x_1, x_2, x_3) \mapsto \sum_{j \in J} x_j$. (Note that $m_\emptyset$ is the zero map, so $m_\emptyset^*\mathscr{L}$ is trivial.)

2.4 Corollary. *Let $\mathscr{L}$ be a line bundle on an abelian variety X.*

(1) *If Y is any k-scheme and f, g, $h\colon Y \to X$ are morphisms then*

$$(f + g + h)^*\mathscr{L} \otimes f^*\mathscr{L} \otimes g^*\mathscr{L} \otimes h^*\mathscr{L} \cong (f + g)^*\mathscr{L} \otimes (f + h)^*\mathscr{L} \otimes (g + h)^*\mathscr{L}$$

as line bundles on Y.

(2) *If x, $y \in X(k)$ then $t_{x+y}^*\mathscr{L} \otimes \mathscr{L} \cong t_x^*\mathscr{L} \otimes t_y^*\mathscr{L}$.* (Theorem of the Square)

(3) *If $n \in \mathbb{Z}$ then*

$$[n]^*\mathscr{L} \cong \mathscr{L}^{n(n+1)/2} \otimes \big([-1]^*\mathscr{L}\big)^{n(n-1)/2} .$$

Proof. For (1), take the pullback of $\Theta(\mathscr{L})$ along the morphism $(f, g, h)\colon Y \to X^3$. For (2), take $f = \mathrm{id}_X$, and let g and h be the constant maps $X \to X$ with images x and y. For (3), first take $f = [n]_X$, $g = \mathrm{id}_X$ and $h = -\mathrm{id}_X$, which gives the relation

$$[n]^*\mathscr{L}^2 \otimes [n+1]^*\mathscr{L}^{-1} \otimes [n-1]^*\mathscr{L}^{-1} \cong (\mathscr{L} \otimes [-1]^*\mathscr{L})^{-1} .$$

Now proceed by upward and downward induction on n, starting from the cases $n = -1, 0, 1$. □

2.5. Corollary 2.4(3) says something that will be important for us. Namely,

- if $\mathscr{L}$ is a *symmetric line bundle*, by which we mean that $[-1]^*\mathscr{L} \cong \mathscr{L}$, then

$$[n]^*\mathscr{L} \cong \mathscr{L}^{n^2} ;$$

  so in this case the effect of $[n]^*$ is *quadratic*;
- if $\mathscr{L}$ is an *antisymmetric line bundle*, by which we mean that $[-1]^*\mathscr{L} \cong \mathscr{L}^{-1}$, then

$$[n]^*\mathscr{L} \cong \mathscr{L}^n ;$$

  so in this case the effect of $[n]^*$ is *linear*.

Moreover, for an arbitrary line bundle $\mathscr{L}$ we can write

$$\mathscr{L}^2 = \bigl(\mathscr{L} \otimes [-1]^*\mathscr{L}\bigr) \otimes \bigl(\mathscr{L} \otimes [-1]^*\mathscr{L}^{-1}\bigr)\,, \tag{2.5.1}$$

in which the first factor is symmetric and the second factor is an antisymmetric line bundle.

We shall later interpret these statements as a first instance of the Beauville decomposition of the Chow group of X; see in particular Example 8.9.

3. Picard varieties

We continue our study of line bundles, now from the perspective of moduli. As we shall see later, every abelian variety arises as (a component of) a moduli space of line bundles on some other variety.

3.1. Let Y be a scheme over a field k. Suppose T is a k-scheme and $\mathscr{L}$ is a line bundle on $Y \times T$. For every point $t \in T(k)$, we get a line bundle $\mathscr{L}_t$ on Y. In this way we may think of $\mathscr{L}$ as a family of line bundles on Y, parametrized by T.

The question that arises is whether there exists a moduli scheme of line bundles. Concretely, one can ask if there exists a k-scheme P and a line bundle $\mathscr{P}$ on $Y \times P$, which is universal in the sense that for every other k-scheme T and line bundle $\mathscr{L}$ on $Y \times T$, there exists a unique morphism $\phi\colon T \to P$ over k such that $\mathscr{L} \cong (\mathrm{id}_Y \times \phi)^*\mathscr{P}$ as line bundles on $Y \times T$. For technical reasons, this is too optimistic and we have to phrase the problem in a slightly different manner. We restrict ourselves to the situation where Y is an irreducible smooth projective k-scheme that has a k-rational point $\varepsilon \in Y(k)$. If T is a k-scheme, we denote by $\varepsilon_T\colon T \to Y \times T$ the section given on points by $t \mapsto (\varepsilon, t)$.

3.2 Definition. Let T be a k-scheme. If $\mathscr{L}$ is a line bundle on $Y \times T$ then by a *rigidification* of $\mathscr{L}$ along $\{\varepsilon\} \times T$ we mean an isomorphism $\alpha\colon \varepsilon_T^*\mathscr{L} \xrightarrow{\sim} \mathscr{O}_T$. A pair $(\mathscr{L}, \alpha)$ of a line bundle and a rigidification is called a *rigidified line bundle* (with respect to the base point ε).

Let $\mathrm{pr}_T\colon Y \times T \to T$ be the projection morphism. We claim that if $\mathscr{L}$ is an arbitrary line bundle on $Y \times T$ then we have a canonical rigidification of the line bundle $\mathscr{L}' = \mathscr{L} \otimes \mathrm{pr}_T^*\varepsilon_T^*\mathscr{L}^{-1}$. Indeed, because $\mathrm{pr}_T \circ \varepsilon_T = \mathrm{id}_T$, we have a natural isomorphism $\varepsilon_T^*\mathscr{L}' \xrightarrow{\sim} \varepsilon_T^*\mathscr{L} \otimes \varepsilon_T^*\mathscr{L}^{-1} = \mathscr{O}_T$. If Y is an irreducible variety then to work with rigidified line bundles is in fact the same as working with line bundles modulo pullbacks of line bundles from the base scheme, see Theorem 3.4(2) below.

3.3. We denote by $\mathrm{Pic}_{Y/k,\varepsilon}(T)$ the group of isomorphism classes of rigidified line bundles on $Y \times T$. The group structure is given by the tensor product: if $(\mathscr{L}, \alpha)$ and $(\mathscr{M}, \beta)$ are rigidified line bundles on $Y \times T$, we define their product to be the pair $(\mathscr{L} \otimes \mathscr{M}, \alpha \otimes \beta)$, where $\alpha \otimes \beta$ is the rigidification given by the composition

$$\varepsilon_T^*(\mathscr{L} \otimes \mathscr{M}) = \varepsilon_T^*(\mathscr{L}) \otimes \varepsilon_T^*(\mathscr{M}) \xrightarrow[\alpha\otimes\beta]{\sim} \mathscr{O}_T \otimes_{\mathscr{O}_T} \mathscr{O}_T \xrightarrow[\mathrm{can}]{\sim} \mathscr{O}_T\,.$$

The group $\mathrm{Pic}_{Y/k,\varepsilon}(T)$ is abelian, and $T \mapsto \mathrm{Pic}_{Y/k,\varepsilon}(T)$ is a contravariant functor from the category Sch_k of k-schemes to the category Ab of abelian groups.

3.4 Theorem. *Let Y be an irreducible smooth projective k-scheme with base point $\varepsilon \in Y(k)$.*

(1) *The functor $\mathrm{Pic}_{Y/k,\varepsilon} \colon \mathsf{Sch}_k^{\mathrm{op}} \to \mathsf{Ab}$ is representable. This means that there exists a k-group scheme $\mathrm{Pic}_{Y/k,\varepsilon}$ and a line bundle $\mathscr{P}$ on $Y \times \mathrm{Pic}_{Y/k,\varepsilon}$ with rigidification $\alpha_{\mathscr{P}}$ along $\{\varepsilon\} \times \mathrm{Pic}_{Y/k,\varepsilon}$ that has the following universal property: for every k-scheme T and rigidified line bundle $(\mathscr{L}, \alpha)$ on $Y \times T$, there exists a unique morphism $\phi \colon T \to \mathrm{Pic}_{Y/k,\varepsilon}$ over k such that $(\mathscr{L}, \alpha) \cong (\mathrm{id}_Y \times \phi)^*(\mathscr{P}, \alpha_{\mathscr{P}})$ as rigidified line bundles on $Y \times T$.*

(2) *If T is a k-scheme, we have a short exact sequence*

$$0 \longrightarrow \mathrm{Pic}(T) \xrightarrow{\mathrm{pr}_T^*} \mathrm{Pic}(Y \times_k T) \xrightarrow{r} \mathrm{Pic}_{Y/k,\varepsilon}(T) \longrightarrow 0\,,$$

where r is the map that sends a line bundle $\mathscr{L}$ on $Y \times T$ to $\mathscr{L} \otimes \mathrm{pr}_T^* \varepsilon_T^* \mathscr{L}^{-1}$ with its canonical rigidification. The map $\mathrm{Pic}_{Y/k,\varepsilon}(T) \to \mathrm{Pic}(Y \times_k T)$ that forgets the rigidifcation is a section of r.

(3) *The connected components of $\mathrm{Pic}_{Y/k,\varepsilon}$ are irreducible projective k-schemes. If $\mathrm{char}(k) = 0$, these components are smooth over k.*

For a detailed discussion, we refer to [13], Chapter 8. Note that over fields of characteristic p, the components of $\mathrm{Pic}_{Y/k,\varepsilon}$ are not necessarily reduced.

The scheme $\mathrm{Pic}_{Y/k,\varepsilon}$ is called the *Picard scheme* of Y. We have presented here the version via rigidified line bundles that requires the existence of a k-rational base point; in full generality the Picard scheme is defined as the fppf sheafification of the naive Picard functor, but in the situation considered above, the result is the same.

3.5 Example. Let C/k be a smooth projective (irreducible) curve of genus g. Assume C has a k-rational point $\varepsilon \in C(k)$. (This assumption is made only to simplify the exposition and is not essential.) We define the *Jacobian* of C to be the identity component $\mathrm{Pic}^0_{C/k,\varepsilon}$, and we use the notation $\mathrm{Jac}(C)$ for it. It can be shown that $\mathrm{Jac}(C)$ is smooth over k and is therefore an abelian variety. Its dimension equals g.

The previous example is of great importance; while not all abelian varieties are Jacobians of curves, it can be shown that every abelian variety over an algebraically closed field is at least a quotient of a Jacobian. Throughout the development of the theory, the link between curves and abelian varieties has played an important role.

As we shall see in Theorem 4.6, every abelian variety is the Pic^0 of some smooth projective variety.

3.6. Let Y and Z be irreducible smooth projective k-schemes and let $f \colon Y \to Z$ be a morphism. Assume we have a k-rational point $\varepsilon_Y \in Y(k)$, and let $\varepsilon_Z = f(\varepsilon_Y)$. We have a universal line bundle $\mathscr{P}_Z$ on $Z \times \mathrm{Pic}_{Z/k,\varepsilon_Z}$ with rigidification α along $\{\varepsilon_Z\} \times \mathrm{Pic}_{Z/k,\varepsilon_Z}$. The pullback $(f \times \mathrm{id})^* \mathscr{P}_Z$ is a line bundle on $Y \times \mathrm{Pic}_{Z/k,\varepsilon_Z}$ with rigidification along $\{\varepsilon_Y\} \times \mathrm{Pic}_{Z/k,\varepsilon_Z}$. By the universal property of the universal rigidified line bundle $\mathscr{P}_Y$, there exists a unique k-morphism

$$P(f) \colon \mathrm{Pic}_{Z/k,\varepsilon_Z} \to \mathrm{Pic}_{Y/k,\varepsilon_Y}$$

such that

$$(f \times \mathrm{id})^* \mathscr{P}_Z \cong \big(\mathrm{id} \times P(f)\big)^* \mathscr{P}_Y$$

as rigidified line bundles on $Y \times \mathrm{Pic}_{Z/k,\varepsilon_Z}$. The morphism $P(f)$ is in fact a *homo*morphism of group schemes. We normally use the notation f^* for it. (Note that, by construction, if $\mathscr{M}$ is a rigidified line bundle on Z that corresponds to a point $u \in \mathrm{Pic}_{Z/k,\varepsilon_Z}(k)$, the image point $f^*(u) \in \mathrm{Pic}_{Y/k,\varepsilon_Y}(k)$ corresponds to the rigidified line bundle $f^*(\mathscr{M})$. So the notation f^* is natural.)

In this way we see that a morphism $f\colon Y \to Z$ with $f(\varepsilon_Y) = \varepsilon_Z$ gives rise to a homomorphism $f^*\colon \mathrm{Pic}_{Z/k,\varepsilon_Z} \to \mathrm{Pic}_{Y/k,\varepsilon_Y}$. Note that $f \rightsquigarrow f^*$ is contravariant: $(g \circ f)^* = f^* \circ g^*$.

4. Duality of abelian varieties

4.1 Theorem. *Let X/k be an abelian variety with origin $e \in X(k)$. Then $\mathrm{Pic}_{X/k,e}$ is smooth over k and the identity component $X^t = \mathrm{Pic}^0_{X/k,e}$ is an abelian variety of the same dimension as X.*

In what follows, we simply write $\mathrm{Pic}_{X/k}$ instead of $\mathrm{Pic}_{X/k,e}$.

4.2 Definition. Let X/k be an abelian variety. Then $X^t = \mathrm{Pic}^0_{X/k}$ is called the *dual abelian variety*. The rigidified line bundle $(\mathscr{P}, \alpha_{\mathscr{P}})$ on $X \times X^t$ as in Theorem 3.4(1) is called the *Poincaré bundle*.

Note that we use the term Poincaré bundle both for the universal (rigidified) line bundle on $X \times \mathrm{Pic}_{X/k,e}$ and for its restriction to $X \times X^t = X \times \mathrm{Pic}^0_{X/k}$. By Theorem 3.4(2) we have $\mathrm{Pic}_{X/k}(k) = \mathrm{Pic}(X)$, and $X^t(k)$ is the subgroup $\mathrm{Pic}^0(X) \subset \mathrm{Pic}(X)$ of line bundles that lie in the identity component of $\mathrm{Pic}_{X/k}$; such line bundles are said to be algebraically trivial.

We shall use the symbol e both for the origin of X and for the origin of the dual abelian variety X^t. This should not lead to confusion.

4.3. If $f\colon X \to Y$ is a homomorphism of abelian varieties, we are in the situation of 3.6, with the origins e_X and e_Y as base points. As explained there, f gives rise to a homomorphism $f^*\colon \mathrm{Pic}_{Y/k} \to \mathrm{Pic}_{X/k}$. This homomorphism maps the identity component $Y^t = \mathrm{Pic}^0_{Y/k}$ to the identity component $X^t = \mathrm{Pic}^0_{X/k}$, and thus gives a homomorphism

$$f^t\colon Y^t \to X^t$$

of the dual abelian varieties, called the *dual of the homomorphism f*. Now consider the diagram

$$\begin{tikzcd} & X \times Y^t \arrow[dl, "f\times\mathrm{id}"'] \arrow[dr, "\mathrm{id}\times f^t"] & \\ Y \times Y^t & & X \times X^t \end{tikzcd}$$

By construction, f^t has the property that

$$(f \times \mathrm{id}_Y)^* \mathscr{P}_Y \cong (\mathrm{id}_X \times f^t)^* \mathscr{P}_X \tag{4.3.1}$$

as rigidified line bundles on $X \times Y^t$, and this identity characterizes f^t.

4.4. Basic properties.

(1) If $g\colon Y \to Z$ is a second homomorphism of abelian varieties, $(g \circ f)^t = f^t \circ g^t$.

(2) The construction of dual homomorphisms is compatible with the addition: if f_1, $f_2\colon X \to Y$ are homomorphisms, $(f_1 + f_2)^t = f_1^t + f_2^t$.

Note that this last assertion is not at all obvious, and in fact, it is only true on the identity components of the Picard schemes. To explain this in more detail, recall that the homomorphisms f_i induce homomorphisms of group schemes $f_i^*\colon \mathrm{Pic}_{Y/k,e} \to \mathrm{Pic}_{X/k,e}$. The remark, then, is that for these pullback homomorphisms it is *not* true that $(f_1+f_2)^*$ equals $f_1^* + f_2^*$. We see this for instance from Corollary 2.4(3), as $[1+1]^*\mathscr{L} = [2]^*\mathscr{L}$ is in general not isomorphic to $\mathscr{L}^2$.

4.5. Duality. Let $\mathscr{P}_X$ be the Poincaré line bundle on $X \times X^t$, which comes equipped with a rigidification α along $\{e\} \times X^t$. There exists a unique rigidification α' of $\mathscr{P}_X$ along $X \times \{e\}$ such that the two rigidifications are the same on (e, e). Let $\sigma\colon X \times X^t \to X^t \times X$ be the morphism that exchanges the two factors. Then $(\sigma^*\mathscr{P}_X, \alpha')$ is a rigidified line bundle on $X^t \times X$. We can think of it as a family of line bundles on X^t parametrized by X. By definition of the double dual abelian variety $X^{tt} = (X^t)^t$, we have a Poincaré bundle $\mathscr{P}_{X^t}$ on $X^t \times X^{tt}$, and its universal property gives us a unique homomorphism $\kappa_X\colon X \to X^{tt}$ such that $\sigma^*\mathscr{P}_X \cong (\mathrm{id}_{X^t}, \kappa_X)^*\mathscr{P}_{X^t}$ as rigidified line bundles on $X^t \times X$.

4.6 Theorem.

(1) *The homomorphism $\kappa_X\colon X \to X^{tt}$ is an isomorphism of abelian varieties.*

(2) *Let $f\colon X \to Y$ be a homomorphism of abelian varieties. Under the identifications $\kappa_X\colon X \xrightarrow{\sim} X^{tt}$ and $\kappa_Y\colon Y \xrightarrow{\sim} Y^{tt}$ we have $f^{tt} = f$.*

The interpretation of this is that the functor $X \mapsto X^t$ really gives a duality theory. In what follows we always identify X and X^{tt} via the canonical isomorphism κ_X.

4.7. It turns out that the dual of an abelian variety X is always isogenous (but in general not isomorphic!) to X. To see this, we use the following construction.

Let $\mathscr{L}$ be a line bundle on an abelian variety X. For every point $x \in X(k)$ we have the translation $t_x\colon X \to X$, and the class of the line bundle $t_x^*\mathscr{L} \otimes \mathscr{L}^{-1}$ lies in $X^t(k) = \mathrm{Pic}^0_{X/k}(k)$. (This is not obvious but it will follow from the arguments below.) By the Theorem of the Square (see Corollary 2.4), the map $x \mapsto [t_x^*\mathscr{L} \otimes \mathscr{L}^{-1}]$ defines a homomorphism $\phi_L\colon X(k) \to X^t(k)$. We claim that it in fact defines a homomorphism of abelian varieties $\phi_L\colon X \to X^t$. One way to prove this is to extend the construction as just described to T-valued points, for arbitrary k-schemes T. Another, perhaps simpler, way to proceed is to consider the line bundle

$$\Lambda(\mathscr{L}) := m^*\mathscr{L} \otimes \mathrm{pr}_1^*\mathscr{L}^{-1} \otimes \mathrm{pr}_2^*\mathscr{L}^{-1}$$

on $X \times X$, where $m\colon X \times X \to X$ is the group law. (This bundle is sometimes called the Mumford line bundle given by $\mathscr{L}$.) Note that the restrictions of $\Lambda(\mathscr{L})$ to $\{e\} \times X$ and to $X \times \{e\}$ are canonically isomorphic to $\mathscr{O}_X$. If we view the first factor X as our abelian variety and the second factor X as a parameter space, then by definition of $\mathrm{Pic}_{X/k}$ there exists a unique morphism of k-schemes $\phi_{\mathscr{L}}\colon X \to \mathrm{Pic}_{X/k}$ such that

$$\Lambda(\mathscr{L}) \cong (\mathrm{id}_X \times \phi_{\mathscr{L}})^*\mathscr{P}$$

as rigidified line bundles. (See Theorem 3.4(1).) Because the restriction of $\Lambda(\mathscr{L})$ to $X \times \{e\}$ is trivial, $\phi_{\mathscr{L}}(e)$ is the origin of $\mathrm{Pic}_{X/k}$, and since X is connected it follows that $\phi_{\mathscr{L}}$ factors through the identity component $X^t = \mathrm{Pic}^0_{X/k}$. The conclusion is that we have a morphism

$$\phi_{\mathscr{L}}\colon X \to X^t\,,$$

which by the facts stated in 1.2 is a *homo*morphism of abelian varieties. Note that the restriction of $\Lambda(\mathscr{L})$ to $X \times \{x\}$ is the line bundle $t_x^*\mathscr{L} \otimes \mathscr{L}^{-1}$, so ϕ_L is indeed the homomorphism given by $x \mapsto [t_x^*\mathscr{L} \otimes \mathscr{L}^{-1}]$.

4.8. Basic facts. Let X be an abelian variety over k.

(1) If $\mathscr{L}$ is a line bundle on X, the homomorphism $\phi_{\mathscr{L}}\colon X \to X^t$ is self-dual, in the sense that $\phi^t_{\mathscr{L}} = \phi_{\mathscr{L}}$. (Here we use the canonical identification $X^{tt} = X$.) The reason for this is simply that the line bundle $\Lambda(\mathscr{L})$ on $X \times X$ is symmetric with respect to the exchange of the two factors.

(2) Writing $\mathrm{Hom}^{\mathrm{sym}}(X, X^t) \subset \mathrm{Hom}(X, X^t)$ for the subgroup of self-dual homomorphisms, we have a short exact sequence of abelian groups

$$0 \longrightarrow X^t(k) \longrightarrow \mathrm{Pic}_{X/k}(k) \xrightarrow{\ \phi\ } \mathrm{Hom}^{\mathrm{sym}}(X, X^t)\,.$$

In other words, the kernel of the map $\mathscr{L} \mapsto \phi_{\mathscr{L}}$ is given by the line bundles $\mathscr{L}$ that are algebraically trivial. If the base field k is algebraically closed then the map $\phi\colon \mathrm{Pic}_{X/k}(k) \to \mathrm{Hom}^{\mathrm{sym}}(X, X^t)$ is surjective.

(3) Let $\mathscr{P}$ be the Poincaré bundle on $X \times X^t$. If $\mathscr{L}$ is a symmetric line bundle on X (i.e., a line bundle with $[-1]^*\mathscr{L} \cong \mathscr{L}$) then $(\mathrm{id}_X, \phi_{\mathscr{L}})^*\mathscr{P} \cong \mathscr{L}^2$.

4.9. Using these facts, we arrive at several different ways to characterize the line bundles $\mathscr{L}$ on X that are algebraically trivial. Namely, if $\mathscr{L}$ is a line bundle on an abelian variety X, the following properties are equivalent:

(a) The class $[\mathscr{L}] \in \mathrm{Pic}_{X/k}(k)$ lies in the identity component $\mathrm{Pic}^0_{X/k} = X^t$, i.e., $\mathscr{L}$ is algebraically trivial.

(b) The associated homomorphism $\phi_{\mathscr{L}}\colon X \to X^t$ is zero.

(c) We have $\Lambda(\mathscr{L}) \cong \mathscr{O}_{X\times X}$.

(d) We have $[-1]_X^*\mathscr{L} \cong \mathscr{L}^{-1}$.

(e) For all $n \in \mathbb{Z}$ we have $[n]_X^*\mathscr{L} \cong \mathscr{L}^n$.

To see this: (a) $\Leftrightarrow$ (b) is contained in the above fact (2). The equivalence (b) $\Leftrightarrow$ (c) follows from the See-saw Principle 2.1, because $\phi_{\mathscr{L}} = 0$ just says that all line bundles $\Lambda(\mathscr{L})|_{X\times\{x\}}$ are trivial. For (c) $\Leftrightarrow$ (d) one uses that $\phi_{[-1]^*\mathscr{L}} = \phi_{\mathscr{L}}$ (exercise!) whereas $\phi_{\mathscr{L}^{-1}} = -\phi_{\mathscr{L}}$. Finally, (d) $\Leftrightarrow$ (e) follows from Corollary 2.4(3).

The implication (a) $\Rightarrow$ (e) gives as conclusion that

$$\big([n]_X\big)^t = [n]_{X^t}$$

for all $n \in \mathbb{Z}$.

The group

$$\mathrm{NS}(X) = \mathrm{Pic}(X)/\mathrm{Pic}^0(X)$$

is called the *Néron–Severi group* of X. For an arbitrary line bundle $\mathscr{L}$ on X we can write Corollary 2.4(3) as

$$[n]_X^*\mathscr{L} \cong \mathscr{L}^{n^2} \otimes \big(\mathscr{L}^{-1} \otimes [-1]^*\mathscr{L}\big)^{\frac{n(n-1)}{2}} ,$$

and because $\mathscr{L}^{-1} \otimes [-1]^*\mathscr{L}$ is algebraically trivial, we see that $[n]_X^*$ acts as multiplication by n^2 on $\mathrm{NS}(X)$. Thus we have a short exact sequence

$$0 \longrightarrow \mathrm{Pic}^0(X) \longrightarrow \mathrm{Pic}(X) \longrightarrow \mathrm{NS}(X) \longrightarrow 0$$

with $[n]_X^*$ acting as multiplication by n on the first term and as multiplication by n^2 on the last term. If we tensor this sequence with $\mathbb{Q}$ (in fact, $\mathbb{Z}\big[\frac{1}{2}\big]$ suffices) then the sequence naturally splits; we shall return to this in Example 8.9.

4.10 Corollary. *Let $\mathscr{P}$ be the Poincaré bundle on $X \times X^t$. Then for every $n \in \mathbb{Z}$ we have*

$$([n]_X, \mathrm{id})^*\mathscr{P} \cong \mathscr{P}^n \cong (\mathrm{id}, [n]_{X^t})^*\mathscr{P} \,. \tag{4.10.1}$$

Proof. By construction, all line bundles $\mathscr{P}|_{X\times\{\xi\}}$, for $\xi \in X^t$, are algebraically trivial. By using (a) $\Leftrightarrow$ (e) from 4.9 it follows that $(\mathrm{id}, [n]_{X^t})^*\mathscr{P}$ and $\mathscr{P}^n$ have the same restrictions to all $X \times \{\xi\}$. As they both restrict to the trivial line bundle on $\{e\} \times X^t$, it follows from the See-saw Principle 2.1 that they are isomorphic. By duality (exchanging the roles of X and X^t) it follows that also $([n]_X, \mathrm{id})^*\mathscr{P} \cong \mathscr{P}^n$. $\square$

As we have just seen, if a line bundle $\mathscr{L}$ on an abelian variety X is algebraically trivial then $\phi_{\mathscr{L}}$ is the zero homomorphism. On the other hand, if we take an ample line bundle $\mathscr{L}$ (which always exists, because an abelian variety over a field is projective) then $\phi_{\mathscr{L}}$ turns out to be an isogeny. When working over an aritrary base field, it makes sense to consider a slightly larger class of homomorphisms $X \to X^t$ that can be characterized in several equivalent ways, as follows.

4.11 Proposition. *Let X/k be an abelian variety. Let $\mathscr{P}$ be the Poincaré bundle on $X \times X^t$ and let $\lambda\colon X \to X^t$ be a homomorphism. Then the following properties are equivalent:*

(a) *λ is a self-dual isogeny and the line bundle $(\mathrm{id}_X, \lambda)^*\mathscr{P}$ on X is ample;*
(b) *there exists a field extension $k \subset K$ and an ample line bundle $\mathscr{L}$ on X_K such that $\lambda_K = \phi_{\mathscr{L}}$;*
(c) *there exists a finite separable field extension $k \subset K$ and an ample line bundle $\mathscr{L}$ on X_K such that $\lambda_K = \phi_{\mathscr{L}}$.*

Note that, for a self-dual homomorphism $\lambda\colon X \to X^t$, there always exists a line bundle $\mathscr{M}$ on X such that $\phi_{\mathscr{M}} = 2\lambda$, namely the bundle $\mathscr{M} = (\mathrm{id}_X, \lambda)^*\mathscr{P}$. However, over an arbitrary base field there does not, in general, exist an $\mathscr{L}$ such that $\lambda = \phi_{\mathscr{L}}$. The proposition says that such a bundle $\mathscr{L}$ does exist after a finite separable extension of the base field.

The isogenies $X \to X^t$ as in the proposition have some important positivity properties, and they merit a special name:

4.12 Definition. A homomorphism $\lambda\colon X \to X^t$ is called a *polarization* of X if it satisfies the equivalent conditions of Proposition 4.11.

In particular, over $k = \bar{k}$ the isogenies are precisely the homomorphisms of the form $\phi_{\mathscr{L}}\colon X \to X^t$ for $\mathscr{L}$ an ample line bundle on X. Note, however, that $\phi_{\mathscr{L}}$ only depends on the connected component of $\mathrm{Pic}_{X/k}$ that contains $[\mathscr{L}]$, so different ample bundles can give the same isogeny.

4.13 Remark. A line bundle $\mathscr{L}$ on X is said to be *nondegenerate* if $\phi_{\mathscr{L}}\colon X \to X^t$ is an isogeny. By the above, every ample line bundle is nondegenerate, but ampleness is a much stronger notion. For instance, if X is a simple abelian variety, it follows from the exact sequence of 4.8(2) that every line bundle that is not algebraically trivial is in fact nondegenerate.

We conclude this section with an important technical result about the cohomology of the Poincaré bundle that we will later use in our discussion of Fourier duality.

4.14 Theorem. *Let X be a g-dimensional abelian variety over a field k. Let $\mathscr{P}$ be the Poincaré bundle on $X \times X^t$, and let $\mathrm{pr}_1\colon X \times X^t \to X$ be the first projection. Let $e\colon \mathrm{Spec}(k) \to X$ be the origin of X. Then*

$$R^n\mathrm{pr}_{1,*}\mathscr{P} \cong \begin{cases} e_*(k) & \text{for } n = g; \\ 0 & \text{otherwise.} \end{cases}$$

Consequently, $H^g(X \times X^t, \mathscr{P}) \cong k$ and $H^n(X \times X^t, \mathscr{P}) = 0$ for all $n \neq g$.

Note: by $e_*(k)$ we mean the skyscraper sheaf at the origin with fibre k.

4.15. Further reading. A great deal about abelian varieties can be learned from Mumford's book [36]. Note, however, that in this book results are proven only over algebraically closed fields; further, Mumford's discussion of the dual abelian variety is unsatisfactory, in that he directly constructs the dual and does not really explain that it is the identity component of the Picard scheme, which for many purposes gives a much more natural picture.

There are several sets of notes on abelian varieties available. Some chapters of an unfinished book manuscript by Van der Geer and the author of these notes can be found at `www.math.ru.nl/~bmoonen/research.html`. There are notes by (Brian) Conrad and notes by Milne, available on their respective websites. A detailed discussion of Picard schemes can be found in [13], Chapter 8.

5. Chow groups

Recall that if k is a field then by a *variety* over k we mean a reduced and irreducible k-scheme that is separated and of finite type over k. Note that this notion of a variety is not stable under extension of scalars; so if X/k is a variety and $k \subset k'$ is a field extension, the k'-scheme $X_{k'}$ that is obtained from X by extension of scalars is not, in general, a variety over k', as it may be non-reduced or reducible. Similarly, the product $X \times_k Y$ of two varieties is not a variety, in general. These problems do not occur if k is algebraically closed. (See [50], Lemmas 020I and 05P3.)

If X is a scheme over k then by a subvariety of X we mean a closed subscheme $Z \subset X$ that itself is a variety over k.

Even though we are mainly interested in Chow groups of varieties, it is useful to work in greater generality when setting up the basic notions.

5.1 Definition. Let X be k-scheme of finite type. Then we define $\mathscr{Z}_j(X)$ to be the free abelian group on the set of subvarieties $Z \subset X$ of dimension j. If all components of X have dimension d then we define $\mathscr{Z}^j(X) = \mathscr{Z}_{d-j}(X)$.

Note that in order to talk about the codimension of a cycle, we should assume X is equidimensional. (Consider the scheme $X \subset \mathbb{A}^3$ given by the equations $xy = xz = 0$; then what is the codimension of the origin $O = (0,0,0)$ in X ?) In geometry, the grading by codimension (the 'cohomological grading') plays an important role, because for a smooth variety X the intersection product makes $\mathrm{CH}^*(X)$ into a graded ring (see below). However, to develop the general theory, allowing fairly general schemes X, one usually works with the grading by the dimension of cycles. (Note, for instance, that in the Stacks Project [50], the grading by codimension is not introduced until Section 0FE2.)

5.2. Rational equivalence. The groups $\mathscr{Z}_j(X)$ are big and do not have much structure. We obtain more interesting groups by dividing out suitable equivalence relations on algebraic cycles, for which there are several choices. The most relevant for us is rational equivalence (notation $\sim_{\mathrm{rat}}$); we here follow [21], Chapter 1.

The definition of rational equivalence relies on the fact that if Y is a variety with function field $k(Y)$, and if $V \subset Y$ is a subvariety of codimension 1, we can define a homomorphism

$$\mathrm{ord}_V \colon k(Y)^* \to \mathbb{Z}$$

such that $\mathrm{ord}_V(f)$ measures the order of vanishing of f along V. The local ring $\mathscr{O}_{Y,V}$ of Y along V is a 1-dimensional domain with fraction field $k(Y)$, and for $0 \neq f \in \mathscr{O}_{Y,V}$ we have $\mathrm{ord}_V(f) = \mathrm{length}\big(\mathscr{O}_{Y,V}/(f)\big)$, the length of $\mathscr{O}_{Y,V}/(f)$ as an $\mathscr{O}_{Y,V}$-module. (If V is not contained in the singular locus of Y then $\mathscr{O}_{Y,V}$ is a dvr and ord_V is the corresponding discrete valuation.) For a given $f \in k(Y)^*$ there are only finitely many subvarieties $V \subset Y$ of codimension 1 such that $\mathrm{ord}_V(f) \neq 0$, which allows to define $\mathrm{div}(f) = \sum_V \mathrm{ord}_V(f) \cdot [V]$.

Let now X be a k-scheme of finite type. For simplicity we assume that all irreducible components of X have dimension d. Two cycles α, $\beta \in \mathscr{Z}_j(X)$ with $j < d$ are defined to be *rationally equivalent* if there exist $(j+1)$-dimensional subvarieties $Y_1, \ldots, Y_t$ of X and rational functions $f_m \in k(Y_m)^*$ $(m = 1, \ldots, t)$ such that

$$\beta - \alpha = \operatorname{div}(f_1) + \cdots + \operatorname{div}(f_t)$$

as cycles on X. Two cycles of dimension d are rationally equivalent if and only if they are equal. (Note that $\mathscr{Z}_d(X)$ is the free abelian group on the set of irreducible components of X.) We refer to [21], Chapter 1 for the basic properties of this notion.

As explained in ibid. Section 1.6, we can also take a more geometric approach. For this, suppose we have a subvariety $V \subset X \times \mathbb{P}^1$ of dimension $(j+1)$ for which the projection map $V \to \mathbb{P}^1$ is dominant. For $P \in \mathbb{P}^1(k)$, the scheme-theoretic fibre $f^{-1}\{P\}$ defines a j-dimensional cycle $V(P)$ on X (cf. 5.5 below), and we may think of the $V(P)$ as a family of j-dimensional cycles on X parametrized by $\mathbb{P}^1$. Rational equivalence on $\mathscr{Z}_j(X)$ is then the equivalence relation that is generated by the relations $V(P) \sim V(Q)$ for any $V \subset X \times \mathbb{P}^1$ as above and any P, $Q \in \mathbb{P}^1(k)$. In fact, the name rational equivalence comes from the fact that we allow cycles to move in a family parametrized by a *rational* (connected) curve. Varying on this idea, we define algebraic equivalence of cycles in the same way, except that we now allow families that are parametrized by an arbitrary connected base curve. See Definition 7.5 for a more precise version.

5.3 Definition. Let X/k be an equidimensional k-scheme of finite type. Then we define

$$\mathrm{CH}^i(X) = \mathscr{Z}^i(X)/{\sim_{\mathrm{rat}}}\,, \qquad \mathrm{CH}_j(X) = \mathscr{Z}_j(X)/{\sim_{\mathrm{rat}}}\,.$$

These are called the *Chow group* of codimension i cycles, resp. of j-dimensional cycles. Further, we define

$$\mathrm{CH}(X) = \bigoplus_{i=0}^{\dim(X)} \mathrm{CH}^i(X) = \bigoplus_{j=0}^{\dim(X)} \mathrm{CH}_j(X)\,.$$

Finally, we define

$$\mathrm{CH}(X)_{\mathbb{Q}} = \mathrm{CH}(X) \otimes_{\mathbb{Z}} \mathbb{Q}\,,$$

(and $\mathrm{CH}^i(X)_{\mathbb{Q}} = \mathrm{CH}^i(X) \otimes \mathbb{Q}$, etc).

If we want to indicate that we consider $\mathrm{CH}(X)$ with its grading by codimension (resp. dimension) of cycles, we use the notation $\mathrm{CH}^*(X)$, resp. $\mathrm{CH}_*(X)$.

5.4 Example. Assume X is a smooth variety over k. Then $\mathrm{CH}^1(X)$ is just the usual divisor class group. By the correspondence between line bundles and divisor classes we have an isomorphism

$$c_1 \colon \mathrm{Pic}(X) \xrightarrow{\sim} \mathrm{CH}^1(X)\,.$$

(We shall return to this in Section 6.)

5.5. If $Z \subset X$ is a subvariety, say of codimension i, we denote by $[Z]$ its class in $\mathrm{CH}^i(X)$. More generally, if $Z \subset X$ is any closed subscheme, not necessarily reduced or irreducible, let $Z_1, \dots, Z_t$ be its irreducible components, and let $\mathscr{O}_{Z,Z_j}$ be the corresponding local rings. (The Z_j correspond to the generic points η_j of the scheme Z, and then $\mathscr{O}_{Z,Z_j}$ is the stalk of $\mathscr{O}_Z$ at η_j.) Then each $\mathscr{O}_{Z,Z_j}$ is a artinian local ring, and we define the cycle class of Z in $\mathrm{CH}(X)$ by the rule

$$[Z] = \sum_{j=1}^{t} \mathrm{length}_{\mathscr{O}_{Z,Z_j}}(\mathscr{O}_{Z,Z_j}) \cdot [Z_j]\,.$$

5.6. Remark. We shall be mostly interested in Chow groups of abelian varieties, which are very interesting objects that are rich in structure. At the same time, these Chow groups are extremely subtle and complicated objects; they can be very big and may have a lot of torsion. Their structure strongly depends on the type of field over which we work. There are many things that we do not yet know about such Chow groups, and many things that we do know have been established only fairly recently, using sophisticated techniques.

In these notes, I focus on Chow groups of smooth projective k-varieties. This will suffice for the purposes of my lectures, though many facts about Chow groups are valid much more generally. Fulton's book [21] is the canonical reference for this, but much of what is discussed there concerns situations that are more difficult to handle than what we shall need. For a somewhat gentler introduction to intersection theory, also [22] is recommended. For an introduction from a much more geometric perspective, see [18].

What follows is a brief summary of some of the main structures on Chow groups that we shall use. (Appendix A of [23] also contains a very useful summary.) The technical details and proofs of the main properties go far beyond what we can summarize in a couple of paragraphs; for this you will need to consult the literature.

5.7. Push-forward. If $f\colon X \to Y$ is a proper morphism of k-varieties, it induces a homomorphism $f_*\colon \mathrm{CH}(X) \to \mathrm{CH}(Y)$ that preserves the grading by dimension of cycles. Concretely, f_* can be described as follows. If $Z \subset X$ is a subvariety, $f(Z) \subset Y$ is again a closed subvariety (because f is proper). If the map $f\colon Z \to f(Z)$ is generically finite of degree d then we define $f_*[Z] = d \cdot \bigl[f(Z)\bigr]$. If $f\colon Z \to f(Z)$ is not generically finite, $f_*[Z] = 0$. The map f_* is obtained by extending this linearly, i.e., $f_*\bigl(\sum\, m_i[Z_i]\bigr) = \sum\, m_i f_*[Z_i]$. One has to prove that this operation on cycles gives rise to a well-defined operation on Chow groups; we refer to [21], Section 1.4 for this, and for further discussion of the basic properties of f_*. If $g\colon Y \to Z$ is a second proper morphism then $(g \circ f)_* = g_* \circ f_*$.

5.8. Pullback and Gysin homomorphisms. Most of the work in intersection theory goes into defining suitable pullback operations, and proving that these have good properties.

The simplest situation to consider is that of a flat morphism $f\colon X \to Y$ of some fixed relative dimension, say n. In that case we have a pullback operation $f^*\colon \mathscr{Z}_j(Y) \to \mathscr{Z}_{j+n}(X)$ already on the level of cycles, given by the rule $f^*(Z) = [f^{-1}(Z)]$ for a subvariety $Z \subset Y$, where by $f^{-1}(Z)$ we mean the scheme-theoretic inverse image. This operation respects rational equivalence ([21], Theorem 1.7) and therefore induces maps $f^*\colon \mathrm{CH}_j(Y) \to \mathrm{CH}_{j+n}(X)$.

The second situation to consider is that of a regular embedding $f\colon X \hookrightarrow Y$ of some fixed codimension d. (A regular embedding $X \to Y$ of codimension d is a closed immersion such that the ideal sheaf of X in Y is locally generated by a regular sequence of length d.) In this situation, we have well-defined homomorphisms $f^*\colon \mathrm{CH}_j(Y) \to \mathrm{CH}_{j-d}(X)$, called the Gysin homomorphism. The construction is much more involved than in the case of a flat pullback; we refer to [21], Chapter 6 for the details. In fact, the pullback operation that we obtain is more general: still with the same assumptions on f, whenever we have a Cartesian diagram

$$\begin{tikzcd} X' \arrow[r, "f'"] \arrow[d, "g"'] & Y' \arrow[d, "h"] \\ X \arrow[r, "f"] & Y \end{tikzcd} \tag{5.8.1}$$

we obtain a well-defined homomorphism $f^!\colon \mathrm{CH}_j(Y') \to \mathrm{CH}_{j-d}(X')$. Note that this really increases the generality, as the morphism f' may no longer be a regular embedding. It is customary to use the notation f^* for the pullback map $\mathrm{CH}_j(Y) \to \mathrm{CH}_{j-d}(X)$ obtained from a regular embedding and to use the notation $f^!$ in the more general setting provided by a diagram (5.8.1). If f' is again a regular embedding of codimension d (which is automatic if h is flat) then $f^!\colon \mathrm{CH}_j(Y') \to \mathrm{CH}_{j-d}(X')$ is the same as $(f')^*$. (Caution: if f' is a regular embedding of codimension $< d$ then $f^!$ is not the same as $(f')^*$; see the Excess Intersection Formula of [21], Theorem 6.3. This also explains the need of a separate notation $f^!$.)

The two previous cases can be combined to define Gysin maps f^* for morphisms $f\colon X \to Y$ that admit a factorization

$$\begin{tikzcd} X \arrow[r, "i"] \arrow[dr, "f"'] & P \arrow[d, "p"] \\ & Y \end{tikzcd}$$

with i a regular embedding of codimension d for some $d \geq 0$ and p smooth of relative dimension $n+d$ for some $n \in \mathbb{Z}$. Such morphisms are called lci morphisms, where lci stands for 'local complete intersection'. (In fact, this is not quite correct: an lci morphism is a morphism that *locally on* Y admits such a factorization. For our purposes it suffices to consider, as in [21], lci morphisms that globally admit a factorization as above.) In this case, we define $f^*\colon \mathrm{CH}_j(Y) \to \mathrm{CH}_{j+n}(X)$ by the rule $f^* = i^* \circ p^*$. The main point of this definition is that the map f^* thus obtained is independent of the choice of the factorization $f = p \circ i$. Again, the construction gives even more: if we have a Cartesian diagram (5.8.1) with f an lci morphism then we obtain Gysin homomorphisms $f^!\colon \mathrm{CH}_j(Y') \to \mathrm{CH}_{j+n}(X')$. If in this diagram either h is flat or f is a flat lci morphism then f' is an lci morphism (see [50], Lemmas 069I, 069K and 01UI) and $f^! = (f')^*$.

If $f\colon X \to Y$ and $g\colon Y \to Z$ are two lci morphisms then so is $g \circ f$, and $(g \circ f)^* = f^* \circ g^*$. Note that if X and Y are smooth over k, every morphism $f\colon X \to Y$ is lci: consider the factorization

$$X \xrightarrow{(\mathrm{id},f)} X \times Y \xrightarrow{\mathrm{pr}_Y} Y\,.$$

5.9. Behaviour in Cartesian squares. Consider a Cartesian square as in (5.8.1)such that f is an lci morphism. If h (and therefore also g) is a proper morphism then for every $\alpha \in \mathrm{CH}(W)$ we have the relation $f^*h_*(\alpha) = g_*f^!(\alpha)$. If either h is flat or f is a flat lci morphism then $f^!(\alpha) = f^{\prime,*}(\alpha)$ and we have the relation $f^*h_*(\alpha) = g_*f^{\prime,*}(\alpha)$.

5.10. Exterior products. Let X and Y be smooth varieties over k. If $A \subset X$ and $B \subset Y$ are closed subschemes, then $A \times B \subset X \times Y$ is again a closed subscheme, and the class $[A \times B] \in \mathrm{CH}(X \times Y)$ only depends on $\alpha = [A] \in \mathrm{CH}(X)$ and $\beta = [B] \in \mathrm{CH}(Y)$. We write $\alpha \times \beta$ for the class $[A \times B]$. Extending this bilinearly, we obtain a map

$$\mathrm{CH}(X) \times \mathrm{CH}(Y) \to \mathrm{CH}(X \times Y)\,, \qquad (\alpha, \beta) \mapsto \alpha \times \beta\,,$$

and $\alpha \times \beta$ is called the *exterior product* of α and β. If $\alpha \in \mathrm{CH}^i(X)$ and $\beta \in \mathrm{CH}^j(Y)$ then $\alpha \times \beta \in \mathrm{CH}^{i+j}(X \times Y)$.

5.11. Intersection product. Let X be a smooth k-variety. Then $\mathrm{CH}^*(X)$ comes equipped with a natural structure of a commutative graded ring. The multiplication is called the intersection product; as the name suggests, it is closely related to intersections of algebraic cycles. If A, $B \subset X$ are subvarieties that intersect transversally, we have

$$[A] \cdot [B] = [A \cap B]\,,$$

where $A \cap B = A \times_X B$ is the scheme-theoretic intersection of A and B, which is a closed subscheme of X.

To give a simple example, if A and B are subvarieties of X then $A \times X = \mathrm{pr}_1^*(A)$ and $X \times B = \mathrm{pr}_2^*(B)$ intersect transversally and the exterior product $[A] \times [B]$ of their classes is the intersection product $(A \times X) \cdot (X \times B)$.

Of course, cycles do not always intersect transversally (just think of the case $A = B$!). One strategy to define the intersection product in general is based on the moving lemma. If A and B do not intersect transversally, we could try to replace them by equivalent cycles A' and B' (i.e., we move A and B within their equivalence classes) in such a manner that A' and B' do intersect transversally. This strategy works, but it requires some work to prove that the intersection product that is obtained is independent of choices. The definition of the intersection product that is given in [21] uses another strategy and exploits the existence of Gysin homomorphisms. Namely, the assumption that X is smooth over k implies that the diagonal morphism $\Delta\colon X \to X \times_k X$ is an lci morphism (see the final remark in 5.8), and this allows to define the *intersection product* of $\alpha \in \mathrm{CH}^i(X)$ and $\beta \in \mathrm{CH}^j(X)$ by

$$\alpha \cdot \beta = \Delta^*(\alpha \times \beta)\,.$$

The fundamental class $[X] \in \mathrm{CH}^0(X)$ is the identity element for the intersection product.

5.12. Projection formula. If $f\colon X \to Y$ is a proper morphism of smooth k-varieties, we have the projection formula

$$f_*\bigl(\alpha \cdot f^*(\beta)\bigr) = f_*(\alpha) \cdot \beta$$

for all $\alpha \in \mathrm{CH}(X)$ and $\beta \in \mathrm{CH}(Y)$.

5.13. Behaviour under change of base field. If $k \subset L$ is a field extension and X/k is a smooth projective variety, we have a natural homomorphism $i\colon \mathrm{CH}(X) \to \mathrm{CH}(X_L)$. The kernel of this homomorphism is torsion. (See for example [11], Appendix to Chap. 1; the argument given there works in general.) If k is algebraically closed then it follows from a result of Lecomte (see [31], Théorème 3.11) that the map i is in fact injective. The map i is in general very far from surjective; we shal return to this in Section 11.

After this general introduction to Chow groups, we now turn to something that is special for abelian varieties. As a general rule, every morphism of (smooth projective) varieties could give interesting structures or relations on Chow groups via the associated push-forwards and pullback operations. For instance, the group law on an abelian variety X, which is a morphism $m\colon X \times X \to X$, gives rise to a *second* ring structure on $\mathrm{CH}(X)$, defined as follows.

5.14 Definition. Let X/k be an abelian variety. Then we define the *Pontryagin product*

$$\star\colon \mathrm{CH}(X) \times \mathrm{CH}(X) \to \mathrm{CH}(X)$$

by the rule

$$\alpha \star \beta = m_*(\alpha \times \beta)\,.$$

The geometric intuition is that if α is the class of a subvariety $Y \subset X$ and β is the class of a subvariety $Z \subset X$ then $\alpha \star \beta$ should be r times the class of the subvariety $W = \big\{y+z \bigm| y \in Y, z \in Z\big\}$, where r is the generic degree of the addition map $Y \times Z \to W$. If this last map is not generically finite, $\alpha \star \beta = 0$; this happens for instance as soon as $\dim(Y) + \dim(Z) > \dim(X)$.

5.15 Proposition. *The Pontryagin product makes* $\mathrm{CH}(X) = \oplus_i \mathrm{CH}_i(X)$ *into a commutative graded ring with respect to the grading by dimension of cycles. The class* $[e_X] \in \mathrm{CH}_0(X)$ *of the origin is the unit element for the Pontryagin product. If* $f\colon X \to Y$ *is a homomorphism of abelian varieties,* $f_*\colon \mathrm{CH}(X) \to \mathrm{CH}(Y)$ *is a homomorphism of rings for the Pontryagin rings structures.*

5.16. Further reading. As already mentioned, Fulton's book [21] is the standard reference for intersection theory, with [22] as a somewhat lighter alternative. Another source is Chapter 0AZ6 of the Stacks Project [50].

6. Chern characters and Grothendieck–Riemann–Roch

If the notions discussed in this section are new for you, skip the details upon first reading.

As before, k denotes a field.

6.1 Definition. Let $\mathscr{L}$ be a line bundle on a smooth k-variety X. Then we define $c_1(\mathscr{L}) \in \mathrm{CH}^1(X)$, the first Chern class of $\mathscr{L}$, as the corresponding divisor class, i.e., $c_1(\mathscr{L}) = [D]$

for any divisor D such that $\mathscr{L} \cong \mathscr{O}_X(D)$. Further, we define the Chern character $\mathrm{ch}(\mathscr{L}) \in \mathrm{CH}(X)_{\mathbb{Q}}$ by

$$\mathrm{ch}(\mathscr{L}) = \exp\bigl(c_1(\mathscr{L})\bigr) = 1 + c_1(\mathscr{L}) + \frac{c_1(\mathscr{L})^2}{2!} + \frac{c_1(\mathscr{L})^3}{3!} + \cdots$$

Note that $c_1(\mathscr{L})^m \in \mathrm{CH}^m(X)_{\mathbb{Q}}$, so that the sum defining $\mathrm{ch}(\mathscr{L})$ is actually finite.

6.2. Grothendieck groups. Suppose C is a small exact category, i.e., an additive category in which we have a well-behaved notion of exact sequences. In almost all cases, we shall actually be dealing with an abelian category. The *Grothendieck group* $K(\mathsf{C})$ is defined as

$$K(\mathsf{C}) = \mathbb{Z}^{(\mathsf{Obj}(\mathsf{C}))}/\!\sim,$$

where $\mathbb{Z}^{(\mathsf{Obj}(\mathsf{C}))}$ is the free abelian group on the set of objects of C, and where $\sim$ is the equivalence relation that is generated by the relations $[B] \sim [A] + [C]$ whenever we have a short exact sequence $0 \longrightarrow A \longrightarrow B \longrightarrow C \longrightarrow 0$. For instance, if R is a ring and R-mod is the category of finitely generated (left) R-modules, $K(R\text{-}\mathsf{mod})$ is the free abelian group on the set of isomorphism classes of simple R-modules of finite type.

Grothendieck groups can also be defined for triangulated categories (such as derived categories). Namely, if D is a triangulated category, define

$$K(\mathsf{D}) = \mathbb{Z}^{(\mathsf{Obj}(\mathsf{D}))}/\!\sim,$$

where now $\sim$ is the equivalence relation that is generated by the relations $[B] \sim [A] + [C]$ whenever we have a distinguished triangle $A \to B \to C \to$. It can be shown that if A is an abelian category, we have a natural isomorphism $K(\mathsf{A}) \xrightarrow{\sim} K\bigl(\mathrm{D^b}(\mathsf{A})\bigr)$. If X is an object of a triangulated category D, we have $\bigl[X[1]\bigr] = -[X]$ in $K(\mathsf{D})$ because we have a distinguished triangle $X \to 0 \to X[1] \to$.

6.3. Let X be a smooth quasiprojective k-variety. We denote by $K^0(X)$ the Grothendieck group of the category of vector bundles on X. For a vector bundle V, let $[V]$ denote its class in $K^0(X)$. By construction, $K^0(X)$ is an abelian group; it has the structure of a commutative ring given by taking tensor products, i.e., $[V] \cdot [W] = [V \otimes W]$.

We can also consider the Grothendieck group $K_0(X)$ of the category $\mathsf{Coh}(X)$ of coherent sheaves on X. There is a natural homomorphism $K^0(X) \to K_0(X)$, sending the class of a vector bundle V to the class of V viewed as a coherent sheaf. This map is an isomorphism; this uses in an essential way that X is regular, as the key idea is that every coherent $\mathscr{O}_X$-module has a finite resolution by vector bundles.

In what follows, we identify $K^0(X)$ and $K_0(X)$, and we simply write $K(X)$ for this ring. By the remark made at the end of 6.2, *$K(X)$ can also be described as the Grothendieck ring of the derived category* $\mathrm{D^b}(X) = \mathrm{D^b}\bigl(\mathsf{Coh}(X)\bigr)$.

A morphism $f\colon X \to Y$ of quasi-projective smooth k-varieties induces a ring homomorphism $f^*\colon K(Y) \to K(X)$. If f is proper it also induces an additive map $Rf_*\colon K(X) \to K(Y)$ by sending the class of a coherent $\mathscr{O}_X$-module $\mathscr{F}$ to

$$Rf_*[\mathscr{F}] = \sum_{i \geq 0} (-1)^i \cdot \bigl[R^i f_*(\mathscr{F})\bigr].$$

(To understand why this is the right definition, use the interpretation $K(X) = K\big(\mathrm{D}^{\mathrm{b}}(X)\big)$.)

6.4. If V is a vector bundle of rank r on a smooth k-variety X, we can define higher Chern classes $c_i(V) \in \mathrm{CH}^i(X)$ and a Chern character $\mathrm{ch}(V) \in \mathrm{CH}(X)_{\mathbb{Q}}$. These classes capture essential geometric properties of V. We first discuss the Chern character. We already know what $\mathrm{ch}(\mathscr{L})$ is, for $\mathscr{L}$ a line bundle. One of the main properties of the Chern character is that it will define a ring homomorphism

$$\mathrm{ch}\colon K(X)_{\mathbb{Q}} \to \mathrm{CH}(X)_{\mathbb{Q}}$$

which for a morphism $f\colon X \to Y$ of smooth quasi-projective varieties is compatible with pullbacks, in the sense that the diagram

$$\begin{tikzcd} K(Y)_{\mathbb{Q}} \arrow[r, "\mathrm{ch}"] \arrow[d, "f^*"'] & \mathrm{CH}(Y)_{\mathbb{Q}} \arrow[d, "f^*"] \\ K(X)_{\mathbb{Q}} \arrow[r, "\mathrm{ch}"] & \mathrm{CH}(X)_{\mathbb{Q}} \end{tikzcd}$$

is commutative. The standard way to define $\mathrm{ch}(V)$ is to pass to a situation where V becomes an iterated extension of line bundles. Namely, given a rank r vector bundle V on X, we can consider the flag variety $\pi\colon \mathrm{Flag}(V) \to X$ whose fibre over a point $x \in X$ is the variety of (complete) flags

$$F_0 = (0) \subsetneq F_1 \subsetneq \cdots \subsetneq F_{r-1} \subsetneq F_r = V(x)\,.$$

By construction, $\pi^*(V)$ then has a filtration

$$\mathscr{F}_0 = (0) \subsetneq \mathscr{F}_1 \subsetneq \cdots \subsetneq \mathscr{F}_{r-1} \subsetneq \mathscr{F}_r = \pi^*(V)$$

by sub-bundles such that each $\mathscr{L}_j = \mathscr{F}_j/\mathscr{F}_{j-1}$ is a line bundle on $\mathrm{Flag}(V)$. According to the properties that we want the Chern character to have, we should have the relation

$$\pi^*\mathrm{ch}(V) = \mathrm{ch}\big(\pi^*(V)\big) = \sum_{j=1}^{r} \mathrm{ch}(\mathscr{L}_j) \tag{6.4.1}$$

in $\mathrm{CH}(\mathrm{Flag}(V))_{\mathbb{Q}}$. (Note that $\pi^*(V) = \sum \mathscr{L}_j$ in $K(\mathrm{Flag}(V))$.) This indeed works: the homomorphism $\pi^*\colon \mathrm{CH}(X)_{\mathbb{Q}} \to \mathrm{CH}(\mathrm{Flag}(V))_{\mathbb{Q}}$ is injective, and one can prove that the class $\sum_{j=1}^{r} \mathrm{ch}(\mathscr{L}_j)$ lies in the image; this then allows to define $\mathrm{ch}(V) \in \mathrm{CH}(X)_{\mathbb{Q}}$ as the unique class that satisfies (6.4.1).

Note that (6.4.1) can be written as the relation

$$\mathrm{ch}(V) = \sum_{j=1}^{r} \exp(\alpha_j)\,,$$

where $\alpha_j = c_1(\mathscr{L}_j)$. This is a relation in $\mathrm{CH}(\mathrm{Flag}(V))_{\mathbb{Q}}$, which contains $\mathrm{CH}(X)_{\mathbb{Q}}$ as a subring (via π^*). The classes α_j are called the *Chern roots* of V. All symmetric expressions in the α_j lie in $\mathrm{CH}(X)_{\mathbb{Q}}$.

While for our purposes the Chern character is the most interesting invariant, let us briefly mention that for a rank r vector bundle we can define *integral* characteristic classes $c_i(V) \in \mathrm{CH}^i(X)$, called the Chern classes of X, from which we can recover the Chern character. (See [21], Example 3.2.3.) With notation as above, the homomorphism $\pi^*\colon \mathrm{CH}(X) \to \mathrm{CH}(\mathrm{Flag}(V))$ (with integral coefficients) is again injective, and the Chern classes $c_i(V)$ are defined by the relation $\pi^*(c_i(V)) = \sigma_i(\alpha_1, \ldots, \alpha_r)$, where σ_i is the ith elementary symmetric polynomial. (In fact, the map $t_j \mapsto \alpha_j$ gives an isomorphism

$$\mathrm{CH}(X)[t_1, \ldots, t_r]/I \xrightarrow{\sim} \mathrm{CH}(\mathrm{Flag}(V)),$$

where $I \subset \mathrm{CH}(X)[t_1, \ldots, t_r]$ is the ideal generated by all elements $\sigma_i(t_1, \ldots, t_r) - c_i(V)$.)

An important insight is that the formation of Chern characters does not, in general, commute with push-forwards. (See however 6.6 for a special case where the two operations do commute.) Grothendieck's version of the Riemann–Roch theorem is the assertion that we do obtain a compatibility with push-forwards if we correct the Chern character by a factor that only depends on the underlying variety, called the *Todd class*. In general, if V is a rank r vector bundle on X with Chern roots $\alpha_1, \ldots, \alpha_r$, the Todd class of V is the class $\mathrm{Td}(V) \in \mathrm{CH}(X)_{\mathbb{Q}}$ defined by

$$\mathrm{Td}(V) = \prod_{j=1}^{r} \frac{\alpha_j}{1 - \exp(-\alpha_j)} = \prod_{j=1}^{r} \Bigl(1 + \tfrac{1}{2}\alpha_j + \sum_{m=1}^{\infty} (-1)^{m-1} \tfrac{B_m}{(2m)!} \alpha_j^{2m}\Bigr),$$

with $B_m = m$th Bernoulli number. Note that the right hand side (which a priori only defines a class in $\mathrm{CH}(\mathrm{Flag}(V))_{\mathbb{Q}}$) is a symmetric expression in the Chern roots, and therefore indeed defines a class in $\mathrm{CH}(X)_{\mathbb{Q}}$. These Todd classes turn out to be important. In particular, if X is a smooth k-variety, say of dimension d, its tangent bundle $\mathscr{T}_X$ is a vector bundle of rank d and we have a Todd class $\mathrm{Td}(\mathscr{T}_X) \in \mathrm{CH}(X)_{\mathbb{Q}}$. With this notation, the Grothendieck–Riemann–Roch theorem is the following result.

6.5 Theorem (Grothendieck–Riemann–Roch)**.** *Let X and Y be smooth projective varieties over k. If $f\colon X \to Y$ is a morphism, the diagram*

$$\begin{tikzcd} K(X)_{\mathbb{Q}} \arrow[r, "\alpha \mapsto \mathrm{ch}(\alpha)\cdot\mathrm{Td}(\mathscr{T}_X)"] \arrow[d, "f_*"'] & \mathrm{CH}(X)_{\mathbb{Q}} \arrow[d, "f_*"] \\ K(Y)_{\mathbb{Q}} \arrow[r, "\beta \mapsto \mathrm{ch}(\beta)\cdot\mathrm{Td}(\mathscr{T}_Y)"'] & \mathrm{CH}(Y)_{\mathbb{Q}} \end{tikzcd}$$

is commutative.

Because abelian varieties have trivial tangent sheaf, the following is an immediate (but for us important) consequence:

6.6 Corollary. *If $f\colon X \to Y$ is a homomorphism of abelian varieties then for every quasi-coherent $\mathscr{O}_X$-module $\mathscr{F}$ we have the relation*

$$f_*\bigl(\mathrm{ch}[\mathscr{F}]\bigr) = \mathrm{ch}\bigl(Rf_*[\mathscr{F}]\bigr).$$

PART 2

Chow groups of abelian varieties

7. Correspondences

Let X and Y be smooth projective varieties over a field k. If $f\colon X \to Y$ is a morphism, we have associated maps f^* and f_* on Chow groups. A very important insight that lies at the basis of the theory of motives, is that we can vastly generalize this, as follows.

7.1 Definition. Let X and Y be smooth projective varieties over k. For $\alpha \in \mathrm{CH}(X \times Y)$, define $\alpha_*\colon \mathrm{CH}(X) \to \mathrm{CH}(Y)$ by the rule

$$\alpha_*(x) = \mathrm{pr}_{Y,*}\bigl(\alpha \cdot \mathrm{pr}_X^*(x)\bigr)\,,$$

where $X \xleftarrow{\mathrm{pr}_X} X \times Y \xrightarrow{\mathrm{pr}_Y} Y$ are the projection maps. Let ${}^{\mathrm{t}}\alpha \in \mathrm{CH}(Y \times X)$ be the transpose of α (i.e., ${}^{\mathrm{t}}\alpha = \sigma^*(\alpha)$, where $\sigma\colon Y \times X \xrightarrow{\sim} X \times Y$ is the isomorphism that exchanges the factors). Then we define $\alpha^*\colon \mathrm{CH}(Y) \to \mathrm{CH}(X)$ to be the map $({}^{\mathrm{t}}\alpha)_*$.

Check for yourself that if we take $\alpha = [\Gamma_f]$ for some morphism $f\colon X \to Y$, we recover the maps f_* and f^* on Chow groups.

7.2. If we think of classes in $\mathrm{CH}(X \times Y)$ as being generalizations of maps from X to Y, we also want to understand how composition of such 'generalized maps' works. It is helpful to first introduce some terminology. If X and Y are smooth projective over k, we call elements of $\mathrm{CH}(X \times Y)$ *correspondences* from X to Y, and we define $\mathrm{Corr}(X,Y) = \mathrm{CH}(X \times Y)$. We shall later usually work with $\mathbb{Q}$-coefficients, writing $\mathrm{Corr}(X,Y)_{\mathbb{Q}}$ for $\mathrm{Corr}(X,Y) \otimes \mathbb{Q}$.

We introduce a grading by setting

$$\mathrm{Corr}^i(X,Y) = \mathrm{CH}^{\dim(X)+i}(X \times Y) = \mathrm{CH}_{\dim(Y)-i}(X \times Y)\,.$$

Note that if $f\colon X \to Y$ is a morphism, the class $[{}^{\mathrm{t}}\Gamma_f]$ of the transposed graph is a correspondence from Y to X of degree 0. The way to remember this is that the induced map $f^* = [{}^{\mathrm{t}}\Gamma_f]_*$ on Chow groups preserves the gradings by codimension of cycles. (See below for the general case.) By definition,

$$\mathrm{Corr}(X,Y) = \bigoplus_{i\in\mathbb{Z}} \mathrm{Corr}^i(X,Y)\,.$$

If Z is a third smooth projective variety over k, we define a composition

$$\circ\colon \mathrm{Corr}(Y,Z) \times \mathrm{Corr}(X,Y) \longrightarrow \mathrm{Corr}(X,Z)$$

by the rule

$$\beta \circ \alpha = \mathrm{pr}_{XZ,*}\bigl(\mathrm{pr}_{XY}^*(\alpha) \cdot \mathrm{pr}_{YZ}^*(\beta)\bigr)$$

(for $\alpha \in \mathrm{Corr}(X,Y)$) and $\beta \in \mathrm{Corr}(Y,Z)$), where the relevant diagram of spaces and maps is

$$\begin{tikzcd} & X \times Y \times Z \arrow[dl, "\mathrm{pr}_{XY}"'] \arrow[d, "\mathrm{pr}_{XZ}"] \arrow[dr, "\mathrm{pr}_{YZ}"] & \\ X \times Y & X \times Z & Y \times Z \end{tikzcd}$$

We shall also need an extension of these definitions to the case where X and Y are smooth projective over k that are not necessarily irreducible. If $X = X_1 \sqcup \cdots \sqcup X_t$ and $Y = Y_1 \sqcup \cdots \sqcup Y_u$ are the decompositions into irreducble components, we then define $\mathrm{Corr}(X,Y) = \oplus_{i,j} \mathrm{Corr}(X_i, Y_j)$. A correspondence $\gamma = (\gamma_{i,j})$ is then said to be homogeneous of degree d if each component $\gamma_{i,j}$ is homogeneous of degree d.

7.3. Basic facts. All varieties that are considered are assumed to be smooth projective over k.

- The composition of correspondences is compatible with the grading, in the sense that it restricts to maps $\mathrm{Corr}^i(Y,Z) \times \mathrm{Corr}^j(X,Y) \longrightarrow \mathrm{Corr}^{i+j}(X,Z)$.
- For $\alpha \in \mathrm{Corr}(X,Y)$ and $\beta \in \mathrm{Corr}(Y,Z)$ we have the relations $(\beta \circ \alpha)_* = \beta_* \circ \alpha_*$ and $(\beta \circ \alpha)^* = \alpha^* \circ \beta^*$.
- If $\alpha \in \mathrm{Corr}^i(X,Y)$ then the induced map on Chow groups is homogeneous of degree i, by which we mean that it restricts to maps $\alpha_* \colon \mathrm{CH}^j(X) \to \mathrm{CH}^{i+j}(Y)$. (Remember that if $f \colon X \to Y$ is a morphism, $f^* = \alpha_*$ with $\alpha = [{}^{\mathrm{t}}\Gamma_\alpha] \in \mathrm{Corr}^0(Y,X)$.)
- The k-vector space $\mathrm{Corr}(X,X)$ with composition as defined above is a graded k-algebra (non-commutative in general). Further, composition makes $\mathrm{Corr}(X,Y)$ a graded $\mathrm{Corr}(X,X)$-$\mathrm{Corr}(Y,Y)$-bimodule.

The following result, which is [17], Proposition 1.2.1, gives some useful formulas that relate compositions of correspondences to the usual push-forward and pullback operations on Chow groups. We state the result as identities between Chow classes. (Recall that $\mathrm{Corr}(X,Y)$ is just another name for $\mathrm{CH}(X \times Y)$.)

7.4 Proposition. *Let X, Y and Z be smooth projective k-varieties.*

(1) *For $\alpha \in \mathrm{CH}(X \times Y)$ and $f \colon Y \to Z$ we have $[\Gamma_f] \circ \alpha = (\mathrm{id}_X \times f)_*(\alpha)$ in $\mathrm{CH}(X \times Z)$.*
(2) *For $\beta \in \mathrm{CH}(Y \times Z)$ and $f \colon X \to Y$ we have $\beta \circ [\Gamma_f] = (f \times \mathrm{id}_Z)^*(\beta)$ in $\mathrm{CH}(X \times Z)$.*
(3) *For $\alpha \in \mathrm{CH}(X \times Y)$ and $g \colon Z \to Y$ we have $[{}^{\mathrm{t}}\Gamma_g] \circ \alpha = (\mathrm{id}_X \times g)^*(\alpha)$ in $\mathrm{CH}(X \times Z)$.*
(4) *For $\beta \in \mathrm{CH}(Y \times Z)$ and $g \colon Y \to X$ we have $\beta \circ [{}^{\mathrm{t}}\Gamma_g] = (g \times \mathrm{id}_Z)_*(\beta)$ in $\mathrm{CH}(X \times Z)$.*

Using the language of correspondences, we can also give a precise definition of algebraic equivalence of Chow classes, which, in a somewhat informal way, was already mentioned in Section 5.2.

7.5 Definition. Let X be a smooth variety over a field k. *Algebraic equivalence*, notation $\sim_{\mathrm{alg}}$, is the equivalence relation on $\mathrm{CH}(X)$ that is generated by all relations $\gamma_*(\alpha) \sim 0$, where $\gamma \in \mathrm{CH}(C \times X)$ for some smooth projective curve C/k and $\alpha \in \mathrm{CH}_0(C)$ is a class of degree 0.

The geometric intuition is that γ is a family of cycles $\gamma(P)$ on X parametrized by the points $P \in C$, and that in the basic case where $\alpha = (P) - (Q)$ for some points P, $Q \in C(k)$, the class $\gamma_*(\alpha)$ is just the difference $\gamma(P) - \gamma(Q)$.

7.6 Proposition. *Let X be a smooth variety over an algebraically closed field k. Then*

$$\mathrm{CH}(X)_{\mathrm{alg}} = \bigl\{\alpha \in \mathrm{CH}(X) \bigm| \alpha \sim_{\mathrm{alg}} 0\bigr\}$$

is a divisible group, i.e., for every $n > 0$ the map $\mathrm{CH}(X)_{\mathrm{alg}} \to \mathrm{CH}(X)_{\mathrm{alg}}$ given by multiplication by n is surjective.

Proof. If C/k is a smooth projective curve, its Jacobian $J = \mathrm{Pic}^0_{C/k}$ is an abelian variety and the group $\mathrm{CH}_0(C)_0$ of 0-cycles of degree 0 is the same as $J(k)$. The assertion now follows from the definition of algebraic equivalence, using that the multiplication by n map $[n]_J \colon J(k) \to J(k)$ is surjective. (Cf. Section 1.5.) $\square$

8. Fourier duality and Beauville decomposition

Throughout, we work over some field k. We consider Chow groups with $\mathbb{Q}$-coefficients; recall that $\mathrm{CH}(X)_{\mathbb{Q}}$ means $\mathrm{CH}(X) \otimes \mathbb{Q}$.

8.1. Let X be an abelian variety with dual X^t. We identify X^{tt} with X via the isomorphism $\kappa \colon X \xrightarrow{\sim} X^{tt}$ of Theorem 4.6. Let $\mathscr{P}$ be the Poincaré bundle of X, which is a line bundle on $X \times X^t$. The Poincaré bundle of X^t is then given by $\mathscr{P}^t = \sigma^*\mathscr{P}$, where $\sigma \colon X^t \times X \xrightarrow{\sim} X \times X^t$ is the isomorphism that switches the factors. Consider the three projection maps

$$\begin{tikzcd}
 & X \times X^t \times X \arrow[dl, "\mathrm{pr}_{12}"'] \arrow[d, "\mathrm{pr}_{13}"] \arrow[dr, "\mathrm{pr}_{23}"] & \\
X \times X^t & X \times X & X^t \times X
\end{tikzcd}$$

and let $(m_{13} \times \mathrm{id}) \colon X \times X^t \times X \to X \times X^t$ be the map $(x_1, \xi, x_3) \mapsto (x_1 + x_3, \xi)$.

8.2 Lemma. *We have the relation*

$$\mathrm{pr}_{12}^*\mathscr{P} \otimes \mathrm{pr}_{23}^*\mathscr{P}^t \cong (m_{13} \times \mathrm{id})^*\mathscr{P}\,.$$

Proof. When restricted to the faces

$$\{0\} \times X^t \times X\,, \qquad X \times \{0\} \times X\,, \qquad X \times X^t \times \{0\}\,,$$

the LHS and the RHS give the same. Now apply Theorem 2.2. $\square$

We now get to the first main result of these notes. This result was first proven by Mukai [34] in the setting of derived categories. The result as stated here, in the setting of Chow groups, is an immediate consequence. Beauville, in [4] and [5], further studied this and gave important applications; see in particular Theorem 8.6 below.

8.3 Theorem (Fourier duality)**.** *Let X/k be a g-dimensional abelian variety. Define*

$$\mathscr{F} = \mathrm{ch}(\mathscr{P})_*\colon \mathrm{CH}(X)_\mathbb{Q} \to \mathrm{CH}(X^t)_\mathbb{Q}\,, \qquad \mathscr{F}^t = \mathrm{ch}(\mathscr{P}^t)_*\colon \mathrm{CH}(X^t)_\mathbb{Q} \to \mathrm{CH}(X)_\mathbb{Q}\,.$$

Then

$$\mathscr{F}^t \circ \mathscr{F} = (-1)^g \cdot [-1]_X^*\,, \qquad \mathscr{F} \circ \mathscr{F}^t = (-1)^g \cdot [-1]_{X^t}^*\,.$$

In particular, $\mathscr{F}$ and $\mathscr{F}^t$ are isomorphisms.

Concretely, for a class $\alpha \in \mathrm{CH}(X)_\mathbb{Q}$ its Fourier transform is given by

$$\mathscr{F}(\alpha) = \mathrm{pr}_{X^t,*}\bigl(\mathrm{pr}_X^*(\alpha)\cdot \mathrm{ch}(\mathscr{P})\bigr)\,,$$

and for $\mathscr{F}^t$ we have the analogous formula. If the context requires it, we use the notation $\mathscr{F}_X$. (So $\mathscr{F}^t = \mathscr{F}_{X^t}$.)

Proof. By symmetry (exchange the roles of X and X^t) it suffices to prove the result about $\mathscr{F}^t \circ \mathscr{F}$. By the basic facts stated in 7.3, the composition $\mathscr{F}^t \circ \mathscr{F}\colon \mathrm{CH}(X)_\mathbb{Q} \to \mathrm{CH}(X)_\mathbb{Q}$ is induced by the correspondence

$$\begin{aligned}
\mathrm{pr}_{13,*}\Bigl(\mathrm{pr}_{12}^*\mathrm{ch}(\mathscr{P})\cdot \mathrm{pr}_{23}^*\mathrm{ch}(\mathscr{P}^t)\Bigr) &= \mathrm{pr}_{13,*}\Bigl(\mathrm{ch}(\mathrm{pr}_{12}^*\mathscr{P})\cdot \mathrm{ch}(\mathrm{pr}_{23}^*\mathscr{P}^t)\Bigr)\\
&= \mathrm{pr}_{13,*}\Bigl(\mathrm{ch}\bigl(\mathrm{pr}_{12}^*\mathscr{P}\otimes \mathrm{pr}_{23}^*\mathscr{P}^t\bigr)\Bigr)\\
&\overset{8.2}{=} \mathrm{pr}_{13,*}\Bigl(\mathrm{ch}\bigl((m_{13}\times \mathrm{id})^*\mathscr{P}\bigr)\Bigr)\\
&= \mathrm{pr}_{13,*}\Bigl((m_{13}\times \mathrm{id})^*\mathrm{ch}(\mathscr{P})\Bigr)\,.
\end{aligned}$$

Because the diagram

$$\begin{tikzcd}
X\times X^t\times X \arrow[r, "m_{13}\times\mathrm{id}"] \arrow[d, "\mathrm{pr}_{13}"'] & X\times X^t \arrow[d, "\mathrm{pr}_1"]\\
X\times X \arrow[r, "m"'] & X
\end{tikzcd}$$

is Cartesian, we find that the given correspondence equals $m^*\mathrm{pr}_{1,*}\bigl(\mathrm{ch}(\mathscr{P})\bigr)$. (Note that $m\colon X\times X\to X$ is a smooth morphism, and in particular is therefore a flat lci morphism; so we can use what was explained in Section 5.9.) At this point we use Corollary 6.6, together with Theorem 4.14. This gives that

$$\mathrm{pr}_{1,*}\bigl(\mathrm{ch}(\mathscr{P})\bigr) = \mathrm{ch}\bigl(R\mathrm{pr}_{1,*}(\mathscr{P})\bigr) = (-1)^g\cdot \mathrm{ch}\bigl(e_*\mathscr{O}_{\mathrm{Spec}(k)}\bigr) = (-1)^g\cdot e_*\bigl[\mathrm{Spec}(k)\bigr]\,.$$

Because the diagram

$$\begin{tikzcd}
\Gamma_{[-1]} \arrow[r, hook] \arrow[d] & X\times X \arrow[d, "m"]\\
\mathrm{Spec}(k) \arrow[r, "e"'] & X
\end{tikzcd}$$

is Cartesian (with $\Gamma_{[-1]}$ = graph of $[-1]_X$ = anti-diagonal in $X\times X$), we find that $\mathscr{F}^t\circ\mathscr{F}$ is given by the correspondence $(-1)^g\cdot m^*e_*\bigl[\mathrm{Spec}(k)\bigr] = (-1)^g\cdot \Gamma_{[-1]}$, as claimed. (Again use that m is a flat lci morphism and note that e is proper; then apply what was explained in Section 5.9.)

For the final assertion, we just have to note that $[-1]_X^*$ is its own inverse. $\square$

The following proposition summarizes some key properties of the Fourier transform.

8.4 Proposition. *Let X be an abelian variety of dimension g.*

(1) *For x, $y \in \mathrm{CH}(X)_\mathbb{Q}$ we have the relations*

$$\mathscr{F}(x \star y) = \mathscr{F}(x) \cdot \mathscr{F}(y)\,, \qquad \mathscr{F}(x \cdot y) = (-1)^g \cdot \mathscr{F}(x) \star \mathscr{F}(y)\,.$$

(2) *If $f\colon X \to Y$ is a homomorphism of abelian varieties with dual $f^t\colon Y^t \to X^t$, we have the relations*

$$(f^t)^* \circ \mathscr{F}_X = \mathscr{F}_Y \circ f_*\colon \mathrm{CH}(X)_\mathbb{Q} \to \mathrm{CH}(Y^t)_\mathbb{Q}$$

and

$$\mathscr{F}_X \circ f^* = (-1)^{\dim(X)-\dim(Y)} \cdot (f^t_*) \circ \mathscr{F}_Y\colon \mathrm{CH}(Y)_\mathbb{Q} \to \mathrm{CH}(X^t)_\mathbb{Q}\,.$$

Proof. (1) Write $X \xleftarrow{\mathrm{pr}_1} X \times X^t \xrightarrow{\mathrm{pr}_2} X^t$ for the projection maps. Consider $X \times X \times X^t$, let π_i ($i = 1, 2, 3$) be the projection onto the ith factor, and let π_{ij} be the projection onto the factors i and j; for instance, $\pi_{13}\colon X \times X \times X^t \to X \times X^t$ is the map $(P_1, P_2, Q) \mapsto (P_1, Q)$. With this notation, the diagram

$$\begin{tikzcd} & X \times X \times X^t \arrow[dl, "\pi_{12}"'] \arrow[dr, "m\times\mathrm{id}"] & \\ X \times X \arrow[dr, "m"'] & & X \times X^t \arrow[dl, "\mathrm{pr}_1"] \\ & X & \end{tikzcd}$$

is Cartesian, so $\mathrm{pr}_1^*(x \star y) = (m \times \mathrm{id})_*\pi_{12}^*(x \times y)$. The projection formula then gives

$$\mathrm{pr}_1^*(x \star y) \cdot \mathrm{ch}(\mathscr{P}) = (m \times \mathrm{id})_*\Big(\pi_{12}^*(x \times y) \cdot (m \times \mathrm{id})^*\mathrm{ch}(\mathscr{P})\Big)\,.$$

As $(m \times \mathrm{id})^*\mathscr{P} \cong \pi_{13}^*\mathscr{P} \otimes \pi_{23}^*\mathscr{P}$ and $\pi_{12}^*(x \times y) = \pi_1^*(x) \cdot \pi_2^*(y)$, this becomes

$$\mathrm{pr}_1^*(x \star y) \cdot \mathrm{ch}(\mathscr{P}) = (m \times \mathrm{id})_*\Big(\pi_1^*(x) \cdot \pi_{13}^*\big(\mathrm{ch}(\mathscr{P})\big) \cdot \pi_2^*(y) \cdot \pi_{23}^*\big(\mathrm{ch}(\mathscr{P})\big)\Big)\,.$$

By definition, $\mathscr{F}(x \star y)$ is the element of $\mathrm{CH}(X^t)$ that we obtain by applying $\mathrm{pr}_{2,*}$ to this. We have $\mathrm{pr}_2 \circ (m \times \mathrm{id}) = \pi_3 = \mathrm{pr}_2 \circ \pi_{13}$; further, $\pi_1^*(x) \cdot \pi_{13}^*\big(\mathrm{ch}(\mathscr{P})\big) = \pi_{13}^*\big(\mathrm{pr}_1^*(x) \cdot \mathrm{ch}(\mathscr{P})\big)$, and $\pi_2^*(y) \cdot \pi_{23}^*\big(\mathrm{ch}(\mathscr{P})\big) = \pi_{23}^*\big(\mathrm{pr}_1^*(y) \cdot \mathrm{ch}(\mathscr{P})\big)$. Again using the projection formula we find

$$\mathscr{F}(x \star y) = \mathrm{pr}_{2,*}\Big(\big(\mathrm{pr}_1^*(x) \cdot \mathrm{ch}(\mathscr{P})\big) \cdot \pi_{13,*}\big(\pi_{23}^*(\mathrm{pr}_1^*(y) \cdot \mathrm{ch}(\mathscr{P}))\big)\Big)$$

Because the diagram

$$\begin{tikzcd} & X \times X \times X^t \arrow[dl, "\pi_{13}"'] \arrow[dr, "\pi_{23}"] & \\ X \times X^t \arrow[dr, "\mathrm{pr}_2"'] & & X \times X^t \arrow[dl, "\mathrm{pr}_2"] \\ & X^t & \end{tikzcd}$$

is Cartesian we have $\pi_{13,*}\big(\pi_{23}^*(\mathrm{pr}_1^*(y)\cdot \mathrm{ch}(\mathscr{P}))\big) = \mathrm{pr}_2^*\big(\mathrm{pr}_{2,*}(\mathrm{pr}_1^*(y)\cdot \mathrm{ch}(\mathscr{P}))\big) = \mathrm{pr}_2^*\big(\mathscr{F}(y)\big)$, and then (once again using the projection formula)

$$\mathscr{F}(x\star y) = \mathrm{pr}_{2,*}\big(\mathrm{pr}_1^*(x)\cdot \mathrm{ch}(\mathscr{P})\big)\cdot \mathscr{F}(y) = \mathscr{F}(x)\cdot \mathscr{F}(y)\ .$$

For the second identity, we may write $x = \mathscr{F}^t(\alpha)$ and $y = \mathscr{F}^t(\beta)$ for some α, $\beta \in \mathrm{CH}(X^t)$, so that $\mathscr{F}(x) = (-1)^g\cdot [-1]_{X^t}^*(\alpha) = (-1)^g\cdot [-1]_{X^t,*}(\alpha)$, and similarly $\mathscr{F}(y) = (-1)^g\cdot [-1]_{X^t,*}(\beta)$. (Note that $[-1]_* = [-1]^*$ because $[-1]$ is its own inverse.) The identity just proven then gives

$$\begin{aligned}\mathscr{F}(x\cdot y) &= (-1)^g\cdot [-1]_{X^t,*}(\alpha\star\beta)\\ &= (-1)^g\cdot \big([-1]_{X^t,*}(\alpha)\big)\star\big([-1]_{X^t,*}(\beta)\big)\\ &= (-1)^g\cdot \mathscr{F}(x)\cdot\mathscr{F}(y)\ .\end{aligned}$$

(2) By (4.3.1) we have $(f,\mathrm{id}_Y)^*\mathrm{ch}(\mathscr{P}_Y) = (\mathrm{id}_X, f^t)^*\mathrm{ch}(\mathscr{P}_X)$ in $\mathrm{CH}(X\times Y^t)_{\mathbb{Q}}$. By Proposition 7.4 we can write this as $\mathrm{ch}(\mathscr{P}_Y)\circ[\Gamma_f] = [{}^{\mathrm{t}}\Gamma_{f^t}]\circ \mathrm{ch}(\mathscr{P}_X)$, which gives the first formula. Using Theorem 8.3, it follows that

$$(-1)^{\dim(X)}\cdot\mathscr{F}_{Y^t}\circ(f^t)^*\circ[-1]_{X^t}^* = (-1)^{\dim(Y)}\cdot[-1]_Y^*\circ f_*\circ\mathscr{F}_{X^t} = (-1)^{\dim(Y)}\cdot[-1]_{Y,*}\circ f_*\circ\mathscr{F}_{X^t}$$

If we apply this to $-f^t$ instead of f, we obtain the second formula. $\square$

A remarkable feature of the Fourier transform—which turns out to be very useful—is that it is *not* compatible with gradings. In other words, it does not, in general, send homogeneous elements in $\mathrm{CH}(X)_{\mathbb{Q}}$ to homogeneous elements in $\mathrm{CH}(X^t)_{\mathbb{Q}}$. To analyze this more precisely, let us set up some notation.

8.5 Definition. Let X be an abelian variety of dimension g. For integers i and s define

$$\mathrm{CH}^i_{(s)}(X) = \big\{x\in \mathrm{CH}^i(X)_{\mathbb{Q}}\ \big|\ [n]_X^*(x) = n^{2i-s}\cdot x \text{ for all } n\in\mathbb{Z}\big\}\,.$$

Further, we define

$$\mathrm{CH}_{i,(s)}(X) = \mathrm{CH}^{g-i}_{(s)}(X) = \big\{x\in \mathrm{CH}_i(X)_{\mathbb{Q}}\ \big|\ [n]_{X,*}(x) = n^{2i+s}\cdot x \text{ for all } n\in\mathbb{Z}\big\}\,.$$

> Whenever we use the notation $\mathrm{CH}^i_{(s)}(X)$ or $\mathrm{CH}_{i,(s)}(X)$, remember that these are subspaces of $\mathrm{CH}^i(X)_{\mathbb{Q}}$, resp. $\mathrm{CH}_i(X)_{\mathbb{Q}}$.

At first sight, the chosen indexing does not seem very convenient. The convenience of the notation will become clearer only once we draw a suitable diagram, as we shall explain in Section 8.8. First, however, we state and prove the second main result of this section, which concerns the Beauville decomposition of Chow groups. This was introduced by Beauville in [5].

8.6 Theorem. *Let X be an abelian variety of dimension g.*

(1) *We have a decomposition (called the Beauville decomposition)*

$$\mathrm{CH}(X)_{\mathbb{Q}} = \bigoplus_{i,s\in\mathbb{Z}} \mathrm{CH}^i_{(s)}(X)\,,$$

and this makes $\mathrm{CH}(X)_{\mathbb{Q}}$ *a bigraded ring with respect to the intersection product, i.e.,*

$$\mathrm{CH}^i_{(s)}(X)\cdot \mathrm{CH}^j_{(t)}(X) \subset \mathrm{CH}^{i+j}_{(s+t)}(X)\,.$$

Viewed as a decomposition

$$\mathrm{CH}(X)_{\mathbb{Q}} = \bigoplus_{i,s\in\mathbb{Z}} \mathrm{CH}_{i,(s)}(X)\,,$$

this makes $\mathrm{CH}(X)_{\mathbb{Q}}$ *a bigraded ring with respect to the Pontryagin product, i.e.,*

$$\mathrm{CH}_{i,(s)}(X)\star \mathrm{CH}_{j,(t)}(X) \subset \mathrm{CH}_{i+j,(s+t)}(X)\,.$$

(2) *The Fourier transform* $\mathscr{F}_X\colon \mathrm{CH}(X)_{\mathbb{Q}} \xrightarrow{\sim} \mathrm{CH}(X^t)_{\mathbb{Q}}$ *restricts to isomorphisms*

$$\mathscr{F}_X\colon \mathrm{CH}^i_{(s)}(X) \xrightarrow{\sim} \mathrm{CH}^{g-i+s}_{(s)}(X^t)\,.$$

(3) *The subspace* $\mathrm{CH}^i_{(s)}(X)$ *is nonzero only for* $i-g\leq s\leq i$*, and if* $i\in\{0,1,g-2,g-1,g\}$ *then* $\mathrm{CH}^i_{(s)}(X)=0$ *for all* $s<0$*.*

As we shall discuss below, it is conjectured that $\mathrm{CH}^i_{(s)}(X)=0$ whenever $s<0$, but apart from the cases mentioned in (3) this is not known in general.

The following lemma gives the key calculation on which the proof of the theorem relies.

8.7 Lemma. *Let X be a g-dimensional abelian variety and $x\in \mathrm{CH}^i(X)_{\mathbb{Q}}$.*

(1) *There exist elements* $y_{(s)}\in \mathrm{CH}^{g-i+s}_{(s)}(X^t)$*, for* $i-g\leq s\leq i$*, such that* $\mathscr{F}(x)=\sum_{s=i-g}^{i} y_{(s)}$*.*

(2) *If* $x\in \mathrm{CH}^i_{(s)}(X)$ *then* $\mathscr{F}(x)\in \mathrm{CH}^{g-i+s}_{(s)}(X^t)$*.*

Proof. Write $\wp = c_1(\mathscr{P})\in \mathrm{CH}^1(X\times X^t)$, so that

$$\mathrm{ch}(\mathscr{P}) = 1+\wp+\tfrac{\wp^2}{2!}+\tfrac{\wp^3}{3!}+\cdots\,.$$

Then

$$\mathscr{F}(x) = \sum_{j\geq 0} y_j\,,\quad \text{with}\quad y_j = \mathrm{pr}_{X^t,*}\bigl(\mathrm{pr}_X^*(x)\cdot \tfrac{\wp^j}{(j!)}\bigr)\in \mathrm{CH}^{i+j-g}(X^t)_{\mathbb{Q}}\,.$$

By Corollary 4.10 we have $(\mathrm{id},[n])^*\wp^j = n^j\cdot\wp^j$, hence $(\mathrm{id},[n])_*\wp^j = n^{2g-j}\cdot\wp^j$. It follows that

$$[n]_*(y_j) = \mathrm{pr}_{X^t,*}\Bigl((\mathrm{id},[n])_*\bigl(\mathrm{pr}_X^*(x)\cdot\tfrac{\wp^j}{(j!)}\bigr)\Bigr) = \mathrm{pr}_{X^t,*}\Bigl(n^{2g-j}\cdot\bigl(\mathrm{pr}_X^*(x)\cdot\tfrac{\wp^j}{(j!)}\bigr)\Bigr) = n^{2g-j}\cdot y_j\,.$$

(For the second equality, note that $\mathrm{pr}_X = \mathrm{pr}_X \circ (\mathrm{id}, [n])$ and use the projection formula.) Hence y_j lies in the subspace $\mathrm{CH}^{i+j-g}_{(2i+j-2g)}(X^t)$. For $s \in \mathbb{Z}$, define $y_{(s)} = y_{2g-2i+s}$. Then we find that

$$\mathscr{F}(x) = \sum_{s=i-g}^{i} y_{(s)}\,, \quad \text{with} \quad y_{(s)} \in \mathrm{CH}^{g-i+s}_{(s)}(X^t)\,,$$

where the indicated range comes from the fact that $\mathrm{CH}^{g-i+s}(X^t)$ can be nonzero only for $i - g \leq s \leq i$.

If $x \in \mathrm{CH}^i_{(r)}(X)$ for some r then it follows from (4.10.1) and Proposition 8.4(2) that

$$[n]^*_{X^t}\big(\mathscr{F}(x)\big) = \mathscr{F}_X\big([n]_{X,*}(x)\big) = \mathscr{F}_X(n^{2g-2i+r} \cdot x) = n^{2g-2i+r} \cdot \mathscr{F}(x)\,,$$

and hence $\mathscr{F}(x) = y_{(r)} \in \mathrm{CH}^{g-i+r}_{(r)}(X^t)$. □

Proof of Theorem 8.6. With notation as in Lemma 8.7(1) we have

$$\begin{aligned} x &= (-1)^g \cdot [-1]^* \mathscr{F}_{X^t}\big(\mathscr{F}_X(x)\big) \\ &= \sum_{s=i-g}^{i} (-1)^g \cdot [-1]^* \mathscr{F}_{X^t}(y_{(s)}) = \sum_{s=i-g}^{i} \mathscr{F}_{X^t}\Big((-1)^g \cdot [-1]_* y_{(s)}\Big)\,. \end{aligned}$$

Further, $[-1]_* y_{(s)} \in \mathrm{CH}^{g-i+s}_{(s)}(X^t)$, so by Lemma 8.7(2) we have $\mathscr{F}_{X^t}\big((-1)^g \cdot [-1]_* y_{(s)}\big) \in \mathrm{CH}^i_{(s)}(X)$. Moreover, it is immediate from Lemma 8.7(2) that the subspaces $\mathrm{CH}^i_{(s)}(X) \subset \mathrm{CH}^i(X)_{\mathbb{Q}}$ are linearly independent. It follows that $\mathrm{CH}^i_{(s)}(X) = \oplus_{s=i-g}^{i} \mathrm{CH}^i(X)_{\mathbb{Q}}$ and that $\mathscr{F}$ induces an isomorphism $\mathrm{CH}^i_{(s)}(X) \xrightarrow{\sim} \mathrm{CH}^{g-i+s}_{(s)}(X^t)$. The stated compatibilities with respect to the two ring structures on $\mathrm{CH}(X)_{\mathbb{Q}}$ follow directly from the definitions. This proves to theorem, except for the last assertion of (3), to which we shall return in 8.10 below. □

8.8. A useful diagram. Thus far, we have kept track of the summands in the Beauville decomposition using the indices i and s, where i is the grading by codimension of cycles. It turns out that in many ways, it is more natural to use $w = 2i - s$ and s, where we refer to the number w as the *weight*. If we draw a diagram with w as a horizontal coordinate and s as a vertical coordinate then we obtain a pyramidal figure, and the beautiful feature of this is that in this diagram Fourier duality acts as a reflection in the central vertical axis.

Figure 1 illustrates this for $g = 7$ (but see below for more on the part with $s < 0$). The Fourier operator $\mathscr{F}$ exchanges the boxes in positions (w, s) and $(2g - w, s)$; for instance, we see that $\mathscr{F}$ gives an isomorphism $\mathrm{CH}^2_{(1)}(X) \xrightarrow{\sim} \mathrm{CH}^6_{(1)}(X^t)$. In this figure, we have only drawn the 'boxes' corresponding to $s \geq 0$. As we have seen, $\mathrm{CH}^i_{(s)}(X) \neq 0$ only for $s \leq i$; this gives the nice triangular shape of the diagram.

8.9 Example. In codimension 1, where we have $\mathrm{CH}^1(X) \cong \mathrm{Pic}(X)$, Beauville's decomposition is nothing but a reformulation of what we have seen in Section 2.5. Namely, if $\mathscr{L}$ is a symmetric line bundle then $[n]^*\mathscr{L} \cong \mathscr{L}^{n^2}$, whereas for an anti-symmetric line bundle $\mathscr{L}$

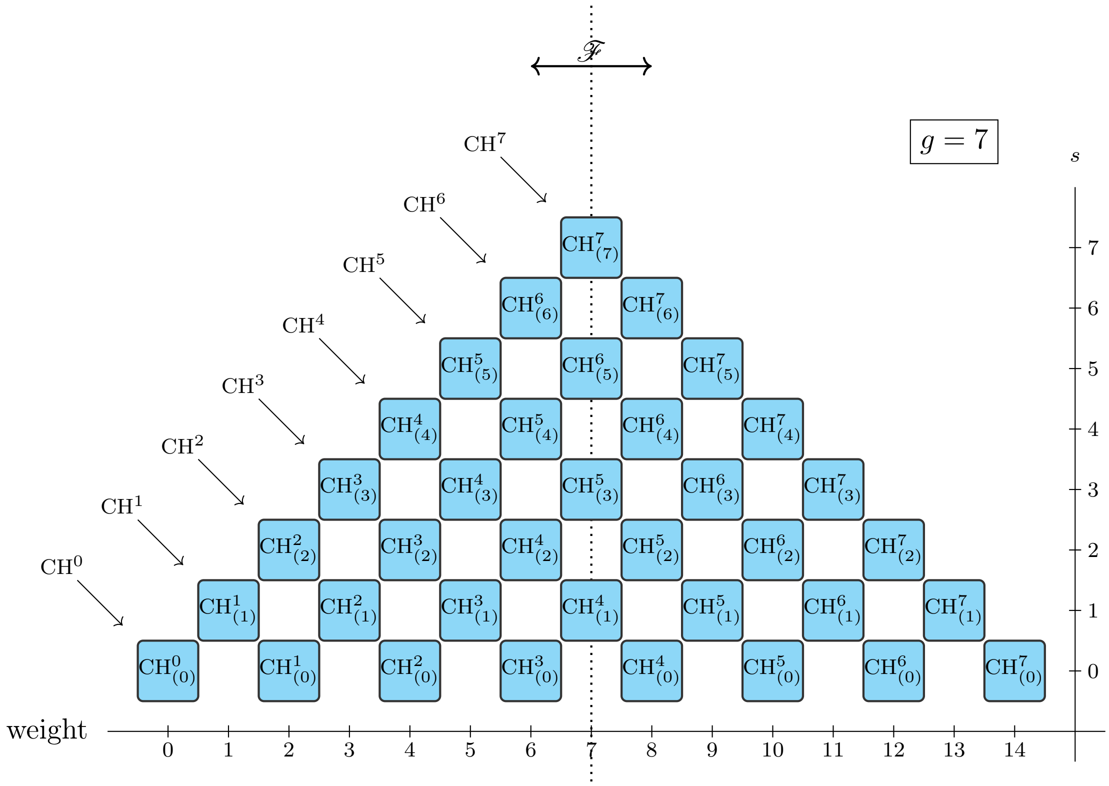


Figure 1: A picture of $\mathrm{CH}(X)_{\mathbb{Q}}$

we have $[n]^*\mathscr{L} \cong \mathscr{L}^n$. We can try to decompose an arbitrary line bundle into a symmetric and an anti-symmetric part by using the relation (2.5.1). If we work with $\mathbb{Q}$-coefficients, we can rewrite this as the identity

$$\gamma = \tfrac{1}{2} \cdot \big(\gamma + [-1]^*\gamma\big) + \tfrac{1}{2} \cdot \big(\gamma - [-1]^*\gamma\big)$$

in $\mathrm{CH}^1(X)_{\mathbb{Q}}$), with $\gamma = c_1(\mathscr{L})$ the class of $\mathscr{L}$. This explains what happens in the decomposition $\mathrm{CH}^1(X)_{\mathbb{Q}} = \mathrm{CH}^1_{(0)}(X) \oplus \mathrm{CH}^1_{(1)}(X)$, and in particular we see that $\mathrm{CH}^1_{(s)} = 0$ for $s \notin \{0, 1\}$.

Note that if $\mathscr{L}$ corresponds to a 2-torsion point of $\mathrm{Pic}_{X/k}$ (and in fact, every torsion point of $\mathrm{Pic}_{X/k}$ lies in $\mathrm{Pic}^0_{X/k} = X^t$), then $\mathscr{L}$ is both symmetric and antisymmetric. We see that there is no Beauville decomposition of $\mathrm{CH}^1(X)$ with integral coefficients; but in codimension 1, to get a decomposition it suffices to work with coefficients in the ring $\mathbb{Z}\big[\frac{1}{2}\big]$. One of the proposed AWS projects is to investigate what denominators are needed to get a Beauville decomposition in general.

A further remark about the codimension 1 case is that, as discussed in Section 4.9, the antisymmetric line bundles are precisely the line bundles that are algebraically trivial, i.e., the line bundles that correspond to points of the dual abelian variety $X^t = \mathrm{Pic}^0_{X/k}$.

For divisor classes, algebraic equivalence is the same as homological equivalence. Thus, for instance, if we consider ℓ-adic cohomology for a prime number ℓ different from $\mathrm{char}(k)$, we have the Chern class map

$$c_1 \colon \mathrm{CH}^1(X)_{\mathbb{Q}} \to H^2\bigl(X_{\bar{k}}, \mathbb{Q}_\ell(1)\bigr)\,,$$

and the kernel of this map is $\mathrm{CH}^1_{(1)}(X) \cong X^t(k) \otimes \mathbb{Q}$. The image of c_1 is isomorphic to $\mathrm{CH}^1_{(0)}(X) \cong \mathrm{NS}(X) \otimes \mathbb{Q}$. (Note that this is in agreement with the fact that $[n]_X^*$ acts as multiplication by n^2 on the cohomology of X in degree 2.) We here see that the summand $\mathrm{CH}^1_{(0)}(X)$ is the part of CH^1 that is visible in cohomology, which is a finite dimensional $\mathbb{Q}$-vector space. On the other hand, $\mathrm{CH}^1_{(1)}(X) \cong X^t(k) \otimes \mathbb{Q}$ is of a geometric nature and its dimension will in general be infinite, and may, depending on the type of base field, even be uncountable. Also observe that $\mathrm{CH}^1_{(1)}(X)$ heavily depends on the base field, and gets bigger if we replace k by a field extension, whereas $\mathrm{CH}^1_{(0)}(X) = \mathrm{NS}(X) \otimes \mathbb{Q}$ will not change under any field extension if the original base field k is separably closed.

8.10. Figure 1 represents all summands $\mathrm{CH}^i_{(s)}(X)$ with $s \geq 0$. Though it is conjectured that all summands $\mathrm{CH}^i_{(s)}$ with $s < 0$ vanish, this is not known to be true. (See 8.11 below for further discussion.) Figure 2 shows the diagram for $g = 9$ including the summands with $s < 0$.

The last assertion of part (3) of Theorem 8.6 can be understood as follows. First one shows that the summands $\mathrm{CH}^0_{(s)}$ and $\mathrm{CH}^1_{(s)}$ with $s < 0$ vanish. In codimension 0 this is clear, simply because $\mathrm{CH}^0(X)_{\mathbb{Q}} = \mathbb{Q} \cdot [X]$ and $[n]^*\bigl([X]\bigr) = [X]$ for all n. In codimension 1 we have seen this in Example 8.9. By (Fourier-)symmetry, we then find that if we draw the full diagram, the outer two layers in the region $s < 0$ are indeed zero. So the diagram should really be drawn as in Figure 2, where we have take $g = 9$.

8.11. The Beauville decomposition is especially relevant in the context of some general deep open conjectures about Chow groups of smooth projective varieties that are due to Beilinson, Bloch and Murre. While it would take us too far to discuss these in detail, let us briefly summarize what they are about.

Conjecturally, for every smooth projective variety X/k, there should exists a descending filtration

$$\mathrm{CH}(X)_{\mathbb{Q}} = \mathrm{Fil}^0 \supset \mathrm{Fil}^1 \supset \mathrm{Fil}^2 \supset \cdots$$

that has a number of good properties. Among these properties are functoriality and the property that $\mathrm{Fil}^i{\cdot}\mathrm{Fil}^j \subseteq \mathrm{Fil}^{i+j}$. The graded pieces $\mathrm{gr}^i = \mathrm{Fil}^i/\mathrm{Fil}^{i+1}$ should be cohomological in nature; this is made precise for instance by the conjectured property that the action of $\mathrm{Corr}^0(X,X)_{\mathbb{Q}}$ on gr^i factors through $\mathrm{Corr}^0(X,X)_{\mathbb{Q}}/\sim_{\mathrm{hom}}$. Further, Fil^1 should be the subspace $\mathrm{CH}(X)_{\mathbb{Q},\mathrm{hom}} \subset \mathrm{CH}(X)_{\mathbb{Q}}$ of classes that are homologically trivial, and Fil^2 should consist of the homologically trivial classes that are in the kernel of all Abel–Jacobi maps.

Different version of these conjectures were formulated by Beilinson [9] and Murre [38] and were later shown to be equivalent. The (conjectural!) theory of mixed motives gives a

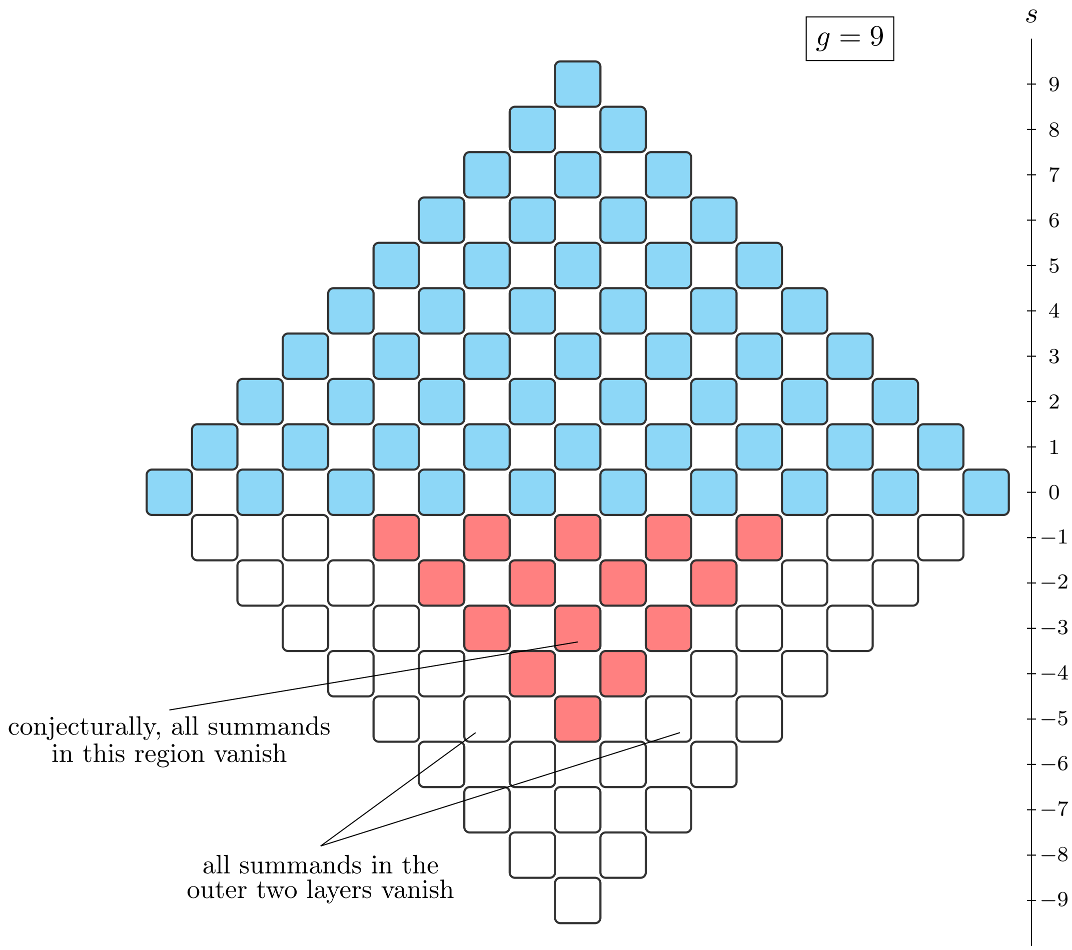


Figure 2: A picture of $\mathrm{CH}(X)_{\mathbb{Q}}$ including the part $s < 0$

very natural explanation for why these conjectures should be true. We refer to the beautiful paper [26] for a detailed discussion.

The Beauville decomposition gives a filtration of $\mathrm{CH}(X)_{\mathbb{Q}}$ in case X is an abelian variety, by setting $\mathrm{Fil}^j = \oplus_{s \geq j} \mathrm{CH}_{(s)}(X)$, where

$$\mathrm{CH}_{(s)}(X) = \bigoplus_{i=0}^{g} \mathrm{CH}^i_{(s)}(X)\,. \tag{8.11.1}$$

This filtration indeed seems to have the right properties, but some of this remains as yet conjectual. For instance, the expected property that $\mathrm{CH}(X)_{\mathbb{Q}} = \mathrm{Fil}^0$ is precisely the statement that $\mathrm{CH}^i_{(s)}(X)$ should be zero whenever $s < 0$.

Clearly, the Beauville decomposition gives even more, as we even have a splitting of the filtration. This is expected to happen only for rather special varieties. For K3 surfaces, such a splitting was given by Beauville and Voisin in [8]. We refer to Beauville's paper [7] for further discussion; as explained there, one expects at least a weak form of such a splitting for hyperkähler varieties. Related to this, Shen and Vial [48] introduced the notion of a *multiplicative Chow–Künneth decomposition*, which turns out to be intimately related to splittings of the Bloch–Beilinson filtration. (We shall briefly return to this in Section 13.5.) One of their conjectures (ibid., Conjecture 4) is that hyperkähler varieties admit such a multiplicative Chow–Künneth decomposition; this has been proven in several special examples. A nice starting point for further reading is the paper [20].

9. The action of the Lie algebra $\mathfrak{sl}_2$ on $\mathrm{CH}(X)_{\mathbb{Q}}$

9.1. In what follows, we simply write $\mathfrak{sl}_2$ for the Lie algebra $\mathfrak{sl}_2$ over the field $\mathbb{Q}$. Explicitly, it is the 3-dimensional vector space $\mathfrak{sl}_2 = \mathbb{Q}\cdot e \oplus \mathbb{Q}\cdot h \oplus \mathbb{Q}\cdot f$, and the Lie bracket is given by the relations

$$[e,h] = 2e\,,\quad [f,h] = -2f\,,\quad [e,f] = h\,. \tag{9.1.1}$$

If W is a $\mathbb{Q}$-vector space (possibly of infinite dimension), a representation of $\mathfrak{sl}_2$ on W is a homomorphism of Lie algebras $\rho\colon \mathfrak{sl}_2 \to \mathrm{End}(W)$. Concretely, such a representation is given by specifying three $\mathbb{Q}$-linear endomorphisms e, h, f of W (the images of the generators of $\mathfrak{sl}_2$ under ρ) that satisfy the commutation relations (9.1.1). As an example, we have the standard 2-dimensional representation

$$\rho_1\colon \mathfrak{sl}_2 \to \mathrm{End}(\mathbb{Q}^2) = M_2(\mathbb{Q})$$

(where $M_2(\mathbb{Q})$ denotes the algebra of 2×2 matrices with rational coefficients, here viewed as a Lie algebra with the usual commutator bracket), given by

$$e \mapsto \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix},\quad h \mapsto \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix},\quad f \mapsto \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}.$$

For $n \in \mathbb{N}$, the $n+1$-dimensional representation $\rho_n = \mathrm{Sym}^n(\rho_1)$ is irreducible. This representation can be realized by taking as underlying vector space

$$W = \mathbb{Q}\cdot w_{-n} \oplus \mathbb{Q}\cdot w_{-n+2} \oplus \mathbb{Q}\cdot w_{-n+4} \oplus \cdots \oplus \mathbb{Q}\cdot w_{n-2} \oplus \mathbb{Q}\cdot w_n\,,$$

with operators given by $h(w_j) = j\cdot w_j$ and

$$e(w_{-n+2i}) = (n-i)\cdot w_{-n+2i+2}\,,\quad f(w_{-n+2i}) = i\cdot w_{-n+2i-2}\,. \tag{9.1.2}$$

It is helpful to think of the action of h as giving a grading $W = \oplus W_j$ with $W_j = \mathbb{Q}\cdot w_j$ of degree j, for $j = -n, -n+2, \ldots, n-2, n$. The commutation relations $[e,h] = 2e$ and $[f,h] = -2f$ then express that e is homogeneous of degree 2 and f has degree -2. One could visualize this representation as a string of 1-dimensional spaces that each have a weight (corresponding to the action of h), with operators e and f of degrees $+2$ and -2, respectively. E.g., for $n = 5$ this gives the picture

$$\begin{tikzcd}
-5 \arrow[r, bend left, "e"] & -3 \arrow[l, bend left, "f"] \arrow[r, bend left, "e"] & -1 \arrow[l, bend left, "f"] \arrow[r, bend left, "e"] & 1 \arrow[l, bend left, "f"] \arrow[r, bend left, "e"] & 3 \arrow[l, bend left, "f"] \arrow[r, bend left, "e"] & 5 \arrow[l, bend left, "f"]
\end{tikzcd}$$

with $e(w_5) = 0$ and $f(w_{-5}) = 0$. (Caution: the picture does not mean to suggest that e and f map base vectors w_j to base vectors w_{j+2}, resp. w_{j-2}; see (9.1.2) for the correct relations.)

Every finite dimensional irreducible representations of $\mathfrak{sl}_2$ is isomorphic to one of the representations ρ_n and every finite dimensional representation of $\mathfrak{sl}_2$ is completely reducible, i.e., is isomorphic to a direct sum of irreducible representations. In some cases this extends to infinite dimensional representations; what we need is the following result.

9.2 Lemma. *Let $\rho\colon \mathfrak{sl}_2 \to \mathrm{End}(W)$ be a representation of $\mathfrak{sl}_2$, with W a $\mathbb{Q}$-vector space, possibly of infinite dimension. Suppose there exists a positive integer N and a grading $W = \oplus_{j=-N}^{N} W_j$ such that $h \in \mathfrak{sl}_2$ acts on W_j as multiplication by j. For $j \geq 0$, let $P_j = \{x \in W_{-j} \mid f(x) = 0\}$. (Note the sign: we define P_j to be a subspace of W_{-j}.) Then we have an isomorphism of $\mathfrak{sl}_2$-representations*

$$W \cong \bigoplus_{j=0}^{N} P_j \otimes \rho_j\,.$$

About the notation: if P is a $\mathbb{Q}$-vector space and $\rho\colon \mathfrak{sl}_2 \to \mathrm{End}(U)$ is a representation of $\mathfrak{sl}_2$ then by $P \otimes \rho$ we mean the vector space $P \otimes_{\mathbb{Q}} U$, viewed as a representation of $\mathfrak{sl}_2$ through the action of $\mathfrak{sl}_2$ on U.

Pictorially, the lemma says that every $\mathfrak{sl}_2$-representation in which h acts semisimply with eigenvalues in a finite range $-N, -N+1, \ldots, N-1, N$, is a direct sum of 'strings' of the type depicted above. A string of length n represents a copy of the representation ρ_n. If we fix n, the leftmost vectors in the strings, which may be thought of as generators of the representation, span a vector space P_n. (Here P is for 'primitive', as the vectors killed by f are usually called the primitive vectors.) The total contribution of all strings of length n can therefore be written as $P_n \otimes \rho_n$.

9.3. We now explain how this is related to Chow groups of abelian varieties. Let X be a g-dimensional abelian variety over a field k, and let $\theta\colon X \to X^t$ be a polarization. We are going to construct an action of $\mathfrak{sl}_2$ on $\mathrm{CH}(X)_{\mathbb{Q}}$. For this we need to specify three operators e, h and f on $\mathrm{CH}(X)_{\mathbb{Q}}$ that satisfy certain commutation relations. The operator h corresponds to a grading on $\mathrm{CH}(X)_{\mathbb{Q}}$, and if we compare our earlier diagrams with the above picture of an $\mathfrak{sl}_2$-representation, it should perhaps not come as a surprise that we define $h\colon \mathrm{CH}(X)_{\mathbb{Q}} \to \mathrm{CH}(X)_{\mathbb{Q}}$ by the rule

$$h = \text{multiplication by } 2i - s - g \text{ on } \mathrm{CH}^i_{(s)}(X).$$

Recall that we refer to $2i-s$ as the weight of the summand $\mathrm{CH}^i_{(s)}(X)$; so the grading that is defined by the operator h is the grading by weight, except that we shift everything by $-g$, so that we obtain a grading in degrees $-g, -g+1, \ldots, g-1, g$. Pictorially: the vertical symmetry axis in the diagrams in Section 8 is now placed in degree 0. To avoid confusion, we shall refer to the new grading of $\mathrm{CH}(X)_{\mathbb{Q}}$ as the h-grading.

What remains to be done is to define operators $e\colon \mathrm{CH}(X)_{\mathbb{Q}} \to \mathrm{CH}(X)_{\mathbb{Q}}$ of degree 2 and $f\colon \mathrm{CH}(X)_{\mathbb{Q}} \to \mathrm{CH}(X)_{\mathbb{Q}}$ of degree -2 such that $[e,f] = h$. This is where the chosen polarization $\theta\colon X \to X^t$ comes in. The degree of a polarization is a square; let $m(\theta)$ be the positive integer such that $\deg(\theta) = m(\theta)^2$. If $\theta = \phi_{\mathscr{L}}$ for some ample line bundle $\mathscr{L}$ then $m(\theta) = h^0(\mathscr{L}) = \dim_k H^0(X,\mathscr{L})$.

Define a class $\ell \in \mathrm{CH}^1(X)_{\mathbb{Q}}$ by

$$\ell = \tfrac{1}{2} \cdot c_1\big((\mathrm{id},\theta)^*\mathscr{P}\big)\,,$$

where $\mathscr{P}$ is the Poincaré bundle on $X \times X^t$. If $\theta = \phi_{\mathscr{L}}$ for some symmetric ample line bundle $\mathscr{L}$ then $\ell = c_1(\mathscr{L})$. (See 4.8(3).) Because such a bundle $\mathscr{L}$ exists after an extension of the base field, it follows that $\ell \in \mathrm{CH}^1_{(0)}(X)$.

Next, define $\lambda \in \mathrm{CH}^{g-1}_{(0)}(X)$ to be the unique class with the property that

$$\mathscr{F}(\ell) = (-1)^g \cdot \theta_*(\lambda)\,.$$

Note that because $\theta\colon X \to X^t$ is an isogeny, the induced maps $\theta_*\colon \mathrm{CH}(X)_{\mathbb{Q}} \to \mathrm{CH}(X^t)_{\mathbb{Q}}$ and $\theta^*\colon \mathrm{CH}(X^t)_{\mathbb{Q}} \to \mathrm{CH}(X)_{\mathbb{Q}}$ are isomorphisms, and by the projection formule we have $\theta_* \circ \theta^* = m(\theta)^2 \cdot \mathrm{id}$. The class $\mathscr{F}(\ell)$ lies in $\mathrm{CH}^{g-1}_{(0)}(X^t)$; so λ is, up to a constant, nothing but the Fourier dual class $\mathscr{F}(\ell)$, 'transported back to X'. It can be shown that

$$\lambda = \frac{\ell^{g-1}}{m(\theta)\cdot(g-1)!}\,.$$

9.4 Theorem. *Let X/k be a g-dimensional abelian variety, and let $\theta\colon X \to X^t$ be a polarization. Define $\ell \in \mathrm{CH}^1_{(0)}(X)$ and $\lambda \in \mathrm{CH}^{g-1}_{(0)}(X)$ as above.*

(1) *The operators e, h and f on $\mathrm{CH}(X)_{\mathbb{Q}}$ given by*

$$\begin{aligned} e(x) &= \ell \cdot x && \textit{(intersection product)}\\ h(x) &= (2i-s-g)\cdot x && \textit{for } x \in \mathrm{CH}^i_{(s)}(X)\\ f(x) &= \lambda \star x && \textit{(Pontryagin product)} \end{aligned}$$

satisfy the commutation relations (9.1.1) *and therefore define an action of the Lie algebra $\mathfrak{sl}_2$ on $\mathrm{CH}(X)_{\mathbb{Q}}$.*

(2) *For every $s \in \mathbb{Z}$, the subspace $\mathrm{CH}_{(s)}(X) \subset \mathrm{CH}(X)_{\mathbb{Q}}$ (see* (8.11.1)*) is preserved under this $\mathfrak{sl}_2$-action.*

(3) *For $s \in \mathbb{Z}$ and $j \in \{0,\ldots,g-|s|\}$ such that $g-s-j = 2i$ is even, define a $\mathbb{Q}$-vector space $P_{j,(s)}$ by*

$$P_{j,(s)} = \big\{x \in \mathrm{CH}^{s+i}_{(s)}(X) \;\big|\; f(x) = 0\big\}\,.$$

Then we have an isomorphism of $\mathfrak{sl}_2$-representations

$$\mathrm{CH}_{(s)}(X) = \bigoplus_{\substack{j\in\{0,\ldots,g-|s|\}\\ j\equiv g-s \bmod 2}} P_{j,(s)} \otimes \rho_j\,.$$

In (3), note that the summand $\mathrm{CH}^{s+i}_{(s)}(X)$ has weight $2(s+i)-s = s+2i$, and for the h-grading therefore has degree $s+2i-g = -j$, in accordance with the notation used in Lemma 9.2.

A picture may help to visualize what is going on. In Figure 3 we draw the same kind of diagram as before, taking $g = 7$ and drawing (for simplicity) only the summands with $s \geq 0$. We use the h-grading as horizontal coordinate, and the parameter s as vertical coordinate.

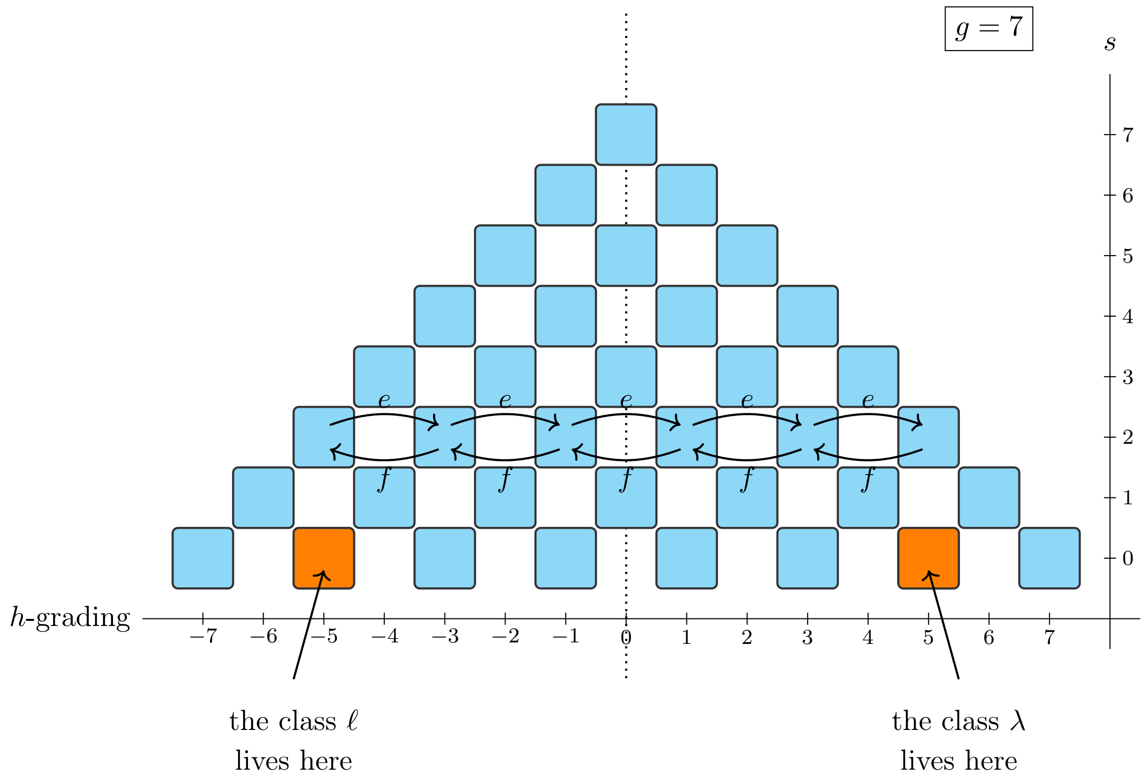
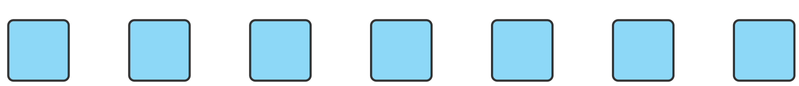


Figure 3: A picture for the $\mathfrak{sl}_2$-action

If we take out one 'horizontal layer', meaning that we fix s and consider $\mathrm{CH}_{(s)}(X) = \oplus_{i=0}^{g} \mathrm{CH}^i_{(s)}(X)$, this is an $\mathfrak{sl}_2$-subrepresentation (pictorially: the operators e and f act horizontally), and $\mathrm{CH}_{(s)}(X)$ is a sum (usually infinite) of copies of the irreducible representations

$$\rho_{g-|s|}\,, \quad \rho_{g-|s|-2}\,, \quad \rho_{g-|s|-4}\,, \quad \cdots$$

For instance, if in the above diagram we consider the layer with $s = 1$, which is



then as a representation of $\mathfrak{sl}_2$ this decomposes as a sum of copies of the irreducible representations

$$
\begin{array}{cccccccc}
\rho_6 & -6 & -4 & -2 & 0 & 2 & 4 & 6 \\
\rho_4 & & -4 & -2 & 0 & 2 & 4 & \\
\rho_2 & & & -2 & 0 & 2 & & \\
\rho_0 & & & & 0 & & &
\end{array}
$$

More precisely, if in this particular example (with $g = 7$ and $s = 1$) we define

$$
P_{6,(1)} = \bigl\{x \in \mathrm{CH}^1_{(1)}(X) \bigm| f(x) = 0\bigr\} \qquad P_{2,(1)} = \bigl\{x \in \mathrm{CH}^3_{(1)}(X) \bigm| f(x) = 0\bigr\}
$$
$$
P_{4,(1)} = \bigl\{x \in \mathrm{CH}^2_{(1)}(X) \bigm| f(x) = 0\bigr\} \qquad P_{0,(1)} = \bigl\{x \in \mathrm{CH}^4_{(1)}(X) \bigm| f(x) = 0\bigr\}
$$

then we find

$$
\mathrm{CH}_{(1)}(X) = \bigl(P_{6,(1)} \otimes \rho_6\bigr) \oplus \bigl(P_{4,(1)} \otimes \rho_4\bigr) \oplus \bigl(P_{2,(1)} \otimes \rho_2\bigr) \oplus \bigl(P_{0,(1)} \otimes \rho_0\bigr)\,.
$$

10. Zero cycles on abelian varieties

In this section we discuss some results about 0-cycles on abelian varieties. Our focus will be on results with *integral* coefficients; but as we shall see, there is a direct connection to the results with $\mathbb{Q}$-coefficients that we have discussed in the previous sections. Throughout, we work over an algebraically closed base field k. (While this restricts generality, it should be recalled that over an arbitrary field k with algebraic closure $\bar{k}$, the kernel of $\mathrm{CH}(X) \to \mathrm{CH}(X_{\bar{k}})$ is torsion; see Section 5.)

10.1. Let X be an abelian variety over a field $k = \bar{k}$. By definition, $\mathrm{CH}_0(X)$ is the group of 0-cycles module rational equivalence. Note that $\mathrm{CH}_0(X)$ is a subring of $\mathrm{CH}(X)$ with respect to the Pontryagin product.

For $P \in X(k)$, we write $(P) \in \mathrm{CH}_0(X)$ for the class of the corresponding 0-cycle. An element of $\mathrm{CH}_0(X)$ can be written as a finite formal sum

$$
Z = \sum_{P \in X} m_P \cdot (P)
$$

where P runs over the k-rational points of X and where the m_P are integers of which only finitely many are nonzero. The first invariant of such a 0-cycle is its degree:

$$
\deg(Z) = \sum_{P \in X} m_P\,.
$$

In what follows, an important role is played by the space

$$
I = \mathrm{Ker}\bigl(\mathrm{CH}_0(X) \xrightarrow{\deg} \mathbb{Z}\bigr)
$$

of 0-cycles of degree 0. Note that $I \subset \mathrm{CH}_0(X)$ is an ideal for the Pontryagin product; as an abelian group it is generated by the classes of the form $(P) - (e)$. For $r \geq 0$ we denote by $I^{\star r}$ the rth power of I with respect to the Pontryagin ring structure.

Clearly, the map $\mathbb{Z} \to \mathrm{CH}_0(X)$ given by $m \mapsto m \cdot (e)$ gives a section of the degree map, so that $\mathrm{CH}_0(X) = \mathbb{Z} \cdot (e) \oplus I$.

As a next step, we can consider the summation map $S\colon \mathrm{CH}_0(X) \to X(k)$, given by

$$S\Big(m_1 \cdot (P_1) + \cdots + m_t \cdot (P_t)\Big) = m_1 P_1 + \cdots + m_t P_t \,,$$

where in the right hand side we use the addition in $X(k)$. Clearly, S factors through $\mathrm{CH}_0(X)/\mathbb{Z} \cdot (e) \cong I$; so all essential information is contained in its restriction $S\colon I \to X(k)$ to the ideal I. The map $X(k) \to I$ given by $P \mapsto (P) - (e)$ gives a section of this map.

10.2 Proposition. *Let X be an abelian variety over an algebraically closed field k.*

(1) *The ideal $I = \mathrm{Ker}(\deg)$ equals the subspace $\mathrm{CH}_0(X)_{\mathrm{alg}}$ of 0-cycles that are algebraically trivial.*

(2) *For all $r \geq 1$, the subgroup $I^{\star r} \subset \mathrm{CH}_0(X)$ is divisible.*

(3) *We have a short exact sequence of abelian groups*

$$0 \longrightarrow I^{\star 2} \longrightarrow I \xrightarrow{S} X(k) \longrightarrow 0\,. \tag{10.2.1}$$

Consequently, $I \cong X(k) \oplus I^{\star 2}$ as an abelian group.

Proof. Part (1) follows from the fact that I is generated, as an additive group, by the classes of the form $(P) - (e)$, as clearly $(P) \sim_{\mathrm{alg}} (e)$. By Proposition 7.6 this gives the case $r = 1$ of (2); the assertion for $r > 1$ is an immediate consequence. For (3), it suffices to show that the inclusion $I^{\star 2} \subset \mathrm{Ker}(S)$ is an equality. We have the relation

$$\begin{aligned} \big((P) - (e)\big) + \big((Q) - (e)\big) &= \big((P+Q) - (e)\big) - \big((P) - (e)\big) \star \big((Q) - (e)\big)\,, \\ &\equiv \big((P+Q) - (e)\big) \bmod I^{\star 2}\,. \end{aligned}$$

Applying this to $P-Q$ and Q we find that $\big((P)-(e)\big)-\big((Q)-(e)\big) \equiv \big((P-Q)-(e)\big) \bmod I^{\star 2}$. Hence every class in $I/I^{\star 2}$ can be represented in the form $\big((P) - (e)\big)$ for some $P \in X(k)$. It follows that $\bar{S}\colon I/I^{\star 2} \to X(k)$ is injective, so $I^{\star 2} = \mathrm{Ker}(S)$. $\square$

The following result is due to Roĭtman [43], with, in case $\mathrm{char}(k) = p > 0$, some results about p-power torsion provided by [33].

10.3 Theorem. *Let X be an abelian variety over an algebraically closed field k. Then the summation map S induces an isomorphism*

$$S\colon \mathrm{CH}_0(X)_{\mathrm{tors}} \xrightarrow{\sim} X(k)_{\mathrm{tors}}\,.$$

Note that $\mathrm{CH}_0(X)_{\mathrm{tors}}$ is contained in I, so in fact $\mathrm{CH}_0(X)_{\mathrm{tors}} = I_{\mathrm{tors}}$.

10.4 Corollary. *For every $r \geq 2$ the subgroup $I^{\star r} \subset \mathrm{CH}_0(X)$ is a $\mathbb{Q}$-vector space.*

Proof. Proposition 10.2 gives that $I^{\star r}$ is divisible. On the other hand, by Roĭtman's theorem together with the exact sequence (10.2.1), $I^{\star 2}$ (and hence also $I^{\star r}$) is torsion-free. Hence $I^{\star r}$ is uniquely divisible and is therefore a $\mathbb{Q}$-vector space. $\square$

10.5. On $\mathrm{CH}_0(X)$ we have the filtration by $\star$-powers of the ideal I:

$$\mathrm{CH}_0(X) = I^{\star 0} \supset I \supset I^{\star 2} \supset I^{\star 3} \supset \cdots ,$$

and as we have seen, $\mathrm{CH}_0(X) \cong \mathbb{Z} \oplus I$ and $I \cong X(k) \oplus I^{\star 2}$. Extending scalars to $\mathbb{Q}$ only has an effect on the first two terms, so we get

$$\mathrm{CH}_0(X)_{\mathbb{Q}} \supset I_{\mathbb{Q}} \supset I^{\star 2} \supset I^{\star 3} \supset \cdots , \tag{10.5.1}$$

of course with $I_{\mathbb{Q}} = I \otimes \mathbb{Q}$.

10.6 Lemma. *Let X be a g-dimensional abelian variety over $k = \bar{k}$.*

(1) *For all $n \in \mathbb{Z}$ the filtration* (10.5.1) *is preserved by the endomorphisms $[n]_*$ and $[n]^*$ of $\mathrm{CH}_0(X)_{\mathbb{Q}}$, and $[n]_*$ induces multiplication by n^r on $I^{\star r}/I^{\star(r+1)}$.*

(2) *For all $r \geq 0$ we have $I_{\mathbb{Q}}^{\star r} \subseteq \oplus_{s\geq r} \mathrm{CH}_{0,(s)}$. In particular, $I^{\star(g+1)} = 0$.*

As we shall show hereafter, in fact $I_{\mathbb{Q}}^{\star r} = \oplus_{s\geq r} \mathrm{CH}_{0,(s)}$ for all $r \geq 0$.

Proof. For $\alpha \in \mathrm{CH}_0(X)_{\mathbb{Q}}$ and $n \in \mathbb{Z}$ we have $\deg\bigl([n]_*\alpha\bigr) = \deg(\alpha)$; hence $[n]_*(I) \subset I$, and because $[n]_*$ is an endomorphism of $\mathrm{CH}(X)_{\mathbb{Q}}$ for the Pontryagin ring structure, it follows that $[n]_*$ preserves the filtration (10.5.1). The same conclusion for the endomorphisms $[n]^*$ follows from the fact that $[n]_* \circ [n]^* = n^{2g} \cdot \mathrm{id}$.

By Fourier duality we have $\mathrm{CH}_{0,(0)}(X) = \mathscr{F}\bigl(\mathrm{CH}^0(X)_{\mathbb{Q}}\bigr) \cong \mathbb{Q}$, and in fact, $\mathrm{CH}_{0,(0)}(X) = \mathbb{Q} \cdot [e]$. If ℓ is a prime number different from $\mathrm{char}(k)$, we have a commutative diagram

$$\begin{tikzcd} \mathrm{CH}_0(X)_{\mathbb{Q}} \arrow[rr, "\deg"] \arrow[drr, "\mathrm{cl}"'] & & \mathbb{Q} \arrow[d, hook] \\ & & H^{2g}\bigl(X_{\bar{k}}, \mathbb{Q}_\ell(g)\bigr) \cong \mathbb{Q}_\ell \end{tikzcd}$$

where cl is the cycle class map in ℓ-adic cohomology. Because $[n]_*$ acts as the identity on $H^{2g}\bigl(X_{\bar{k}}, \mathbb{Q}_\ell(g)\bigr)$, it follows that $I_{\mathbb{Q}} = \oplus_{s\geq 1} \mathrm{CH}_{0,(s)}$, and hence $I_{\mathbb{Q}}^{\star r} \subseteq \oplus_{s\geq r} \mathrm{CH}_{0,(s)}$. The last assertion now follows from Theorem 8.6(3). ☐

10.7 Theorem. *Let X be an abelian variety of dimension $g > 0$ over an algebraically closed field k. Then for all $r \geq 0$ we have*

$$I_{\mathbb{Q}}^{\star r} = \bigoplus_{s\geq r} \mathrm{CH}_{0,(s)}(X) .$$

To avoid confusion: $I_{\mathbb{Q}}^{\star 0} = \mathrm{CH}_0(X)_{\mathbb{Q}}$, and by Corollary 10.4, $I_{\mathbb{Q}}^{\star r} = I^{\star r}$ for all $r \geq 2$.

Proof. To simplify notation, write $\mathrm{CH}_{0,(s)}$ for $\mathrm{CH}_{0,(s)}(X)$. If $V \subset \mathrm{CH}_0(X)_{\mathbb{Q}}$ is a nonzero subspace, let $r(V)$ be the smallest index r such that $V \not\subset I_{\mathbb{Q}}^{\star(r+1)}$; note that by the previous lemma we always have $0 \leq r(V) < g$. If $s \geq 0$ and $r = r\bigl(\mathrm{CH}_{0,(s)}\bigr)$ then $\mathrm{CH}_{0,(s)} \subset I^{\star r}$ and the induced map $\mathrm{CH}_{0,(s)} \to I^{\star r}/I^{\star(r+1)}$ is nonzero. By the lemma, it follows that

$r = s$. Further, the intersection $V = \mathrm{CH}_{0,(s)} \cap I^{\star(s+1)}$ is zero, for otherwise we get a nonzero map $V \to I^{\star r}/I^{\star(r+1)}$ with $r = r(V) > s$, which is impossible. In this way we obtain a well-defined injective map $\alpha\colon \mathrm{CH}_{0,(s)} \hookrightarrow I^{\star s}/I^{\star(s+1)}$. On the other hand, the inclusions $I_{\mathbb{Q}}^{\star r} \subseteq \oplus_{s\geq r} \mathrm{CH}_{0,(s)}$ give us a homomorphism $\beta\colon I^{\star s}/I^{\star(s+1)} \to \mathrm{CH}_{0,(s)}(X)$, and $\alpha \circ \beta$ is the identity. Hence α is an isomorphism for every s, and because the filtration (10.5.1) is finite, the theorem follows. □

11. Small and big Chow groups

The following result is due to Künnemann [29], based on a weight argument from [49]. The proof makes essential use of the theory of Chow motives.

11.1 Theorem. *Let X be an abelian variety over a field k which is a subfield of $\overline{\mathbb{F}}_p$ for some prime number p. Then $\mathrm{CH}^i_{(s)}(X) = 0$ for all i and all $s \neq 0$, so that*

$$\mathrm{CH}(X)_{\mathbb{Q}} = \bigoplus_{i=0}^{g} \mathrm{CH}^i_{(0)}(X)\,.$$

For instance, this applies when k is finite, or $k = \overline{\mathbb{F}}_p$. In this result it is essential that we work with $\mathbb{Q}$-coefficients, and thereby kill all torsion. E.g., we have seen that $\mathrm{CH}^1_{(1)}(X) \cong X^t(k) \otimes \mathbb{Q}$, and while $X^t(k)$ is of course not trivial, it is a torsion group if $k \subset \overline{\mathbb{F}}_p$. (The point is that all classes are defined over some finite field and $X^t(k)$ is a finite group if k is finite.)

With X/k as in the theorem, it is conjectured that (for ℓ a prime number different from p) the cycle class maps

$$\mathrm{cl}_\ell\colon \mathrm{CH}^i_{(0)}(X) \otimes \mathbb{Q}_\ell \to H^{2i}\bigl(X_{\bar{k}}, \mathbb{Q}_\ell(i)\bigr)$$

are injective. (See [15], Section 4.) This would imply that $\mathrm{CH}(X)_{\mathbb{Q}}$ is a finite dimensional $\mathbb{Q}$-vector space.

For 0-cycles this gives a nice result with integral coefficients:

11.2 Corollary. *Let X be an abelian variety over $\overline{\mathbb{F}}_p$. Then $\mathrm{CH}_0(X) \cong \mathbb{Z} \oplus X^t(\overline{\mathbb{F}}_p)$.*

Proof. By the results discussed in 10.1 and Proposition 10.2 we have

$$\mathrm{CH}_0(X) \cong \mathbb{Z} \oplus I \cong \mathbb{Z} \oplus X^t(\overline{\mathbb{F}}_p) \oplus I^{\star 2}\,.$$

But Theorems 10.7 and 11.1 give $I^{\star 2} = I^{\star 2}_{\mathbb{Q}} = \oplus_{s\geq 2} \mathrm{CH}_{0,(s)}(X) = 0$. □

11.3. The above results give the picture that over fields $k \subseteq \overline{\mathbb{F}}_p$, Chow groups are fairly small. It should be clear, though, that the situation is very different for other base fields. E.g., even if X is defined over a subfield k of $\overline{\mathbb{F}}_p$, if $k \subset K = \overline{K}$ then we have seen that $\mathrm{CH}_{0,(1)}(X_K) \cong X(K) \otimes \mathbb{Q}$, which is uncountable if K is.

It is interesting to note that the 0th layer $\mathrm{Fil}^0/\mathrm{Fil}^1$ of the conjectural Bloch–Beilinson filtration should be the image of $\mathrm{CH}(X)_{\mathbb{Q}}$ in cohomology, which should be a finite dimensional $\mathbb{Q}$-vector space that is (almost) independent of the base field, in the sense that it does not change if we extend from an algebraically closed field k to a bigger field. By contrast, already by looking at $\mathrm{CH}_{0,(1)}(X)$, we see that other layers in the filtration will depend in size on k.

11.4. We shall next discuss some results that imply that, in general, Chow groups are very large, at least if we work over an uncountable base field. While it is clear that, except for special cases, we cannot expect $\mathrm{CH}(X)_{\mathbb{Q}}$ to be a finite dimensional $\mathbb{Q}$-vector space, this in itself does not say much about its size. To make more precise what 'very large' should mean, we focus on 0-cycles. Consider a g-dimensional smooth projective variety X (not necessarily an abelian variety) over an algebraically closed field k. Let $\mathrm{CH}_0(X)_{\mathrm{hom}} = \mathrm{Ker}(\deg\colon \mathrm{CH}_0(X)_{\mathbb{Q}} \to \mathbb{Q})$ be the group of 0-cycles of degree 0. (If $Z \in \mathrm{CH}_0(X)_{\mathbb{Q}}$ then the condition $\deg(Z) = 0$ is equivalent to saying that Z is homologically trivial, which for 0-cycles is in turn the same as saying that $Z \sim_{\mathrm{alg}} 0$.)

Let us formulate some properties, each of which expresses that $\mathrm{CH}_0(X)_{\mathbb{Q}}$ is small, in a suitable sense. To begin with, every 0-cycle of degree 0 is of the form $(P_1+P_2+\cdots+P_m)-(Q_1+\cdots+Q_m)$ for some closed points $P_1,\ldots,P_m,Q_1,\ldots,Q_m$ of X, and we may ask if the value of m can be bounded.

Property (A). There exists an integer m such that every $\xi \in \mathrm{CH}_0(X)_{\mathrm{hom}}$ can be written in the form $\xi = [(P_1+P_2+\cdots+P_m)-(Q_1+\cdots+Q_m)]$ for some points P_i and Q_j.

The 0-cycles of degree m are the closed points of the mth symmetric power $\mathsf{S}^m(X) = X^m/\mathfrak{S}_m$, where the symmetric group $\mathfrak{S}_m$ acts on the m-fold power X^m of X by permutation of the coordinates. This $\mathsf{S}^m(X)$ is a variety of dimension gm, which is singular, except when $g \leq 1$ or $m \leq 1$. In what follows we interpret points of $\mathsf{S}^m(X)$ as 0-cycles of degree m on X. We then have the map

$$\gamma_m\colon \mathsf{S}^m(X) \times \mathsf{S}^m(X) \to \mathrm{CH}_0(X)_{\mathrm{hom}}$$

given by $\gamma(Z_1,Z_2) = Z_1 - Z_2$. Even though the target is not an algebraic variety, we can give some meaning to what the dimension of the image of γ_m should be. This uses that the fibres of γ_m are countable unions of closed irreducible subsets of $\mathsf{S}^m(X)\times\mathsf{S}^m(X)$; we may therefore define the 'dimension of the image of γ_m' as the number $d_m = 2gm - r$, where r is the largest integer that occurs as the dimension of a component of a fibre.

Property (B). The function $m \mapsto d_m$ is bounded.

If C/k is a smooth projective curve and J is its Jacobian, we have $\mathrm{CH}_0(X)_{\mathrm{hom}} = J(k)$. So 0-cycles on curves are quite manageable.

Property (C). There exists a nonsingular projective curve $j\colon C \hookrightarrow X$ such that the homomorphism $j_*\colon \mathrm{CH}_0(C)_{\mathbb{Q}} \to \mathrm{CH}_0(X)_{\mathbb{Q}}$ is surjective.

For the next property, fix a base point $x_0 \in X(k)$. (The choice of base point is not essential but it makes it easier to state the results.) The Albanese variety $\mathrm{Alb}(X)$ of (X,x_0) is an abelian variety that comes with a morphism $a\colon X \to \mathrm{Alb}(X)$ such that $\mathrm{alb}(x_0) = 0$, and which has the following universal property: whenever $f\colon X \to B$ is a morphism to an abelian variety with $f(x_0) = 0$, there is a unique homomorphism $h\colon \mathrm{Alb}(X) \to B$ such that $f = h \circ \mathrm{alb}$. (In particular, if X itself is an abelian variety then $\mathrm{Alb}(X) = X$.) We then have an induced homomorphism

$$\mathrm{alb}\colon \mathrm{CH}_0(X) \to \mathrm{Alb}(X)(k)$$

given by

$$[P_1 + \cdots + P_r - Q_1 - \cdots - Q_s] \mapsto a(P_1) + \cdots + a(P_r) - a(Q_1) - \cdots - a(Q_s)\,,$$

where in the right hand side we mean the result of addition and subtraction of points of $\mathrm{Alb}(X)$. If X itself is an abelian variety, this is the summation map S that we have used in Section 10. Because $\mathrm{alb}[x_0] = 0$, we may in fact restrict the map alb to the subgroup $\mathrm{CH}_0(X)_{\mathrm{hom}}$ without loosing any information.

Property (D). The map $\mathrm{alb}\colon \mathrm{CH}_0(X)_{\mathrm{hom}} \to \mathrm{Alb}(X)(k)$ is an isomorphism.

It turns out that if the base field k is big, the above properties are all equivalent. This uses results of Mumford and Roitman. An excellent reference is Section 1 of Jannsen's paper [26].

11.5 Theorem. *Let X be a smooth projective variety over an algebraically closed field k which is uncountable. Then the above properties* (A)–(D) *are all equivalent.*

All hopes that Chow groups might be 'small' were shattered by a result of Mumford [35] (which was inspired by work of Severi) which says that for a complex surface X, properties (A)–(D) do not hold if $p_g(X) > 0$. (Recall that the geometric genus $p_g(X)$ of a surface X/k is the k-dimension of $H^0(X, \Omega^2_{X/k})$, i.e., the Hodge number $h^{2,0}$.) The following result gives a generalization to higher dimensional varieties; for technical reasons we here restrict to $k = \mathbb{C}$ as base field, but closely related results are true more generally. The theorem as stated here is very close to Roitman's generalization [42] of Mumford's theorem.

11.6 Theorem. *Let X be a smooth projective complex variety of dimension g such that the above (equivalent) properties* (A)–(D) *hold. Then $H^0(X, \Omega^i_{X/\mathbb{C}}) = 0$ for all $i \geq 2$.*

An elegant proof of this result can be given using techniques due to Bloch and Srinivas [12]. A good reference is for instance [54], Chapter 10.

We now return to abelian varieties. Theorem 11.1 shows that for abelian varieties over $\overline{\mathbb{F}}_p$, the bound $I^{\star(g+1)} = 0$ from Theorem 10.7 is far from sharp in general. After the preceding discussion, it should not come as surprise that the situation is different over bigger base fields, as the following result of Bloch [10] shows.

11.7 Theorem. *Let k be an uncountable algebraically closed field of characteristic* 0*. If X is a g-dimensional abelian variety over k, we have $I^{\star g} \neq 0$.*

It should be noted that the analogous statement over (uncountable and algebraically closed) fields of characteristic $p > 0$ is false, in general. For instance, by the results of Fakhruddin in [19], $I^{\star g} = 0$ for every g-dimensional supersingular abelian variety.

11.8 Remark. By using the $\mathfrak{sl}_2$-action on $\mathrm{CH}(X)$, we see that $I_{\mathbb{Q}}^{\star s}$ injects into $\mathrm{CH}^i_{(s)}(X)$, for every $i \in \{0, \ldots, s\}$. Hence the results about 0-cycles also have implications for the Chow groups in other codimensions.

11.9. Further reading. There are many interesting further topics related to the themes of the previous sections. One that should certainly be mentioned is the Kimura–O'Sullivan theory of 'finite dimensional motives'. (We shall give the definition in 12.10.) The fascinating idea here is that, even though Chow groups tend to be very large, from a categorical perspective Chow motives of smooth projective varieties should behave as if they were finite dimensional objects. To read about this, we recommend [2], [28] and [25]. Related to this is a conjecture of Voevodsky that, on a smooth projective variety X, every cycle $\alpha \in \mathrm{CH}^i(X)$ which is numerically trivial should be *smash nilpotent*, which means that there exists a positive integer N such that $\alpha \times \cdots \times \alpha$ (N factors) is zero in $\mathrm{CH}^{iN}(X^N)$. By Voevodsky [52] and Voisin [53], this is known to hold for classes α that are algebraically trivial. While it is known that Chow motives of abelian varieties are indeed finite dimensional in the sense of Kimura–O'Sullivan (we shall briefly return to this in Section 13), Voevodsky's smash nilpotence conjecture for abelian varieties is known in general only for cycles of dimension ≤ 1, see [47]. It could be argued that Kimura–O'Sullivan finite dimensionality and Voevodsky's smash nilpotence conjecture are assertions about the complexity of Chow groups, more than about their size.

Another beautiful topic that should be mentioned is the study of tautological classes on Jacobians. Let C/k be a smooth projective curve of genus $g \geq 1$ with base point $x_0 \in C(k)$, and let J be its Jacobian. We have an embedding $C \hookrightarrow J$, given by $x \mapsto \mathscr{O}_C(x - x_0)$. This defines a 1-cycle $[C]$ on J, and we can consider the components $[C]_{(s)} \in \mathrm{CH}_{1,(s)}(J)$ of this class in the Beauville decomposition. The *tautological ring* $\mathscr{T}(J) \subset \mathrm{CH}(J)_{\mathbb{Q}}$ is defined as the smallest $\mathbb{Q}$-subalgebra for both ring structures (intersection product and Pontryagin product) that contains all classes $[C]_{(s)}$. If we work modulo rational equivalence, the ring that is obtained is independent of the chosen base point, and by a result of Beauville [6] it has finite $\mathbb{Q}$-dimension. By a famous result of Ceresa [14], for a very general complex curve C of genus $g \geq 3$ the class $C - C^-$ on J (where $C^- = [-1]^*(C)$), which is twice the sum of the terms $[C]_{(s)}$ with s odd, is *not* algebraically trivial, and it can be shown that in fact $[C]_{(1)} \not\sim_{\mathrm{alg}} 0$. For arbitrary C, there are subtle relations between the Brill-Noether properties of C and the vanishing of the classes $[C]_{(s)}$; a first instance of this is a result of Colombo and van Geemen [16]; this was later refined by Herbaut [24]. If we work in $\mathrm{CH}(J)_{\mathbb{Q}}$ rather than modulo algebraic equivalence, the tautological ring does depend on the chosen base point, but still turns out to be rich in structure. Extensive results on this were obtained by Polishchuk (see [40], [41]). Yin [55] extended many of these ideas to families of curves, and established a close relation to the tautological ring of the moduli space of curves.

Thus far in these notes, we have focused on Chow groups with $\mathbb{Q}$-coefficients, thereby killing all torsion. While the author is not aware of any specific conjectures about the torsion subgroup of $\mathrm{CH}(X)$, it is known that for a very general principally polarized complex abelian threefold, $\mathrm{CH}^2(X)/\ell$ is infinite for all prime numbers ℓ. First examples where $\mathrm{CH}(X)/n$ is infinite were given by Schoen [46]; for the very general complex abelian threefold it was shown by Rosenschon and Srinivas [44] that $\mathrm{CH}(X)^2/\ell$ is infinite for almost all ℓ, and this was extended to all ℓ by Totaro [51]. The proof uses that the very general principally polarized abelian threefold is the Jacobian of a curve, which is no longer true in dimension $g \geq 4$; it seems plausible, though, that a similar result is true for very general abelian varieties of higher dimension.

PART 3

Chow motives

12. The category of Chow motives

We here present a very brief introduction to the theory of Chow motives. We cannot do justice to this rich theory within a couple of pages, so the reader will have to read some of the available longer texts for a more detailed account. We hope, though, that the quick summary of basic facts presented here will make it easier to get an idea of what Chow motives are. For further reading we recommend [39] and [45], or the more panoramic overview given in [1].

12.1. Let k be a field. Let us first give a very informal explanation of what Chow motives are about. The idea is that every smooth projective variety Y/k should have an associated motive, which we denote by $\mathfrak{h}(Y)$, and that the (contravariant) functor $Y \mapsto \mathfrak{h}(Y)$ should behave like a universal cohomology theory, which means that every other (Weil) cohomology theory should factor through this functor. As discussed in Section 7, if Z is another smooth projective variety over k, the elements of $\mathrm{Corr}(Y,Z)_{\mathbb{Q}} := \mathrm{CH}(Y \times Z)_{\mathbb{Q}}$, called correspondences, may be viewed as generalized maps from Y to Z, and correspondences can be composed. In particular, $\mathrm{Corr}(Y,Y)_{\mathbb{Q}} = \mathrm{CH}(Y \times Y)_{\mathbb{Q}}$ becomes a ring under composition, and this ring naturally acts on $\mathrm{CH}(Y)_{\mathbb{Q}}$. The class $[\Delta_Y]$ of the diagonal (which is the graph of the identity morphism) is the identity element in this ring.

If $\pi \in \mathrm{Corr}(Y,Y)_{\mathbb{Q}}$ is an idempotent (meaning that $\pi \circ \pi = \pi$) then so is $\pi' = [\Delta_Y] - \pi$, and using these projectors,[1] we can decompose the motive $\mathfrak{h}(Y)$ as a direct sum of two pieces, namely the images of π and π'. In this way, we obtain new motives; these are not simply smooth projective varieties but 'summands' of such. It is as if smooth projective varieties are molecules and motives are the atoms of which they are composed.

In any (Weil) cohomology theory H we have, for a d-dimensional smooth projective variety Y/k, a decomposition $H(Y) = H^0(Y) \oplus H^1(Y) \oplus \cdots \oplus H^{2d}(Y)$. Analogously, we expect that there exist mutually orthogonal projectors π^j, for $j = 0, \ldots, 2d$ in $\mathrm{Corr}(Y,Y)$ such that $[\Delta_Y] = \pi^0 + \pi^1 + \cdots + \pi^{2d}$ and such that the corresponding motivic decomposition $\mathfrak{h}(Y) = \mathfrak{h}^0(Y) \oplus \mathfrak{h}^1(Y) \oplus \cdots \oplus \mathfrak{h}^{2d}(Y)$ is the analogue of the decomposition in cohomology. See Section 12.7 below. Unfortunately, the existence of such a decomposition is not known in general—but for abelian varieties this *is* known, and this in fact gives a deeper interpretation of the Beauville decomposition. We shall discuss this in the next section.

Before we give a more formal definition, there is one more idea from cohomology theory that is important. Namely, part of a Weil cohomology theory is that there are Tate twists; this is intimately related to the idea that objects have a weight. In the theory of Chow motives, we shall have such Tate twists as well.

12.2 Definition. Let k be a field. The category $\mathsf{CHM}(k)$ of Chow motives over k is obtained as follows:

(1) The objects of $\mathsf{CHM}(k)$ are triples (X, π, m), where X is a smooth projective k-scheme, $\pi \in \mathrm{Corr}(X,X)_{\mathbb{Q}}$ is an idempotent, and $m \in \mathbb{Z}$.

[1] we use the words idempotent and projector as synonyms

(2) For Chow motives (X, π, m) and (Y, ρ, n), the morphisms $\phi\colon (X, \pi, m) \to (Y, \rho, n)$ in $\mathsf{CHM}(k)$ are the elements of the subspace

$$\rho \circ \mathrm{Corr}^{n-m}(X, Y)_{\mathbb{Q}} \circ \pi \subset \mathrm{Corr}^{n-m}(X, Y)_{\mathbb{Q}}\,,$$

i.e., the correspondences that are of the form $\rho \circ \gamma \circ \pi$ with $\gamma \in \mathrm{Corr}^{n-m}(X, Y)_{\mathbb{Q}}$.

(3) Composition of morphisms is given by the composition of correspondences, as in 7.2.

In this definition, recall that $\mathrm{Corr}^i(X, Y)_{\mathbb{Q}} = \mathrm{CH}^{\dim(X)+i}(X \times Y)_{\mathbb{Q}}$. The following is no more than an exercise in unravelling the definitions:

12.3 Proposition. *Let $\mathsf{SmProj}(k)$ denote the category of smooth projective k-schemes. Then we have a functor*

$$\mathfrak{h}\colon \mathsf{SmProj}(k)^{\mathrm{op}} \to \mathsf{CHM}(k)$$

that sends a smooth projective X/k to the motive $\mathfrak{h}(X) = (X, [\Delta_X], 0)$ and sends a morphism $f\colon X \to Y$ to the morphism $f^*\colon \mathfrak{h}(Y) \to \mathfrak{h}(X)$ given by the correspondence $[{}^{\mathrm{t}}\Gamma_f] \in \mathrm{Corr}^0(Y, X)_{\mathbb{Q}}$.

Instead of (X, π, m) one often uses the more suggestive notation $\pi\mathfrak{h}(X)\big(m\big)$. Here $\mathfrak{h}(X)\big(m\big) = (X, [\Delta_X], m)$ should be read as 'the motive of X, Tate-twisted by m', and $\pi\mathfrak{h}(X)\big(m\big)$ is then the direct summand that is obtained as the image of the projector π.

If $M = (X, \pi, m)$ is a motive then $\pi \circ [\Delta_X] \circ \pi \in \mathrm{Corr}^0(X, X)_{\mathbb{Q}}$ is the identity element in $\mathrm{End}_{\mathsf{CHM}(k)}(M)$.

12.4. Basic facts. The category of Chow motives over a field k has a number of agreeable properties. Here are a couple of them.

(1) The category $\mathsf{CHM}(k)$ is a $\mathbb{Q}$-linear category. In this category we have finite direct sums. If (X, π, m) and (Y, ρ, n) are two objects and $m = n$ then the direct sum is given by

$$(X, \pi, m) \oplus (Y, \rho, n) = (X \sqcup Y, \pi + \rho, m)\ .$$

(Slogan: the direct sum of motives comes from the disjoint union of schemes.) If $m \neq n$, the construction is slightly more involved; we shall return to it in Example 12.8.

(2) The category $\mathsf{CHM}(k)$ is Karoubian, which means that whenever M is an object and $\pi \in \mathrm{End}_{\mathsf{CHM}(k)}(M)$ is an idempotent, we have a decomposition $M = M_1 \oplus M_2$ such that π is the composition $M \twoheadrightarrow M_1 \hookrightarrow M$ and $\mathrm{id}_M - \pi$ is $M \twoheadrightarrow M_2 \hookrightarrow M$. (In other words: $M_1 = \pi M$ is the summand that is cut out by π and M_2 is the image of $\mathrm{id}_M - \pi$.) Note, however, that $\mathsf{CHM}(k)$ is certainly not an abelian category, unless possibly when k is contained in the algebraic closure of a finite field; this is explained for instance in [45], Section 3.

(3) The category $\mathsf{CHM}(k)$ has a tensor product, given by

$$(X, \pi, m) \otimes (Y, \rho, n) = (X \times_k Y, \pi \otimes \rho, mn)\,;$$

here $\pi \otimes \rho \in \mathrm{Corr}(X \times Y, X \times Y)_{\mathbb{Q}}$ is given by the rule $\pi \otimes \rho = \mathrm{pr}_{13}^*(\pi) \cdot \mathrm{pr}_{24}^*(\rho)$, where $\mathrm{pr}_{13}\colon X \times Y \times X \times Y \to X \times X$ and $\mathrm{pr}_{24}\colon X \times Y \times X \times Y \to Y \times Y$ are the projections.

(Slogan: the tensor product of motives comes from the product of schemes.) This makes $\mathsf{CHM}(k)$ into a symmetric monoidal category.

(4) The motive $\mathbf{1} = \mathfrak{h}\big(\mathrm{Spec}(k)\big)$ is the identity for the tensor product. For $n \in \mathbb{Z}$ we define $\mathbf{1}(n) = \big(\mathrm{Spec}(k), [\Delta], n\big)$; these objects satisfy $\mathbf{1}(m) \otimes \mathbf{1}(n) = \mathbf{1}(m+n)$. If $M = (X, \pi, m)$ is a Chow motive and $n \in \mathbb{Z}$, we define $M(n) = (X, \pi, m+n) = M \otimes \mathbf{1}(n)$.

12.5. Chow groups of motives. If M is a Chow motive, we define its Chow groups by

$$\mathrm{CH}^i(M) = \mathrm{Hom}_{\mathsf{CHM}(k)}\big(\mathbf{1}(-i), M\big) = \mathrm{Hom}_{\mathsf{CHM}(k)}\big(\mathbf{1}, M(i)\big)\,.$$

As a sanity check, note that for $M = \mathfrak{h}(Y)$ this gives

$$\mathrm{CH}^i\big(\mathfrak{h}(Y)\big) = \mathrm{Corr}^i(\mathrm{Spec}(k), Y)_{\mathbb{Q}} = \mathrm{CH}^i(Y)_{\mathbb{Q}}\,.$$

Further note that $\mathrm{CH}^i\big(M(n)\big) = \mathrm{CH}^{i+n}(M)$, so the effect of a Tate twist is to shift the grading.

12.6. Cohomological realizations. Let H be a Weil cohomology theory on smooth projective varieties over k with coefficient field Q. In particular, this means that H is a functor $H\colon \mathsf{SmProj}(k)^{\mathrm{op}} \to \mathsf{GrVec}(Q)$ (the category of graded Q-vector spaces). Then H factors via the functor $\mathfrak{h}$ of Proposition 12.3, i.e., there exists a functor $\bar{H}\colon \mathsf{CHM}(k) \to \mathsf{GrVec}(Q)$ such that $H = \bar{H} \circ \mathfrak{h}$. (In practice, we again write H instead of $\bar{H}$ for this new functor.)

In many situations we have something stronger. The classical Weil cohomology theories all take values in a more interesting category. For instance, de Rham cohomology takes values in the category of filtered k-vector spaces, and singular cohomology (for $k = \mathbb{C}$) takes values in the category of polarizable $\mathbb{Q}$-Hodge structures. If ℓ is a prime number different from the characteristic of the base field, ℓ-adic cohomology takes values in the category of $\mathbb{Q}_\ell$-vector spaces equipped with a continuous action of $\mathrm{Gal}(k^{\mathrm{sep}}/k)$. In each of these examples, the functor $\bar{H}$ lifts to a functor with values in the relevant target category.

12.7. Chow–Künneth decompositions. It was conjectured by Murre [38] that for every smooth projective variety X/k, say of dimension d, there exist mutually orthogonal projectors

$$\pi^i \in \mathrm{End}_{\mathsf{CHM}(k)}\big(\mathfrak{h}(X)\big) = \mathrm{Corr}^0(X, X)_{\mathbb{Q}}\,, \qquad i = 0, \ldots, 2d,$$

such that:

- $[\Delta_X] = \pi^0 + \pi^1 + \cdots + \pi^{2d}$, so that we obtain a decomposition

  $$\mathfrak{h}(X) = \mathfrak{h}^0(X) \oplus \mathfrak{h}^1(X) \oplus \cdots \oplus \mathfrak{h}^{2d}(X)\,,$$

  where $\mathfrak{h}^i(X) = \pi^i\mathfrak{h}(X)$;
- for any Weil cohomology theory H, the induced decomposition of $H(X) = H\big(\mathfrak{h}(X)\big)$ is the usual cohomological decomposition $H(X) = H^0(X) \oplus \cdots \oplus H^{2d}(X)$.

Such a decomposition of $\mathfrak{h}(X)$—if it exists—is called a *Chow–Künneth decomposition*, sometimes abbravtiated to CK decomposition. Murre's conjecture about the existence of such a Chow–Künneth decomposition is completely open in general. (A notable exception is the case where the base field is contained in $\overline{\mathbb{F}}_p$ for some p.) Note that if there exists a CK decomposition of $\mathfrak{h}(X)$, it will usually be non-unique; see below.

While the existence of CK decompositions is one of the major open problems in the theory, there are some small pieces of the puzzle that we do have. For instance, suppose X has a k-rational point $x_0 \in X(k)$. Then

$$\pi^0 = [\{x_0\} \times X]\,, \quad \pi^{2d} = [X \times \{x_0\}]\,, \quad \pi' = [\Delta_X] - \pi^0 - \pi^{2d}$$

are mutually orthogonal idempotents in $\mathrm{Corr}^0(X,X)$ that define a decomposition $\mathfrak{h}(X) = \mathfrak{h}^0(X) \oplus \mathfrak{h}'(X) \oplus \mathfrak{h}^{2d}(X)$ whose cohomological realization (in any Weil cohomology theory H) is

$$H^0(X) \oplus \Big[\oplus_{i=1}^{2d-1} H^i(X)\Big] \oplus H^{2d}(X)\,.$$

Already here we see that we should expect CK projectors to be non-unique: in general, a different choice of base point x_0 gives different projectors π^0 and π^{2d}. The motives $\mathfrak{h}^0(X)$ and $\mathfrak{h}^{2d}(X)$ are easy to understand, in fact,

$$\mathfrak{h}^0(X) \cong \mathbf{1}\,, \qquad \mathfrak{h}^{2d}(X) \cong \mathbf{1}(-d)\,.$$

To see this, note that $x_0\colon \mathrm{Spec}(k) \to X$ gives rise to a morphism $x_0^*\colon \mathfrak{h}(X) \to \mathbf{1}$ and also to a morphism $x_{0,*}\colon \mathbf{1}(-d) \to \mathfrak{h}(X)$, whereas the structural morphism $a\colon X \to \mathrm{Spec}(k)$ induces morphisms $a^*\colon \mathbf{1} \to \mathfrak{h}(X)$ and also $a_*\colon \mathfrak{h}(X) \to \mathbf{1}(-d)$. Because $a \circ y_0$ is the identity on $\mathrm{Spec}(k)$ we get $a^* \circ x_0^* = \mathrm{id}$ and $x_{0,*} \circ a_* = \mathrm{id}$; on the other hand, it is easy to verify that $x_0^* \circ a^* = \pi^0$ and $a_* \circ x_{0,*} = \pi^{2d}$.

Going one step further, it is known by the work of Murre [37] how to define projectors π^1 and π^{2d-1} such that (for some fixed $x_0 \in X(k)$) the projectors

$$\pi^0\,, \quad \pi^1\,, \quad \pi'' = [\Delta_X] - \pi^0 - \pi^1 - \pi^{2d-1} - \pi^{2d}\,, \quad \pi^{2d-1}\,, \quad \pi^{2d}$$

are mutually orthogonal, and such that the resulting decomposition

$$\mathfrak{h}(X) = \mathfrak{h}^0(X) \oplus \mathfrak{h}^1(X) \oplus \mathfrak{h}''(X) \oplus \mathfrak{h}^{2d-1}(X) \oplus \mathfrak{h}^{2d}(X)$$

lifts the corresponding decomposition in cohomology, with $H\big(\mathfrak{h}''(X)\big) = \oplus_{i=2}^{2d-2} H^i(X)$.

12.8 Example. Take a point $x_0 \in \mathbb{P}^1(k)$. The resulting projectors $\pi^0 = [\{x_0\} \times \mathbb{P}^1]$ and $\pi^2 = [\mathbb{P}^1 \times \{x_0\}]$ are independent of the choice of x_0 because any two k-rational points on $\mathbb{P}^1$ are rationally equivalent. We have $[\Delta] = \pi^0 + \pi^2$ in $\mathrm{Corr}^0(\mathbb{P}^1,\mathbb{P}^1)_{\mathbb{Q}} = \mathrm{CH}^1(\mathbb{P}^1 \times \mathbb{P}^1)_{\mathbb{Q}}$. (Use that $\mathrm{Pic}(\mathbb{P}^1 \times \mathbb{P}^1) = \mathbb{Z}^2$ with the classes of $\{x_0\} \times \mathbb{P}^1$ and $\mathbb{P}^1 \times \{x_0\}$ as generators.) This gives us a decomposition

$$\mathfrak{h}(\mathbb{P}^1) = \mathfrak{h}^0(\mathbb{P}^1) \oplus \mathfrak{h}^2(\mathbb{P}^1)$$

with $\mathfrak{h}^0(\mathbb{P}^1) \cong \mathbf{1}$ and $\mathfrak{h}^2(\mathbb{P}^1) \cong \mathbf{1}(-1)$.

With this, we can also describe direct sums of motives in general. As in 12.4(1), suppose we have two Chow motives $M = \pi\mathfrak{h}(X)(m) = (X,\pi,m)$ and $N = \rho\mathfrak{h}(Y)(n) = (Y,\rho,n)$, say with $m \leq n$. We have already seen how to describe the direct sum in case $m = n$. Assume then that $m < n$. The trick is to use that $M \cong \bigl[\pi\mathfrak{h}(X) \otimes \mathfrak{h}^2(\mathbb{P}^1)^{\otimes(n-m)}\bigr](n)$, so that $M \oplus N \cong \bigl[\bigl(\pi\mathfrak{h}(X) \otimes \mathfrak{h}^2(\mathbb{P}^1)^{\otimes(n-m)}\bigr) \oplus \rho\mathfrak{h}(Y)\bigr](n)$.

12.9. Let X/k be a smooth projective variety. The diagonal morphism $\delta\colon X \to X \times_k X$ defines a morphism of Chow motives

$$\delta^*\colon \mathfrak{h}(X) \otimes \mathfrak{h}(X) \to \mathfrak{h}(X)\,.$$

The induced map on Chow rings is the intersection product; in other words: if $\alpha \in \mathrm{CH}^i(X)_\mathbb{Q}$ and $\beta \in \mathrm{CH}^j(X)_\mathbb{Q}$ correspond to the morphisms $\alpha\colon \mathbf{1}(-i) \to \mathfrak{h}(X)$ and $\beta\colon \mathbf{1}(-j) \to \mathfrak{h}(X)$ then the composition

$$\mathbf{1}(-i-j) \cong \mathbf{1}(-i) \otimes \mathbf{1}(-j) \xrightarrow{\alpha\otimes\beta} \mathfrak{h}(X) \otimes \mathfrak{h}(X) \xrightarrow{\delta^*} \mathfrak{h}(X)$$

corresponds to the class $\alpha \cdot \beta \in \mathrm{CH}^{i+j}(X)_\mathbb{Q}$. Similarly, the induced map in cohomology is the cup-product.

A Chow–Künneth decomposition as in 12.7 is said to be *multiplicative* if δ^* restricts to morphisms $\mathfrak{h}^i(X)\otimes\mathfrak{h}^j(X) \to \mathfrak{h}^{i+j}(X)$ for all i, $j \geq 0$. As we shall discuss in the next section, an abelian variety has such a multiplicative CK decomposition. Perhaps surprisingly, it is known that for general smooth projective varieties X/k there does not exist a multiplicative CK decomposition. For a very nice discussion of this, we refer to the paper [20].

12.10. The tensor product of motives makes $\mathsf{CHM}(k)$ into a $\mathbb{Q}$-linear symmetric monoidal category. As a result, notions such as symmetric and exterior products, and more general Schur functors, make sense in $\mathsf{CHM}(k)$. To define $\mathrm{Sym}^n(M)$ and $\wedge^n(M)$, for M a Chow motive and $n \geq 0$, consider the action of the symmetric group $\mathfrak{S}_n$ on the motive $M^{\otimes n} = M \otimes \cdots \otimes M$ by permutation of the factors. Then

$$s_n = \tfrac{1}{n!} \sum_{\sigma\in\mathfrak{S}_n} \sigma$$

is an idempotent in $\mathrm{End}_{\mathsf{CHM}(k)}(M)$, and $\mathrm{Sym}^n(M)$ is defined as the image of this projector. Similarly, $\wedge^n(M)$ is defined as the image of the projector $\lambda_n = \frac{1}{n!}\sum_{\sigma\in\mathfrak{S}_n} \mathrm{sgn}(\sigma)\cdot\sigma$.

A motive M is said to be *evenly finite dimensional* if there exists an integer $n \geq 1$ such that $\wedge^n(M) = 0$, and *oddly finite dimensional* if there exists an integer $n \geq 1$ such that $\mathrm{Sym}^n(M) = 0$. Finally, M is said to be *finite dimensional* in the sense of Kimura–O'Sullivan if there exists a decomposition $M = M^+ \oplus M^-$ such that M^+ is evenly finite dimensional and M^- is oddly finite dimensional. It is known that the class of finite dimensional motives is closed under direct factors, direct sums and tensor products, and contains all motives of abelian varieties (see Corollary 13.5 below). Beyond that, the Kimura–O'Sullivan conjecture that all Chow motives are finite dimensional is widely open.

12.11 Remark. An additive category may be viewed as a 'ring with many objects'. (This idea was developed notably by Mitchell and Street.) This is a very useful perspective for the theory of motives. For a detailed discussion, see for instance the paper [3] by André and Kahn. It was shown by Jannsen in [27] that adequate equivalence relations on algebraic cycles correspond to $\otimes$-ideals in $\mathsf{CHM}(k)$. Jannsen also showed that each step Fil^i in the conjecturel filtration of Bloch and Beilinson (see 8.11) may be viewed as a $\otimes$-ideal, and that if the Bloch–Beilinson conjecture and the Lefschetz type standard conjecture are true, Fil^i equals the ith power if the ideal $\mathscr{I}_{\mathrm{hom}}$ that corresponds to the relation 'homological equivalence'. See also [2], Théorème 2.13.

12.12. Variant: the covariant (homological) category of Chow motives. For some purposes, it is more convenient to work in a category of motives that plays the role of a homology theory, rather than a cohomology theory. This means we want to construct a category $\mathsf{CHM}_\bullet(k)$ (we use a dot to distinguish it from the category $\mathsf{CHM}(k) = \mathsf{CHM}^\bullet(k)$ as above), together with a covariant functor $\mathfrak{h}_\bullet \colon \mathsf{SmProj}(k) \to \mathsf{CHM}_\bullet(k)$. The construction is almost the same as for the contravariant version; we here only give a summary.

If X and Y are smooth projective over k with $\dim(X) = d$, define $\mathrm{Corr}_i(X,Y) = \mathrm{CH}_{d+i}(X \times Y)$. Composition of correspondences defines maps $\mathrm{Corr}_i(X,Y) \times \mathrm{Corr}_j(Y,Z) \to \mathrm{Corr}_{i+j}(X,Z)$. The objects of $\mathsf{CHM}_\bullet(k)$ are triples (X,p,m) with X/k smooth projective, $p \in \mathrm{Corr}_0(X,X)$ an idempotent (i.e., $p \circ p = p$), and $m \in \mathbb{Z}$. If (Y,q,n) is a second such object, we define

$$\mathrm{Hom}_{\mathsf{CHM}_\bullet(k)}\big((X,p,m),(Y,q,n)\big) = q \circ \mathrm{Corr}_{m-n}(X,Y) \circ p\,,$$

which is a subspace of $\mathrm{Corr}_{m-n}(X,Y)$, and composition of morphisms is given by composition of correspondences.

We have a covariant functor $\mathfrak{h}_\bullet$ that sends X to $\mathfrak{h}_\bullet(X) = (X,[\Delta],0)$ and that sends $f \colon X \to Y$ to the morphism $f_* \colon \mathfrak{h}_\bullet(X) \to \mathfrak{h}_\bullet(Y)$ given by $[\Gamma_f] \in \mathrm{Corr}_0(X,Y)$.

We define $\mathbf{1}(n) = \big(\mathrm{Spec}(k),\mathrm{id},n\big)$, and then the Chow groups of a motive M are defined by $\mathrm{CH}_n(M) = \mathrm{Hom}_{\mathsf{CHM}_\bullet(k)}(\mathbf{1}(n),M)$. As a sanity check: for $M = \mathfrak{h}_\bullet(X)$ this gives

$$\mathrm{Hom}_{\mathsf{CHM}_\bullet(k)}\big(\mathbf{1}(n),\mathfrak{h}_\bullet(X)\big) = \mathrm{Corr}_n(\mathrm{Spec}(k),X) = \mathrm{CH}_n(X)\,,$$

which is what we want.

The category $\mathsf{CHM}_\bullet(k)$ thus obtained is isomorphic to the opposite of the 'cohomological' category of Chow motives $\mathsf{CHM}(k) = \mathsf{CHM}^\bullet(k)$, via the functor $D \colon \mathsf{CHM}_\bullet(k)^{\mathrm{op}} \to \mathsf{CHM}^\bullet(k)$ that sends an object (X,p,m) of $\mathsf{CHM}_\bullet(k)$ to the object $(X,{}^{\mathrm{t}}p,-m)$ of $\mathsf{CHM}^\bullet(k)$ and that sends a morphism

$$q \circ \alpha \circ p \quad \in \quad q \circ \mathrm{Corr}_{m-n}(X,Y) \circ p = \mathrm{Hom}_{\mathsf{CHM}^\bullet(k)}\big((X,p,m),(Y,q,n)\big)$$

to

$${}^{\mathrm{t}}p \circ {}^{\mathrm{t}}\alpha \circ {}^{\mathrm{t}}q \quad \in \quad {}^{\mathrm{t}}p \circ \mathrm{Corr}^{-m+n}(Y,X) \circ {}^{\mathrm{t}}q = \mathrm{Hom}_{\mathsf{CHM}^\bullet(k)}\big((Y,{}^{\mathrm{t}}q,-n),(X,{}^{\mathrm{t}}p,-m)\big)\,.$$

As special cases of this, $D\big(\mathfrak{h}_\bullet(X)\big) = \mathfrak{h}^\bullet(X)$, and for a morphism $f \colon X \to Y$ we have $D(f_*) = f^*$. Further note that D sends $\mathbf{1}(n) \in \mathsf{CHM}_\bullet(k)$ to $\mathbf{1}(-n) \in \mathsf{CHM}^\bullet(k)$

13. Chow motives of abelian varieties

The following result is due to Deninger and Murre [17].

13.1 Theorem. *Let X be a g-dimensional abelian variety over a field k. Then there exist a Chow–Künneth decomposition*

$$[\Delta_X] = \pi^0 + \pi^1 + \cdots + \pi^{2g}$$

of the diagonal as a sum of mutually orthogonal projectors, such that the resulting decomposition of the Chow motive

$$\mathfrak{h}(X) = \mathfrak{h}^0(X) \oplus \mathfrak{h}^1(X) \oplus \cdots \oplus \mathfrak{h}^{2g}(X)$$

is stable under all endomorphisms $[n]^*$ for $n \in \mathbb{Z}$, and such that $[n]^*$ induces multiplication by n^i on $\mathfrak{h}^i(X)$. This decomposition is multiplicative, in the sense that the restriction of $\delta^*\colon \mathfrak{h}(X) \otimes \mathfrak{h}(X) \to \mathfrak{h}(X)$ to $\mathfrak{h}^i(X) \otimes \mathfrak{h}^j(X)$ factors through $\mathfrak{h}^{i+j}(X)$ for all i, $j \geq 0$. The induced decomposition of $\mathrm{CH}(X)_{\mathbb{Q}}$ is the same as Beauville's decomposition.

Deninger and Murre in fact prove the analogous statement in the setting of abelian schemes over some smooth quasi-projective base scheme. This level of generality is needed even if one is interested only in abelian varieties over fields. The sought-for Chow–Künneth decomposition is a decomposition of the class $[\Delta_X] \in \mathrm{CH}^g(X \times_k X)_{\mathbb{Q}}$. A natural idea is to consider the Beauville decomposition of $\mathrm{CH}(X \times X)_{\mathbb{Q}}$ and to use this to 'decompose' the class of the diagonal. This, however, gives nothing of interest, as it is not hard to show that $[n]^*_{X\times X}$ acts as multiplication by n^{2g} on the class of the diagonal, so the class $[\Delta_X]$ already lies in one of the summands in the Beauville decomposition. Instead, one wants to consider the endomorphisms $(n, \mathrm{id})\colon (x_1, x_2) \mapsto (nx_1, x_2)$ of $X \times X$ and use these to decompose the Chow group; but this goes beyond the setting of Beauville's decomposition. To remedy this, one should view $X \times X$ as an abelian scheme over X via the second projection, because then (n, id) becomes the usual multiplication by n endomorphism of this abelian scheme. To make this work requires a generalization of Fourier duality and Beauville's decomposition to the relative setting. This is precisely what Deninger and Murre do in their paper.

13.2 Remark. The components $\mathfrak{h}^0(X)$ and $\mathfrak{h}^{2g}(X)$ are the same as in the discussion of Section 12.7 (here applied with $d = g$), taking the origin of X as base point. In particular, $\mathfrak{h}^0(X) \cong \mathbf{1}$ and $\mathfrak{h}^{2g}(X) \cong \mathbf{1}(-g)$.

Explicit formulas for all projectors π^i can be found in [30]. There is also another concrete way to describe these. Namely, consider the matrix M indexed by $\{0, \ldots, 2g\}^2$ with coefficients $M_{rs} = r^s$. By Vandermonde this matrix is invertible. If Q is the inverse matrix, the projectors π^i are given by $\pi^i = \sum_{n=0}^{2g} q_{in} \cdot [n]^*_X$.

The next result, due to Künnemann [29], expresses the motives $\mathfrak{h}^n(X)$ in terms of $\mathfrak{h}^1(X)$. If X is a g-dimensional abelian variety and $n \in \{0, \ldots, 2g\}$ then by $\mathfrak{h}^n(X)$ we mean the motives as in the Deninger–Murre decomposition. For $n > 2g$ we define $\mathfrak{h}^n(X) = 0$.

13.3 Theorem. *Let X be an abelian variety over a field k, and let n be a natural number. Then the morphism $\mathfrak{h}^1(X)^{\otimes n} \to \mathfrak{h}(X)$ given by the diagonal morphism $X \to X^n$ induces an isomorphism* $\mathrm{Sym}^n\big(\mathfrak{h}^1(X)\big) \xrightarrow{\sim} \mathfrak{h}^n(X)$.

In any Weil cohomology theory we have $H^n(X) \cong \wedge^n H^1(X)$, and one might therefore think that the correct assertion should be that $\mathfrak{h}^n(X)$ is isomorphic to $\wedge^n \mathfrak{h}^1(X)$. (In fact, this is the way the result is stated in [29].) However, the cohomology ring $H(X)$ is a supercommutative algebra (i.e., it is graded-commutative, meaning that for homogeneous elements x and y of degrees p and q we have $y \cup x = (-1)^{pq} \cdot x \cup y$). In the setting of superalgebras, because $H^1(X) \subset H(X)$ is an odd subspace, $\mathrm{Sym}^n\big(H^1(X)\big)$ has $\wedge^n H^1(X)$ as its underlying vector space.

It should be noted that the theorem has no immediate consequences for Chow groups. For a motive M, we have homomorphisms

$$\mathrm{CH}^{i_1}(M) \otimes \cdots \otimes \mathrm{CH}^{i_n}(M) \to \mathrm{CH}^{i_1+\cdots+i_n}(M^{\otimes n}),$$

but in general $\mathrm{CH}^i(M^{\otimes n})$ is not spanned by the images of such maps. Therefore, there is in general no way to express $\mathrm{CH}(M^{\otimes n})$ (of which $\mathrm{CH}(\mathrm{Sym}^n(M))$ is a direct factor) in terms of $\mathrm{CH}(M)$.

13.4 Remark. Künnemann's theorem shows that the entire Chow motive of X can be expressed in terms of $\mathfrak{h}^1(X)$. This is analogous to the fact that in any cohomology theory, $H(X)$ is the exterior algebra on $H^1(X)$. (Note that, though we present Chow motives as some kind of 'cohomology theory', in many ways the theory of Chow motives is far more subtle. Chow motives are not simply some kind of vector spaces with additional structure.)

A useful fact is that also the endomorphism algebra of X can be recovered from the motive $\mathfrak{h}^1(X)$, in the sense that we have a canonical isomorphism

$$\mathrm{End}^0(X) \xrightarrow{\sim} \mathrm{End}_{\mathsf{CHM}(k)}\big(\mathfrak{h}^1(X)\big).$$

(See [45], Corollary 5.10.)

13.5 Corollary. *If X/k is a g-dimensional abelian variety,* $\mathrm{Sym}^{2g+1}(\mathfrak{h}^1(X)) = 0$*, and* $\mathfrak{h}(X)$ *is a finite dimensional motive.*

Proof. The first assertion follows from the fact that $\mathfrak{h}^{2g+1}(X) = 0$. The second assertion then follows from the theorem together with the fact that the class of finite dimensional motives is stable under taking direct factors, direct sums and tensor products. □

To conclude, we mention the result, also due to Künnemann [29], that the $\mathfrak{sl}_2$-action that we have discussed in Section 9 is motivic.

13.6 Theorem. *Let (X, θ) be a g-dimensional polarized abelian variety over a field k. Then there is an action of the Lie algebra $\mathfrak{sl}_2$ on $\mathfrak{h}(X)$ such that $h \in \mathfrak{sl}_2$ acts on $\mathfrak{h}^i(X)$ as multiplication by $i - g$, and such that the induced $\mathfrak{sl}_2$-action on* $\mathrm{CH}(X)_{\mathbb{Q}}$ *is the one of* Theorem 9.4.

13.7. Further reading. The main results discussed in these notes are several decades old by now, and it should be clear that there is a lot of more recent research that goes beyond the scope of these notes. As a very inspiring example of how the story continues, let me mention the recent work of Maulik, Shen and Yin [32] in which they extend Fourier duality and the Beauville decomposition (on a motivic level) to families of abelian varieties with degenerate fibres; they then use this to give a new proof of the so-called 'P=W conjecture' in nonabelian Hodge theory.

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